REMARKS ON A PAPER OF TIENARI
CONCERNING QUASICONFORMAL
CONTINUATION

BY

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Remarks on a paper of Tienari concerning quasiconformal continuation

1. Introduction. All sets considered in this paper are assumed to lie in the extended complex plane. Let \( \gamma \) be either an open Jordan arc\(^1\) or a closed Jordan curve. We say that \( \gamma \) is quasiconformal at a point \( z \in \gamma \) if there exists an open subarc \( \gamma_z \) of \( \gamma \) and a quasiconformal mapping of a domain \( G \) such that \( z \in \gamma_z \subset G \) and the image of \( \gamma_z \) is a line segment. \( \gamma \) is called quasiconformal if it is quasiconformal at each point.

Quasiconformal arcs and curves have been studied recently by Tienari [4] who called them »curves which allow a quasiconformal continuation«; see also Pfluger [3]. Tienari showed that the »local« quasiconformality defined above implies quasiconformality in the large, i.e., a closed quasiconformal curve can be mapped onto a circle and every compact subarc of an open quasiconformal arc can be mapped onto a line segment by a quasiconformal mapping of the whole plane.

Tienari arrived at the class of quasiconformal arcs and curves by considering the possibility of extending a quasiconformal mapping over a boundary arc. He considered only simply connected domains. We shall generalize his results for multiply connected domains in Section 3. Section 2 deals with general properties of quasiconformal arcs.

2. We first establish

**Theorem 1.** Let \( f \) be a real-valued function, defined on the real interval \( a < x < b \) and satisfying a Lipschitz condition

\[
|f(x_1) - f(x_2)| \leq M |x_1 - x_2|.
\]

Then the graph of \( f \) is a quasiconformal arc.

**Proof.** Let \( G \) be the strip domain \( a < \text{Re}z < b \). Define a mapping \( \varphi: G \to G \) by

\[
\varphi(x + iy) = x + i(y - f(x)).
\]

\( \varphi \) is clearly a homeomorphism. By (1), \( \varphi \) is absolutely continuous on every horizontal and vertical line. At a point \( z = x + iy \) where \( \varphi \) is differentiable, the dilatation has the value

\(^1\) An open arc is an arc minus its endpoints.
\[ \frac{1}{2} (2 + f'(x)^2 + \sqrt{4 + f'(x)^2}) \leq \frac{1}{2} (2 + M^2 + \sqrt{4 + M^2}). \]

By the analytic definition of quasiconformal mappings, \( \varphi \) is quasiconformal (see e.g. Pfluger [2]). Since \( \varphi \) maps the graph of \( f \) onto the segment \( a < x < b \), the theorem follows.

Tienari [4] proved that an arc is quasiconformal if it has a continuous curvature. With the aid of Theorem 1 this result can be sharpened as follows.

**Theorem 2.** If an arc \( \gamma \) has a continuous tangent, it is quasiconformal.

**Proof.** We must prove the quasiconformality of \( \gamma \) at an arbitrary point \( z_0 \in \gamma \). We may assume that \( z_0 = 0 \) and the tangent of \( \gamma \) at \( z_0 \) is the real axis. Because of the continuity of the tangent, there exists a subarc \( \gamma_0 \) of \( \gamma \) which has the representation \( z = x + if(x), \ |x| < a \), where \( f \) is continuously differentiable for \( |x| \leq a \). By Theorem 1, \( \gamma_0 \) is quasiconformal at \( z_0 \), q.e.d.

**Remark.** Using Theorem 16 of [4] one can weaken the condition of Theorem 2 and allow a finite number of exceptional points where \( \gamma \) has only two half-tangents which make an angle \( \neq 0 \).

3. In this section we shall discuss the extension of a quasiconformal mapping over a boundary arc or curve. Let \( G_1 \) and \( G_2 \) be two disjoint domains such that \( G_2 \) is simply connected and there exists an open quasiconformal arc \( \gamma \) which is a free boundary arc of both \( G_1 \) and \( G_2 \). Let \( G_1', G_2' \) and \( \gamma' \) satisfy the same conditions. Assume that \( f \) is a quasiconformal mapping of \( G_1 \) onto \( G_1' \) which carries \( \gamma \) onto \( \gamma' \). Tienari's result ([4], Theorem 16) can then be formulated as follows: If \( G_1 \) is also simply connected and if \( \gamma_0 \) is a compact subarc of \( \gamma \) with image \( \gamma_0' \), then \( f \) can be extended to a quasiconformal mapping \( f'^* : G_1 \cup \gamma_0 \cup G_2 \rightarrow G_1' \cup \gamma_0' \cup G_2' \). However, it is almost trivial to show that the simple connectedness of \( G_1 \) is an unnecessary restriction. For, since \( \gamma \) is a free boundary arc of \( G_1 \), we can find a simply connected subdomain \( D \) of \( G_1 \) such that \( D \) has a free open boundary arc \( \gamma_1 \), \( \gamma_0 \subset \gamma_1 \subset \gamma \), and then extend the mapping \( f|D \) to \( D \cup \gamma_0 \cup G_2 \).

The case of a closed curve is slightly more complicated.

**Theorem 3.** Let \( G \) be a domain bounded by a closed quasiconformal curve \( \gamma \) and a compact set which does not meet \( \gamma \). Denote by \( D \) the complementary domain of \( \gamma \) which does not meet \( G \). Let \( G', \gamma', D' \) satisfy the same conditions. Then every quasiconformal mapping \( f : G \to G' \) which maps \( \gamma \) onto \( \gamma' \) can be extended to a quasiconformal mapping \( f'^* : G \cup \gamma \cup D \to G' \cup \gamma' \cup D' \).

The proof is based on the following
Lemma. Let \( \alpha \) and \( \beta \) be two disjoint closed quasiconformal curves. Then there exists a quasiconformal mapping of the whole plane which maps \( \alpha \) and \( \beta \) onto concentric circles.

Proof. We first map the complementary domains of \( \alpha \) onto the upper and lower half-planes by conformal mappings \( S_1 \) and \( S_2 \), respectively, so that \( S_1^{-1}(\infty) = S_2^{-1}(\infty) \) and \( \beta \) is mapped into the upper half-plane. Let \( G \) be the doubly connected domain bounded by \( \alpha \) and \( \beta \). We next map \( G \) by a conformal mapping \( f \) onto a domain bounded by the real axis and a circle \( C \) in the upper half-plane so that \( f^{-1}(\infty) = S_1^{-1}(\infty) \). The mapping \( fS_1^{-1} \) is conformal and maps the real axis onto itself and \( \infty \) into \( \infty \). By the reflection principle, the derivative of \( fS_1^{-1} \) is bounded from 0 and \( \infty \) on the real axis. Consequently, \( fS_1^{-1} \) satisfies the so-called \( \varphi \)-condition (see [1]). By Theorem 3 of [4], the mapping \( S_1S_2^{-1} \) of the real axis also satisfies a \( \varphi \)-condition. Thus the composite mapping \( (fS_1^{-1})(S_1S_2^{-1}) = fS_2^{-1} \) satisfies a \( \varphi \)-condition. By a well-known result of Beurling and Ahlfors [1], there exists a quasiconformal mapping \( T \) of the lower half-plane onto itself such that \( T = fS_2^{-1} \) on the real axis. The mappings \( f \) and \( TS_2 \) define a quasiconformal mapping of the \( \alpha \)-component of the complement of \( \beta \) onto the unbounded complementary component of \( C \). Since \( \beta \) is quasiconformal and Theorem 3 holds for simply connected domains ([4], Theorem 5), this mapping can be extended to a quasiconformal mapping \( h \) of the whole plane. The desired mapping is then \( gh \) where \( g \) is the linear mapping which maps \( C \) and the real axis onto two concentric circles.

Proof for Theorem 3. Let \( h \) be a quasiconformal mapping of the whole plane such that \( h \) maps \( D \) onto the unit disc \( |z| < 1 \). \( h \) maps \( G \) onto a domain which contains an annulus \( 1 < |z| < R \). Let \( \alpha' \) be the image of the circle \( |z| = R \) under the mapping \( fh^{-1} \). By the above Lemma, there exists a quasiconformal mapping \( g \) of the whole plane which maps \( \gamma' \) and \( \alpha' \) onto two concentric circles \( |z| = 1 \) and \( |z| = R' > 1 \), respectively. Then \( gfh^{-1} \) is a quasiconformal mapping of \( 1 < |z| < R \) onto \( 1 < |z| < R' \). By repeated reflections, it can be extended to a quasiconformal mapping \( \varphi \) of \( |z| < R \) onto \( |z| < R' \). Then \( g^{-1}qh \) maps \( D \) onto \( D' \) and \( g^{-1}qh(z) = f(z) \) for \( z \in \gamma \). We have thus obtained the desired extension of \( f \).

Corollary 1. Let \( G \) and \( G' \) be two domains, each bounded by a finite number of disjoint closed quasiconformal curves. Then every quasiconformal mapping of \( G \) onto \( G' \) can be extended to a quasiconformal mapping of the whole plane.

Corollary 2. Let \( G \) and \( G' \) be two subdomains of the upper half-plane, each bounded by the real axis and a compact subset of the upper half-plane. Let \( f : G \to G' \) be a quasiconformal mapping which maps the real axis onto itself and \( \infty \) into \( \infty \). Then \( f \) satisfies a \( \varphi \)-condition on the real axis.
References


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