ON THE GRAM DETERMINANT AND LINEAR TRANSFORMATIONS OF HILBERT SPACE

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1. Introduction. Recently Everitt proved two theorems concerning inequalities for Gram determinants of $L^2$-integrable functions [2]. The first of these theorems can be found — as the author remarks — in [1].

Moppert generalized these inequalities in [3]. He replaced the partial map by an arbitrary orthogonal projection of a Hilbert space. This was based on the following remark: Let $E_1$ and $E_2$ be two measurable sets and $E_1 \subset E_2$. Let $L^2(E_i)$ be the space of $L^2$-integrable functions in $E_i$ ($i = 1, 2$). The space $L^2(E_1)$ can then be considered as a subspace of $L^2(E_2)$. Let $\varphi_{E_1}$ be the characteristic function of the set $E_1$. Then the map $f \rightarrow \varphi_{E_1}f$ is an orthogonal projection of the space $L^2(E_2)$ on the subspace $L^2(E_1)$.

In the present paper a generalization of the results of Everitt and Moppert will be given. The projections are here replaced by arbitrary bounded linear transformations. It is shown how the (generalized) Courant-Hilbert inequality can be deduced as a consequence of (generalized) Everitt inequality. A necessary and sufficient condition for equality in (generalized) Courant-Hilbert inequality will be given and finally a simple «geometric» interpretation is pointed out.

2. The Gram determinant. Let $H$ be a Hilbert space. The inner product of $x$ and $y$ is denoted by the symbol $(x, y)$ and the norm of vector $x$ by $|x|$. The Gram determinant of vectors $x_1, \ldots, x_n$ is

$$G(x_1, x_2, \ldots, x_n) = \begin{vmatrix} (x_1, x_1) & (x_1, x_2) & \cdots & (x_1, x_n) \\ (x_2, x_1) & (x_2, x_2) & \cdots & (x_2, x_n) \\ \cdots & \cdots & \cdots & \cdots \\ (x_n, x_1) & (x_n, x_2) & \cdots & (x_n, x_n) \end{vmatrix}$$

Obviously $G(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = G(x_1, x_2, \ldots, x_n)$, where $\sigma$ is any permutation of numbers $1, 2, \ldots, n$. It is easy to see that $G(x_1, x_2, \ldots, x_n) \geq 0$ and $G(x_1, x_2, \ldots, x_n) = 0$ if and only if the vectors $x_i$ are linearly dependent.

1) The scalars may be real or complex. We do not assume that $H$ is separable.
We denote the linear subspace generated by vectors \( y_1, y_2, \ldots, y_k \) by \( L(y_1, y_2, \ldots, y_k) \).

Let now \( x_1, x_2, \ldots, x_n \) be arbitrary vectors in \( H \). We set

\[
x_n = x_n' + x_n'' ,
\]

where \( x_n'' \) is the orthogonal projection of \( x_n \) on the space \( L(x_1, x_2, \ldots, x_{n-1}) \).

In other words: \( (x_i', x_i) = 0 \), when \( i = 1, 2, \ldots, n-1 \), and \( x_n'' \in L(x_1, x_2, \ldots, x_{n-1}) \). It is well-known that \( x_n' \) and \( x_n'' \) are unique. By bilinearity of the inner product and elementary properties of determinants we have

\[
G(x_1, x_2, \ldots, x_n) = G(x_1, x_2, \ldots, x_{n-1}) |x_n'|^2 .
\]

Repeating in the same manner we set \( x_h = x_h' + x_h'' (h = n-1, n-2, \ldots, 2) \), where \( x_h'' \) is the orthogonal projection of \( x_h \) on the subspace \( L(x_1, x_2, \ldots, x_{h-1}) \), and we thus have

\[
G(x_1, x_2, \ldots, x_n) = |x_1|^2 |x_2'|^2 \cdots |x_n'|^2 .
\]

If the vectors \( x_i \) are linearly independent we have \( x_1 \neq 0 \) and \( x_i' \neq 0 \) (\( i = 2, \ldots, n \)) and then \( G(x_1, x_2, \ldots, x_n) > 0 \).

3. Inequalities. From (1) we get immediately the inequality

\[
G(x_1, x_2, \ldots, x_n) \leq G(x_1, x_2, \ldots, x_{n-1}) |x_n|^2 ,
\]

where equality holds if and only if one (or both) of the following conditions is satisfied: 1) The vectors \( x_1, x_2, \ldots, x_{n-1} \) are linearly dependent. 2) \( x_n = x_n' \), this condition being equivalent to \( |x_n'| = |x_n| \).

Let \( T \) be a bounded linear transformation of \( H \). We consider \( n \) vectors \( x_1, x_2, \ldots, x_n \) and from now on we assume always that the vectors \( x_i \) are linearly independent. The vectors \( Tx_1, Tx_2, \ldots, Tx_n \) may be linearly independent or not. We split the vector \( Tx_n \) in two orthogonal components \( Tx_n = (Tx_n)' + (Tx_n)''' \), where \( (Tx_n)''' \) is the orthogonal projection of \( Tx_n \) on the subspace \( L(Tx_1, \ldots, Tx_{n-1}) \). Let \( x_n = x_n' + x_n'' \) as before. \( Tx_n = Tx_n' + Tx_n'' \) and \( Tx_n'' \in L(Tx_1, \ldots, Tx_{n-1}) \). By the minimal property of the normal vector \( (Tx_n)' \) we thus have

\[
| (Tx_n)' | \leq | Tx_n' | ,
\]

where the sign of equality holds if and only if \( (Tx_n)' = Tx_n' \). Since \( Tx_n = Tx_n' + Tx_n'' = (Tx_n)' + (Tx_n)''' \) this condition is equivalent to \( (Tx_n)''' = Tx_n'' \).

Then we have by (1), (2) and (3)
\[
G(Tx_1, \ldots, Tx_n) = G(Tx_1, \ldots, Tx_{n-1}) \frac{\lvert (Tx_n)' \rvert^2}{\lvert (Tx_n)' \rvert^2}
\]

\[
\leq G(Tx_1, \ldots, Tx_{n-1}) \lvert Tx_n' \rvert^2 \leq \lvert T \rvert^2 G(Tx_1, \ldots, Tx_{n-1}) \lvert x_n' \rvert^2
\]

\[
= \lvert T \rvert^3 G(Tx_1, \ldots, Tx_{n-1}) \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_{n-1})}.
\]

We thus obtain the

**Theorem 1:** If the vectors \( x_1, x_2, \ldots, x_n \) are linearly independent and the
linear transformation \( T \) is bounded we have the following inequality

\[
G(Tx_1, \ldots, Tx_n) \leq \lvert T \rvert^3 G(Tx_1, \ldots, Tx_{n-1}) \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_{n-1})},
\]

where the equality holds if and only if one (or both) of the following conditions is satisfied: 1) The vectors \( Tx_1, \ldots, Tx_{n-1} \) are linearly dependent (in which case both sides in (4) are 0). 2) \( Tx_n = (Tx_n)' \) and \( \lvert Tx_n' \rvert = \lvert T \rvert \lvert x_n' \rvert \).

By repeated use of (4) we get

\[
G(Tx_1, \ldots, Tx_n) \leq \lvert T \rvert^4 G(Tx_1, \ldots, Tx_{n-2}) \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_{n-2})},
\]

and so on. Finally we have

\[
G(Tx_1, \ldots, Tx_n) \leq \lvert T \rvert^{2n-2} \frac{\lvert Tx_1 \rvert^2}{\lvert x_1 \rvert^2} G(x_1, \ldots, x_n).
\]

Since \( G \) is symmetric we can replace the index 1 by any index \( i \) and we thus have

\[
G(Tx_1, \ldots, Tx_n) \leq \lvert T \rvert^{2n-2} \min_{1 \leq i \leq n} \frac{\lvert Tx_i \rvert^2}{\lvert x_i \rvert^2} G(x_1, \ldots, x_n).
\]

From this we get immediately the less sharp inequality

\[
G(Tx_1, \ldots, Tx_n) \leq \lvert T \rvert^{2n} G(x_1, \ldots, x_n).
\]

4. Special cases. Let \( T = P \) = orthogonal projection, i.e. \( P^2 = P = P^* \).
| \( Px \) | = \( | x \) | if and only if \( Px = x \), thus the condition 2) in theorem 1 now reads: \( (Px_n)' = Px_n' = x_n' \). We assert that this condition is equivalent to

\[
(Px_n)' = Px_n' = x_n'.
\]

Assume that the latter is the case. We have by (3): \( \lvert (Px_n)' \rvert \leq \lvert Px_n' \rvert \leq \lvert x_n' \rvert \) and hence \( \lvert (Px_n)' \rvert = \lvert Px_n' \rvert = \lvert x_n' \rvert \), but this implies \( (Px_n)' = Px_n' = x_n' \).

We thus have the following result: When \( T = P \) the inequality (4) has the form

\[
G(Px_1, \ldots, Px_n) \leq G(Px_1, \ldots, Px_{n-1}) \frac{G(x_1, \ldots, x_n)}{G(x_1, \ldots, x_{n-1})},
\]
where equality holds if and only if one (or both) of the following conditions is satisfied: 1) The vectors \( Px_1, \ldots, Px_{n-1} \) are linearly dependent. 2) \((Px_n)' = x_n'\).

The inequality (7) is due to Everitt and Moppert (cf. [2] and [3]).

When \( T = P \) the inequality (6) becomes

\[
G(Px_1, \ldots, Px_n) \leq G(x_1, \ldots, x_n),
\]

where equality holds (by inequality (5)) if and only if \( Px_i = x_i \) (\( i = 1, 2, \ldots, n \)). This is the Courant-Hilbert inequality (cf. [1] pp. 107-108).

5. Investigation of inequality (6). Assume that \( |T| > 0 \), that is, \( T \) is not identically \( = 0 \). Then, by (5), a necessary condition for equality in (6) is \( |Tx_i| = |T||x_i| \) (\( i = 1, 2, \ldots, n \)). We shall see that this condition is also sufficient. For this purpose we first prove the following

Lemma: Let \( x_1, x_2, \ldots, x_n \) be linearly independent vectors in the Hilbert space \( H \) and let \( T \) be a bounded linear transformation of \( H \) and \( |T| > 0 \). If \( |Tx_i| = |T||x_i| \) (\( i = 1, 2, \ldots, n \)), so \( T = |T|V \), where \( V \) maps the subspace \( L = L(x_1, x_2, \ldots, x_n) \) isometrically on the subspace \( T(L) \).

Proof. Let \( E \) be the orthogonal projection of \( H \) on the subspace \( L \) and let \( x \) and \( y \) be in \( L \). We have

\[
(ET*Tx, y) = (T^*Tx, Ey) = (T^*Tx, y) = (x, T^*Ty) = (Ex, T^*Ty)
= (x, ET*Ty).
\]

Thus the restriction \((ET*T)_L \) of transformation \( ET*T \) to the subspace \( L \) is self-adjoint (and completely continuous, the dimension of \( L \) being finite). We have for \( x_i \) (\( i = 1, 2, \ldots, n \))

\[
|T|^2|x_i|^2 = |Tx_i|^2 = (Tx_i, Tx_i) = (T^*Tx_i, x_i) = (T^*Tx_i, Ey_i) = (ET^*Tx_i, x_i) \leq |ET^*Tx_i||x_i| \leq |ET^*T||x_i|^2 \leq |ET^*T||x_i|^2 = |T|^2|x_i|^2.
\]

It follows that there is equality everywhere in the above inequalities and thus we have \( |ET^*T| = |T|^2 \) and \( |ET^*Tx_i| = |ET^*T||x_i| \). As the transformation \((ET^*T)_L \) is self-adjoint and completely continuous this implies (cf. [4] p. 229) that every vector \( x_i \) is an eigenvector of \((ET^*T)_L \) and \( ET^*Tx_i = |T|^2x_i \). Since the vectors \( x_i \) form a base of \( L \) it follows that \((ET^*T)_L = |T|^2I_L \) where \( I_L \) denotes the identity mapping of \( L \).

Let now \( x \) and \( y \) be in \( L \). We have

\[
(Tx, Ty) = (T^*Tx, y) = (T^*Tx, Ey) = (ET^*Tx, y) = |T|^2(x, y).
\]

The restriction of the transformation \( \frac{T}{|T|} \) to \( L \) thus maps \( L \) isometrically on the subspace \( T(L) \). Q.E.D.

By this lemma we are now able to prove the following
Theorem 2: Let the vectors \( x_1, x_2, \ldots, x_n \) be linearly independent and \( |T| > 0 \). Then \( G(Tx_1, \ldots, Tx_n) = |T|^{2n}G(x_1, \ldots, x_n) \) if and only if \( |Tx_i| = |T||x_i| (i = 1, 2, \ldots, n) \).

As already stated, the necessity follows from (5).

Sufficiency. By lemma we have \( T = |T|V \), where the restriction of \( V \) to \( L = L(x_1, x_2, \ldots, x_n) \) maps \( L \) isometrically on \( T(L) \). We set again \( x_i = x'_i + x''_i \), where \( x''_i \) is the orthogonal projection of \( x_i \) on the subspace \( L(x_1, x_2, \ldots, x_{i-1}) \) \( (i = 2, \ldots, n) \). We have
\[
G(Tx_1, \ldots, Tx_n) = G(Tx_1', Tx_2', \ldots, Tx_n') = |Tx_1|^2|Tx_2'|^2 \cdots |Tx_n'|^2
= |T|^{2n}|x_1|^2|x_2'|^2 \cdots |x_n'|^2 = |T|^{2n}G(x_1, x_2', \ldots, x_n')
= |T|^{2n}G(x_1, x_2, \ldots, x_n).
\]
The proof is thus completed.

6. A remark. The above lemma may have some interest in itself. Let e.g. \( A \) be a contraction of \( n \)-dimensional \( (n \) is finite) linear space \( R^n \), that is, \( |Ax| \leq |x| \) for every \( x \in R^n \). If then \( |Ax_i| = |x_i| \) \( (i = 1, 2, \ldots, n) \) and the vectors \( x_i \) are linearly independent, so, by lemma, \( A \) is a unitary transformation of \( R^n \).

7. A geometric interpretation. The determinant
\[
G(x_1, \ldots, x_n; y_1, \ldots, y_n) = \det (x_i, y_k) \quad (1 \leq i, k \leq n)
\]
defines an inner product in the \( n \)-th exterior power of \( H \) (cf. [5] p. 20). The above considerations can be properly interpreted in terms of exterior powers of linear transformations. Thus, for example, the exterior power of an orthogonal projection is an orthogonal projection (in the metric defined by the determinant \( G \), cf. [5] p. 36). Interpreted in this way, the Courant-Hilbert inequality (8) states only the simple fact that the norm of any orthogonal projection of a vector is at most equal to the norm of the vector.
Bibliography


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