CHARACTERIZATION OF THE SET OF VALUES APPROACHED BY A MEROMORPHIC FUNCTION ON SEQUENCES OF JORDAN CURVES

BY

FREDERICK BAGEMIHL

HELSINKI 1963
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1963.328
Communicated 14 September 1962 by F. Nevanlinna and Olli Lehto

KESKUSKIRJAPAINO
HELSEINKI 1963
§ 1. Introduction

1. Let $\Omega$ denote the extended complex plane (or the Riemann sphere), and let $C$ be the unit circle and $D$ be the open unit disk.

A sequence $A$ of distinct Jordan curves $J_1, J_2, \ldots, J_n, \ldots$ in $D$ will be called an annulation if

1) $J_n$ lies in the interior of $J_{n+1}$ ($n = 1, 2, 3, \ldots$)
and
2) given any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ such that $n > n_0$ implies that $J_n$ lies in the region $1 - \varepsilon < |z| < 1$.

If, furthermore,
3) every $J_n$ is a circle with the origin as center, then the sequence $A$ will be called a strict annulation.

Suppose that the function $f(z)$ is meromorphic in $D$. Then we define the sets $\Theta(f)$ and $\Theta^\circ(f)$ as follows: The point $c \in \Omega$ belongs to $\Theta(f)$, resp. $\Theta^\circ(f)$, provided that, for some annulation, resp. strict annulation, $A = \{J_n\}$, given any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ such that $n > n_0$ implies that, for every $z \in J_n$, we have $|f(z) - c| < \varepsilon$ or $|1/f(z)| < \varepsilon$ according as $c$ is finite or the point at infinity. Under these conditions we say that the function $f$ tends to the value $c$ on the annulation $A$.

2. An immediate consequence of the foregoing definition is that $\Theta^\circ(f) \subseteq \Theta(f)$. In § 6 we shall give an example of a function $f$ for which $\Theta^\circ(f) \neq \Theta(f)$; in fact, we shall show that $\Theta^\circ$ and $\Theta$ can be prescribed arbitrarily except for the natural restrictions that they be closed sets and that $\Theta \subseteq \Theta^\circ$.

The main question that arises at once, however, is: What kind of point sets are $\Theta^\circ$ and $\Theta$? It is again obvious that $\Theta^\circ$ and $\Theta$ are closed sets. We are going to establish the following characterization:

**Theorem 1.** Let $F$ be an arbitrary closed subset of $\Omega$. Then there exists a function $f(z)$, meromorphic in $D$, such that $\Theta^\circ(f) = \Theta(f) = F$.

Moreover, if $F$ is not empty, and $A$ is a given annulation, then there exists an $f$ such that $\Theta(f) = F$ and every value $c \in F$ is approached by $f$ on a subannulation of $A$. 
§ 2. Relations between $R(f)$ and $\Theta(f)$

3. It is well known that although a function $f(z)$ that is holomorphic in $D$ cannot tend uniformly to infinity as $z$ approaches $C$, nevertheless there exist holomorphic functions $f(z)$ in $D$ for which $\{ \infty \} = \Theta^2(f) = \Theta(f)$ (see, e.g., [1]). For any holomorphic function $f$, we have $\infty \in R(f)$, where, as is customary, $R(f)$ stands for the range of values of $f$ in $D$ [5, p. 48].

Now suppose that $f(z)$ is meromorphic in $D$ and that $R(f)$ omits at least three values of $\Omega$. Then (see [4, p. 97, Theorem 6] or [5, pp. 52–53]) $\Theta(f)$ is empty if $f(z)$ is not identically constant.

If $R(f)$ omits just two values of $\Omega$, then (see [4, p. 112, Theorem 9, (ii)] or [5, pp. 50–51, Theorem 1, (ii)]) $f(z)$ possesses at least two asymptotic values, and hence $\Theta(f)$ is empty.

If the complement of $R(f)$ consists of the sole value $c \in \Omega$, then either, as in the preceding case, $\Theta(f)$ is empty, or $c$ is an asymptotic value of $f$, so that if $\Theta(f)$ is not empty, then $\Theta(f) = \{ c \}$. For example, let $h(z) = \cos \frac{1}{1 - z}$, and set $f(z)$ equal to $h(z)$, $1/h(z)$, $(1/h(z)) + b$ according as $c$ is equal to $\infty$, $0$, $b$, where $b \in \Omega - \{ 0, \infty \}$. Then $f(z)$ is meromorphic in $D$, $R(f)$ omits the sole value $c$, and $\Theta(f)$ is empty. On the other hand, let $h(z)$ be the function described as an infinite product in [2, p. 79], and then define $f(z)$ as in the preceding sentence. The function $f(z)$ is meromorphic in $D$, $R(f)$ omits the sole value $c$ (cf. [5, p. 75, Remark, (v)]) , and $\Theta^2(f) = \Theta(f) = \{ c \}$.

The reason for making a distinction between $\Theta^2(f)$ and $\Theta(f)$ is that, although it may be known for a specific $f$ that $c \in \Theta(f)$, it may be by no means an easy matter to determine whether $c \in \Theta^2(f)$ — and a strict annulation is, after all, the neatest.

§ 3. The skeleton $S$ and the continuous function $g(z)$ on $S$

4. To prove the theorem formulated in § 1, we shall assume that the given closed subset $F$ of $\Omega$ is not empty, because if $F$ is empty, the assertion of the theorem is obviously true.

Let the annulation $A = \{ J_n \}$ be given. We shall show that there exists a meromorphic function $f(z)$ in $D$ such that $\Theta(f) = F$ and every value $c \in F$ is approached by $f$ on a subannulation of $A$; this will prove the second part of the theorem. If we then take $A$ to be a strict annulation, we obtain the first part of the theorem.
5. To accomplish this, we first define a point set $S$ in $D$, where $S \supset \bigcup_{n=1}^{\infty} J_n$. Let $a_{00}$ and $a_{10}$ be distinct points in the interior of $J_1$, and take $B_0$, $B_1$ to be mutually exclusive Jordan arcs with initial points $a_{00}$ resp. $a_{10}$, and terminal points $1$ resp. $-1$; we regard the initial points, but not the terminal points, as belonging to these arcs. We further require $B_0$ and $B_1$ to lie in $D$, and each of them to intersect every $J_n$ in precisely one point, $a_{0n}$ resp. $a_{1n}$, so that

$$B_0 \cap J_n = \{a_{0n}\}, \quad B_1 \cap J_n = \{a_{1n}\} \quad (n = 1, 2, 3, \ldots).$$

We put

$$S = B_0 \cup B_1 \cup \bigcup_{n=1}^{\infty} J_n$$

and call $S$ the skeleton.

6. Our next step is to define a certain continuous function $g(z)$ on $S$. Whereas the definition of the skeleton depended on the given annulation $A$, the definition of $g(z)$ depends on the given (nonempty) closed set $F$.

Suppose that $F$ contains $\infty$ as an isolated point. Then there exists a finite point $\zeta \in F$. The image of $F$ under the transformation $z' = \frac{1}{z - \zeta}$ is a set $F'$ that does not contain $\infty$. If we can prove that there exists a meromorphic function $h(z)$ in $D$ such that $\Theta(h) = F'$ and every value in $F'$ is approached by $h$ on a subannulation of $A$, then the function $f(z) = \frac{1}{h(z)} + \zeta$ is meromorphic in $D$, $\Theta(f) = F$, and every value in $F$ is approached by $f$ on a subannulation of $A$. We may therefore assume, in what follows, that $F$ does not contain $\infty$ as an isolated point.

7. Let

$$c_1, \ c_2, \ldots, \ c_n, \ldots$$

be an infinite sequence of finite complex numbers in $F$ with the property that $\{c_n\}$ is everywhere dense in $F$ in the sense that every isolated point of $F$ occurs infinitely often as a term of the sequence.

We put

$$g(z) \equiv c_n \quad (z \in J_n; \ n = 1, 2, 3, \ldots).$$

For every natural number $n$, let $A_{jn}$ ($j = 0, 1$) be the subarc of $B_i$ extending from $a_{j, n-1}$ to $a_{jn}$, including the two end points.

We set

$$g(z) \equiv c_1 \quad (z \in A_{01} \cup A_{11}).$$
For every \( n > 1 \), consider \( c_n \) and \( c_{n-1} \). If \( c_n = c_{n-1} \), we put

\[ g(z) \equiv c_n \quad (z \in A_{0n} \cup A_{1n}). \]

But if \( c_n \neq c_{n-1} \), we take the circle whose diameter is the rectilinear segment joining \( c_n \) and \( c_{n-1} \), and denote one of the semicircular arcs of this circle extending from \( c_{n-1} \) to \( c_n \), including the end points, by \( A_{0n}^* \), and the other one, also extending from \( c_{n-1} \) to \( c_n \), by \( A_{1n}^* \). We now define \( g(z) \) on \( A_j \) (\( j = 0, 1 \)) to be a homeomorphism of \( A_j \) onto \( A_j^* \) such that \( g(a_{j, n-1}) = c_{n-1} \) and \( g(a_{jn}) = c_n \).

This completes the definition of \( g(z) \) on \( S \); \( g(z) \) is obviously a continuous function on \( S \).

§ 4. The relation between \( g(z) \) and \( f(z) \)

8. For every natural number \( n \), choose a point \( b_{jn} (j = 0, 1) \) on \( B_j \) between \( a_{jn} \) and \( a_{jn+1} \), and let \( B_{jn} \) be the subarc of \( B_j \) that extends from \( b_{jn-1} \) to \( b_{jn} \), including the end points, where we set \( b_{j0} = a_{j0} \). Then put

\[ S_n = B_{0n} \cup B_{1n} \cup J_n \quad (n = 1, 2, 3, \ldots). \]

We propose to demonstrate in § 5 the existence of a meromorphic function \( f(z) \) in \( D \), with no poles on the skeleton, such that

\[ \lim_{n \to \infty} \max_{z \in S_n} |f(z) - g(z)| = 0. \]

9. Suppose for a moment that this has already been accomplished. We wish to show how the conclusion of our theorem follows.

We first prove that \( F \subseteq \Theta(f) \). Let \( c \in F \). Then there exists an infinite subsequence \( \{c_n\} \) of \( c \) such that \( \lim_{k \to \infty} c_n = c \). Let \( \varepsilon > 0 \) be given.

If \( c \) is finite, then there exists a \( k_1 = k_1(\varepsilon) \) such that, for every \( k > k_1 \),

\[ |c_n - c| < \frac{\varepsilon}{2}. \]

According to (1), for every \( z \in J_{nk} \) we have

\[ g(z) \equiv c_{nk} \quad (z \in J_{nk}). \]

It follows from (5) and (6) that, for every \( k > k_1 \),

\[ |g(z) - c| < \frac{\varepsilon}{2} \quad (z \in J_{nk}). \]
By (3) and (4) there exists a \( k_2 = k_2(\varepsilon) \) such that, for every \( k > k_2 \),

\[
|f(z) - g(z)| < \frac{\varepsilon}{2} \quad (z \in J_{n_k}).
\]

Now for every \( k > \max (k_1, k_2) \), (8) and (7) yield

\[
|f(z) - c| < \varepsilon \quad (z \in J_{n_k}).
\]

This implies that \( f(z) \) tends to \( c \) along the subannulation \( \{J_{n_k}\} \) of \( A \).

In case \( c = \infty \), replace inequalities (5), (7), (8), (9) by \( |c_{n_k}| > \frac{2}{\varepsilon} \),

\[
|g(z)| > \frac{2}{\varepsilon}, \quad |f(z) - g(z)| < \frac{1}{\varepsilon}, \quad \left| \frac{1}{f(z)} \right| < \varepsilon.
\]

10. It remains to be proved that \( \Theta(f) \subseteq F \). This is accomplished by showing that if \( c \in F \) then \( c \notin \Theta(f) \).

If \( c \) is finite, it is a positive distance \( d \) from the nonempty closed set \( F \). Suppose that \( c \in \Theta(f) \). Then there exists an annulation \( A^* = \{J_n^*\} \) on which \( f(z) \) tends to \( c \). Take \( \varepsilon = \frac{d \sqrt{2}}{2 \sqrt{2 + 4}} \). According to (4), (3), and (1), there exists an \( n_0 \) such that, for every \( n > n_0 \),

\[
|f(z) - g(z)| < \varepsilon \quad (z \in S_n),
\]

and hence, in particular,

\[
|f(z) - c_n| < \varepsilon \quad (z \in J_n).
\]

Let \( \delta \) be the positive distance between \( J_{n+1} \) and \( C \). By 2) in the definition of annulation, there exists an \( m_1 \) such that \( J_m^* \) lies in the region \( 1 - \delta < |z| < 1 \) for every \( m > m_1 \). There also exists an \( m_2 \) such that, for every \( m > m_2 \),

\[
|f(z) - c| < \varepsilon \quad (z \in J_m^*).
\]

If \( m_0 = \max (m_1, m_2) \), then \( m > m_0 \) implies that \( J_m^* \) intersects no \( J_n \) with \( n > n_0 \), for otherwise it would follow from (11) and (12) that

\[
|c - c_n| < d \frac{2 \sqrt{2}}{2 \sqrt{2 + 4}} < d, \quad \text{contradicting the definition of } d.
\]

Therefore there exists an \( n_1 > n_0 \) such that \( J_{n_1} \) is in the interior of \( J_m^* \) and \( J_m^* \) is in the interior of \( J_{n_1+1} \). Consequently \( J_m^* \) intersects \( A_{0,n_1+1} \) in at least one (interior) point \( t_0 \) and \( A_{1,n_1+1} \) in at least one point \( t_1 \). Let

\[
g(t_j) = t_j^* \quad (j = 0, 1).
\]
11. Suppose first that $c_{n_1} \neq c_{n_1 + 1}$. It then follows from the definition of $g(z)$ that

\[ t_j^* \in A_{j, n_1 + 1} \quad (j = 0, 1). \]

In view of (3) and (10) for $n = n_1, n_1 + 1$, we have

\[ |f(t_j) - t_j^*| < \varepsilon \quad (j = 0, 1). \]

From (12), for $z = t_j (j = 0, 1)$, and (14), we obtain

\[ |t_j^* - c| < 2\varepsilon \quad (j = 0, 1), \]

and hence

\[ |t_0^* - t_1^*| < 4\varepsilon = d \frac{4\sqrt{2}}{2\sqrt{2} + 4}. \]

Now

\[ |t_j^* - c_{n_1}| > d - 2\varepsilon \quad (j = 0, 1), \]

because otherwise, in view of (15), we should have $|c - c_{n_1}| < d$, contrary to the definition of $d$. Likewise we must have

\[ |t_j^* - c_{n_1 + 1}| > d - 2\varepsilon \quad (j = 0, 1). \]

Because of (17), (18), and (13), $|c_{n_1} - c_{n_1 + 1}| > (d - 2\varepsilon)\sqrt{2}$ and

\[ |t_0^* - t_1^*| > (d - 2\varepsilon)\sqrt{2}. \]

According to (16) and (19),

\[ (d - 2\varepsilon)\sqrt{2} < d \frac{4\sqrt{2}}{2\sqrt{2} + 4}, \]

which implies that $\varepsilon > \frac{d\sqrt{2}}{2\sqrt{2} + 4}$, contrary to our choice of $\varepsilon$. Hence, if $c_{n_1} \neq c_{n_1 + 1}$, then $c \notin \Theta(f)$.

12. Suppose next that $c_{n_1} = c_{n_1 + 1}$. It then follows from (2) that

\[ t_j^* = c_{n_1 + 1} \quad (j = 0, 1), \]

and from (3) and (10) that

\[ |f(t_j) - c_{n_1 + 1}| < \varepsilon \quad (j = 0, 1). \]

On the other hand, (12) implies that

\[ |f(t_j) - c| < \varepsilon \quad (j = 0, 1). \]
From (20) and (21) we infer that \( |c - c_{n+1}| < 2\epsilon < d \), contradicting the definition of \( d \), and hence \( c \in \Theta(f) \).

This disposes of the case that \( c \) is finite.

13. Now suppose that \( c = \infty \). Then \( F \) is a bounded set. It follows from this and the way in which \( g(z) \) was defined, that \( g(z) \) is a bounded function on \( S \), and (4) now implies that \( c \in \Theta(f) \).

§ 5. The construction of \( f(z) \)

14. To complete the proof of the theorem all, that remains is the demonstration of the existence of a function \( f(z) \), meromorphic in \( D \), satisfying (4). This is accomplished by means of approximation and interpolation by rational functions. We use a modification of a method devised in [3].

Let \( K_0 \) be a Jordan curve in the interior of \( J_1 \) having no point in common with \( B_{q1} \cup B_{11} \). For every natural number \( n \), let \( K_n \) be a Jordan curve containing \( J_n \) in its interior and contained in the interior of \( J_{n+1} \), such that

\[
K_n \cap A_{0,n+1} = \{ b_{0n} \}, \quad K_n \cap A_{1,n+1} = \{ b_{1n} \}.
\]

Denote by \( D_n \) (\( n = 0, 1, 2, \ldots \)) the set of all points lying either on \( K_n \) or in the interior of \( K_n \), and put

\[
E_n = D_n \cup S_{n+1}.
\]

15. We now define, by induction on \( n \), a function \( \varphi_n(z) \) on \( E_n \) and a rational function \( r_n(z) \).

Let

\[
\varphi_0(z) = \begin{cases} 0, & z \in D_0; \\ g(z), & z \in S_1. \end{cases}
\]

The function \( \varphi_0(z) \) is evidently continuous on \( E_0 \) and holomorphic at all interior points of \( E_0 \). Because of the nature of \( E_0 \), there exists (cf. [6, pp. 260—261, 313]) a rational function \( r_0(z) \) with no poles on \( D_0 \cup S \), such that

\[
|r_0(z) - \varphi_0(z)| < 1 \quad (z \in E_0),
\]

and

\[
r_0(b_{j1}) = g(b_{j1}) \quad (j = 0, 1).
\]

Now suppose that \( n > 0 \) and that rational functions \( r_0(z), r_1(z), \ldots, r_{n-1}(z) \) have been determined so that \( r_0(z) + r_1(z) + \ldots + r_{n-1}(z) \) has no poles on \( S \) and
\[ r_{n-1}(b_{jn}) = g(b_{jn}) - [r_0(b_{jn}) + r_1(b_{jn}) + \ldots + r_{n-2}(b_{jn})] \quad (j = 0, 1), \]
where the expression in brackets is missing in case \( n = 1 \).

Let
\[
q_n(z) = \begin{cases} 
0, & z \in D_n; \\
g(z) - [r_0(z) + r_1(z) + \ldots + r_{n-1}(z)], & z \in S_{n+1}.
\end{cases}
\]

It follows from (23), (24), and (25) that \( q_n(z) \) is continuous on \( E_n \) and holomorphic at all interior points of \( E_n \). As before, there exists a rational function \( r_n(z) \) with no poles on \( D_n \cup S \), such that
\[
|q_n(z) - q_n(z)| < \frac{1}{2^n} \quad (z \in E_n),
\]
and
\[
r_n(b_{j,n+1}) = g(b_{j,n+1}) - [r_0(b_{j,n+1}) + r_1(b_{j,n+1}) + \ldots + r_{n-1}(b_{j,n+1})] \quad (j = 0, 1).
\]

This completes the induction.

16. Set
\[
f(z) = \sum_{n=0}^{\infty} r_n(z).
\]

Suppose that \( z \in D_n \). It follows from (24) and (26) that
\[
|r_n(z) - q_n(z)| < \frac{1}{2^n + k} \quad (k = 0, 1, 2, \ldots).
\]

Since \( r_{n+k}(z) \) \((k = 0, 1, 2, \ldots)\) is holomorphic on \( D_n \), (27) implies that \( \sum_{k=0}^{\infty} r_{n+k}(z) \) is holomorphic in the interior of \( D_n \), and hence \( f(z) \) is meromorphic in the interior of \( D_n \). Every point of \( D \), however, is in the interior of \( D_n \) for some value of \( n \); this is so because of 2) in the definition of annulation, and because of the way in which \( K_n \) was defined. Hence, \( f(z) \) is meromorphic in \( D \).

17. Let \( z \in S_n \), where \( n > 1 \). Then by (26) and (22), we have
\[
|r_{n-1}(z) - q_{n-1}(z)| < \frac{1}{2^{n-1}}.
\]

According to (25),
\[
q_{n-1}(z) = g(z) - [r_0(z) + r_1(z) + \ldots + r_{n-2}(z)].
\]

Combining (28) and (29), we obtain
Now (30), (27), and the definition of $f(z)$ yield

$$|f(z) - g(z)| \leq |r_0(z) + r_1(z) + \ldots + r_{n-1}(z) - g(z)| + |r_n(z)| + |r_{n+1}(z)| + \ldots$$

$$< \sum_{m=n-1}^{\infty} \frac{1}{2^m},$$

which implies (4).

§ 6. A function $f(z)$ with prescribed $\Theta^0(f)$ and $\Theta(f)$

18. As we remarked in § 1, for every function $f(z)$ that is meromorphic in $D$, $\Theta^0(f)$ and $\Theta(f)$ are closed subsets of $\Omega$ such that $\Theta^0(f) \subseteq \Theta(f)$. We are going to show that $\Theta^0$ and $\Theta$ are subject to no further restrictions. Specifically, we shall prove the following:

**Theorem 2.** Let $F$ and $F_0$ be closed subsets of $\Omega$ such that $F_0 \subseteq F$. Then there exists a function $f(z)$, meromorphic in $D$, such that $\Theta^0(f) = F_0$ and $\Theta(f) = F$.

19. If $F_0 = F$, then Theorem 2 reduces to the first part of Theorem 1. We shall therefore assume that $F_0 \subset F$.

As in § 2.6, we may assume that $F$ does not contain $\infty$ as an isolated point.

If $F$ contains $\infty$, but not as an isolated point, whereas $F_0$ contains $\infty$ as an isolated point, we can reduce this case to the case that neither $F$ nor $F_0$ contains $\infty$ as an isolated point by taking a finite point $\zeta \notin F_0$ such that $\zeta$ is not an isolated point of $F$ and proceeding as in § 2.6.

20. We define an annulation $A$ and, in case $F_0$ is not empty, a sub-annulation $A_0$, as follows.

Let

$$0 < a_0 < a_1 < \ldots < a_{n-1} < a_n < \ldots < 1,$$

and

$$\lim_{n \to \infty} a_n = 1.$$

Put

$$\beta_n = \frac{a_{n-1} + a_n}{2} \quad (n = 1, 2, 3, \ldots).$$
If \( F_0 \) is not empty, we take \( J_{2n-1} \) \((n = 1, 2, 3, \ldots)\) to be the circle with center at the origin and radius \( \beta_{2n-1} \), and \( J_{2n} \) \((n = 1, 2, 3, \ldots)\) to be the ellipse whose major axis extends from \( a_{2n} e^{\frac{\pi i}{4}} \) to \( a_{2n} e^{\frac{3\pi i}{4}} \) and whose minor axis extends from \( a_{2n-1} e^{\frac{\pi i}{4}} \) to \( a_{2n-1} e^{\frac{5\pi i}{4}} \). We then set

\[
A = \{ J_n \}, \quad A_0 = \{ J_{2n-1} \} \quad (n = 1, 2, 3, \ldots).
\]

In case \( F_0 \) is empty, we set

\[
A = \{ J_{2n} \} \quad (n = 1, 2, 3, \ldots).
\]

21. Our next step is the definition of a skeleton \( S \).

Let \( \beta_0 = \frac{\alpha_0}{2} \), and denote by \( B_0 \) resp. \( B_1 \) the rectilinear segment extending from \( \beta_0 \) resp. \( \beta_0 \) to 1 resp. \(-1\); the initial point is included, the terminal point not.

We take \( T_0, 2n-1 \) resp. \( T_1, 2n-1 \) \((n = 1, 2, 3, \ldots)\) to be the closed rectilinear segment extending from \( \beta_{2n-1} e^{\frac{\pi i}{4}} \) resp. \( \beta_{2n-1} e^{\frac{3\pi i}{4}} \) to \( \beta_{2n-2} e^{\frac{\pi i}{4}} \) resp. \( \beta_{2n-2} e^{\frac{3\pi i}{4}} \); we similarly define \( T_0, 2n \) resp. \( T_1, 2n \) \((n = 1, 2, 3, \ldots)\) to extend from \( \beta_{2n} e^{\frac{\pi i}{4}} \) resp. \( \beta_{2n} e^{\frac{3\pi i}{4}} \) to \( \beta_{2n-1} e^{\frac{\pi i}{4}} \) resp. \( \beta_{2n-1} e^{\frac{3\pi i}{4}} \).

If \( F_0 \) is not empty, put

\[
S = B_0 \cup B_1 \cup \bigcup_{n=1}^{\infty} (\bigcup J_n) \cup (\bigcup T_{0n}) \cup (\bigcup T_{1n}).
\]

In case \( F_0 \) is empty, set

\[
S = B_0 \cup B_1 \cup \bigcup_{n=1}^{\infty} (\bigcup J_{2n}) \cup (\bigcup T_{0n}) \cup (\bigcup T_{1n}).
\]

22. We now show how to define a continuous function \( g(z) \) on \( S \).

In case \( F_0 \) is not empty, we take

\[
c_1, \ c_3, \ c_5, \ldots, \ c_{2n-1}, \ldots
\]

to be an infinite sequence of finite complex numbers in \( F_0 \) such that \( \{c_{2n-1}\} \) is everywhere dense in \( F_0 \), and we let

\[
c_2, \ c_4, \ c_6, \ldots, \ c_{2n}, \ldots
\]

be a similar sequence in \( F - F_0 \) that is everywhere dense in \( F - F_0 \).
If $F_0$ is empty, we consider merely a sequence
\[ c_2, c_4, c_6, \ldots, c_{2n}, \ldots \]
in $F$ that is everywhere dense in $F$.
If $F_0$ is not empty, set
\[
g(z) = \begin{cases} 
  c_{2n-1}, & z \in J_{2n-1} \cup T_{0,2n-1} \cup T_{1,2n-1} \cup T_{0,2n} \cup T_{1,2n}; \\
  c_{2n}, & z \in J_{2n};
\end{cases} 
(n = 1, 2, 3, \ldots).
\]
But if $F_0$ is empty, put
\[
g(z) = \begin{cases} 
  0, & z \in \bigcup_{n=1}^{\infty} T_{0n}; \\
  1, & z \in \bigcup_{n=1}^{\infty} T_{1n}; \\
  c_{2n}, & z \in J_{2n} (n = 1, 2, 3, \ldots).
\end{cases}
\]
On the segments of $B_0$ and $B_1$ where $g(z)$ has not yet been defined, we define $g(z)$ by means of homeomorphisms like those described in §2.7, and thus obtain a continuous function $g(z)$ on $S$.

23. Now it is not difficult to see that a function $f(z)$, meromorphic in $D$, can be constructed after the pattern of §5 so that
\[
\lim_{n \to \infty} \max_{a_n < |z| < a_{n+1}} |f(z) - g(z)| = 0.
\]
If $F_0$ is empty, then clearly $\Theta(f) = F$. Furthermore, $\Theta^2(f)$ is empty, because every circle $Q$ in $D$ with the origin as center and a sufficiently large radius, intersects both $T_{0n}$ and $T_{1n}$ for a suitable $n$. By the definition of $g(z)$ on $T_{jn}$, it follows from (31) that every $Q$ sufficiently near $C$ contains a point at which $f(z)$ is very close to zero as well as a point at which $f(z)$ is very close to one, and hence $f(z)$ cannot tend to a limit on any sequence of such circles $Q$ tending to $C$.

If $F_0$ is not empty, it is again clear that $\Theta(f) = F$, and that every value $c \in F_0$ is approached by $f$ on the annulation $A_0$. The way in which $g(z)$ was defined on $T_{jn}$ in this case evidently guarantees that no value in the complement of $F_0$ can be approached by $f$ on a sequence of circles $Q$ with the origin as center and tending to $C$, and hence $\Theta^2(f) = F_0$.

24. Remark. We have also succeeded in characterizing the well-known sets $\Phi(f, e^{\Theta})$ and $\Phi(f)$ for a function $f(z)$ meromorphic in $D$. Our results in this direction will appear elsewhere (Math. Z. 80 (1962), 230—238).
References


Wayne State University
Detroit, Michigan, U.S.A.

Printed February 1963