ON THE COMPLETION OF UNIFORM SPACES

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One of the principal results in the theory of uniform spaces is the possibility of imbedding an arbitrary uniform space in a complete uniform space as a dense subspace. This problem has been treated by Bourbaki ([1], [2]) in two different ways. To the present author it seems that the situation becomes more lucid, if the construction of Bourbaki is modified by using a trick of Kowalsky ([3], p. 159). This modification yields the following, more precise result:

**Theorem:** For every uniform space $E$ there exists a complete uniform space $\tilde{E}$, containing $E$ as a dense subspace, and such that every point of $\tilde{E} - E$ is $T_0$-separated from every other point of $\tilde{E}$. Two complete spaces $\tilde{E}_1$ and $\tilde{E}_2$, satisfying these conditions, are always uniformly isomorphic and even in such a way that the identity mapping $E \rightarrow E$ has a unique extension to a uniform isomorphism $\tilde{E}_1 \rightarrow \tilde{E}_2$.

**Corollary:** For every uniform $T_0$-space $E$ there exists a complete uniform $T_0$-space $\tilde{E}$, containing $E$ as a dense subspace. If $\tilde{E}_1$ and $\tilde{E}_2$ are two complete $T_0$-spaces, containing $E$ as a dense subspace, then the identity mapping $E \rightarrow E$ has a unique extension to a uniform isomorphism $\tilde{E}_1 \rightarrow \tilde{E}_2$.

The purpose of this paper is to prove the theorem stated above and deduce the corollary from it.

If $(E, \omega)$ is a uniform space, $\omega$ being its uniformity, we denote the uniform topology of $(E, \omega)$ by $\tau_\omega$ and the neighborhood filter of a point $x$ of the topological space $(E, \tau_\omega)$ by $\tau_\omega(x)$. If $u$ is a filter of $E$ and $M$ a subset of $E$, we denote the trace of $u$ in $M$ by $u_M$. The induced uniformity of $M$ is, however, denoted by $\omega_M$ instead of $\omega_{M \times M}$. If $W$ is a set of $u$ and $x$ a point of $E$, the set of all points $y$ in $E$ such that $(x, y) \in W$ is denoted by $W[x]$.

1. **Minimal Cauchy filters.** A filter $u$ of a uniform space $(E, \omega)$ is called open, if it has a base every set of which is open in the topology $\tau_\omega$. 
It is called a Cauchy filter, if for every set $W$ of $\mathcal{W}$ there exists a set $A$ of $\mathcal{U}$ such that $A \times A \subseteq W$. If $\mathcal{U}$ is a Cauchy filter of $(E, \mathcal{W})$, then its trace $\mathcal{U}_M$ in an arbitrary subspace $(M, \mathcal{W}_M)$ is a Cauchy filter of this subspace.

A uniform space $(E, \mathcal{W})$ is said to be complete, if every Cauchy filter of $(E, \mathcal{W})$ is convergent in the uniform topology $\tau_\mathcal{W}$, i.e. if it contains, as a subfamily, at least one neighborhood filter $\tau_\mathcal{W}(x)$.

The family of all Cauchy filters of a uniform space $(E, \mathcal{W})$ is partially ordered by inclusion. The maximal element of this partially ordered family is the filter that consists of all subsets of $E$, including the empty set $\emptyset$. We denote this filter by $\mathcal{O}$ and call it the zero filter of $E$. The minimal elements of the partially ordered family of all Cauchy filters are called minimal Cauchy filters. In other words, a Cauchy filter $\mathcal{U}$ of $(E, \mathcal{W})$ is minimal, if from the relation $\mathcal{U} \subseteq \mathcal{V}$ it necessarily follows that the Cauchy filter $\mathcal{V}$ is $= \mathcal{U}$.

Every neighborhood filter $\tau_\mathcal{W}(x)$ is a minimal Cauchy filter and every minimal Cauchy filter is open. Furthermore, for every Cauchy filter $\mathcal{U} \neq \mathcal{O}$ there exists a unique minimal Cauchy filter $\mathcal{V}$ such that $\mathcal{V} \subseteq \mathcal{U}$. It is readily seen that if $a$ and $b$ are two minimal Cauchy filters, such that $A \cap B \neq \emptyset$ for every $A$ in $a$ and every $B$ in $b$, then $a = b$. In fact, the family $\mathcal{C} = \{A \cup B : A \in a, B \in b\}$ is then a Cauchy filter, satisfying $c \subseteq a$ and $c \subseteq b$, and from this it follows that $a = c = b$. Finally we observe that in a complete uniform space $(E, \mathcal{W})$ the family of all minimal Cauchy filters coincides with the family of all neighborhood filters $\tau_\mathcal{W}(x)$.

2. The construction of $\tilde{E}$. Let $(E, \mathcal{W})$ be an arbitrary uniform space.

We denote by $\tilde{E}$ the set whose elements are

(a) the points of $E$,

(b) those minimal Cauchy filters of $(E, \mathcal{W})$ that do not coincide with any neighborhood filter $\tau_\mathcal{W}(x)$.

We define a mapping $q$ of $E$ onto the family of all minimal Cauchy filters of $(E, \mathcal{W})$ by setting $q(x) = \tau_\mathcal{W}(x)$ for every $x$ in $E$ and $q(\tilde{x}) = \tilde{x}$ for every $\tilde{x}$ in the class (b).

With every set $W$ of $\mathcal{W}$ we associate the set $\tilde{W}$ consisting of all pairs $(\tilde{x}, \tilde{y})$, for which there exists a set $A \in q(\tilde{x}) \cap q(\tilde{y})$ such that $A \times A \subseteq W$.

It is easy to see that

$$\tilde{V} \cap \tilde{W} = \tilde{V} \cap \tilde{W}$$

for every pair $(V, W)$ of sets of $\mathcal{W}$. From this it immediately follows that the family
is a base of a filter $\tilde{\nu}$ in $\tilde{E} \times \tilde{E}$. We will show that $\tilde{\nu}$ is a uniformity for $\tilde{E}$.

Since each $\phi(x)$ is a Cauchy filter, every pair $(\tilde{x}, \tilde{y})$ belongs to every set of the family $\widehat{\mathcal{B}}$. Thus every set of $\widehat{\mathcal{B}}$ contains the diagonal of $\tilde{E} \times \tilde{E}$.

From the definition of the sets $\tilde{W}$ it immediately follows that each of these sets is symmetric.

Finally, let $\tilde{W}$ be an arbitrary set of $\mathfrak{B}$, and let $V$ be a set of $\nu$ such that $V \circ V \subseteq W$. We consider an arbitrary element $(\tilde{x}, \tilde{y})$ of $\tilde{V} \circ \tilde{V}$. If $z$ is an element of $\tilde{E}$ such that $(\tilde{x}, z)$ and $(\tilde{y}, z)$ belong to $\tilde{V}$, then there exists a set $A$ in $q(\tilde{x}) \cap q(\tilde{z})$ and a set $B$ in $q(\tilde{z}) \cap q(\tilde{y})$ such that $A \times A \subseteq V$ and $B \times B \subseteq V$. Since $A \cap B$ belongs to $q(z) \neq 0$, $A \cap B$ is non-empty. From this it readily follows that $A \times B \subseteq V \circ V \subseteq W$ and, consequently,

$$(A \cup B) \times (A \cup B) \subseteq W,$$

if $W$ is symmetric. Since, on the other hand, $A \cup B$ belongs to $q(\tilde{x}) \cap q(\tilde{y})$, the pair $(\tilde{x}, \tilde{y})$ thus belongs to $\tilde{W}$, and we have

$$\tilde{V} \circ \tilde{V} \subseteq \tilde{W}.$$ 

The filter $\tilde{\nu}$ is therefore a uniformity for $\tilde{E}$.

3. $E$ as a subspace of $\tilde{E}$. We consider the trace $\tilde{\nu}_E$ of the uniformity $\tilde{\nu}$ in $E \times E$. Let $W$ be a set of $\nu$. If $(x, y)$ belongs to $\tilde{W} \cap (E \times E)$, then there exists a set $A$ in $q(x) \cap q(y) = \tau_w(x) \cap \tau_w(y)$ such that $A \times A \subseteq W$. The points $x$ and $y$ then belong to $A$ and, consequently, $(x, y)$ belongs to $A \times A \subseteq W$. Thus we have

$$(1) \quad \tilde{W} \cap (E \times E) \subseteq W$$

for every set $W$ in $\nu$.

Conversely, let $(x, y)$ be an arbitrary element of $W$. Since $\tau_w(x)$ and $\tau_w(y)$ are Cauchy filters, there exists a set $A$ in $\tau_w(x)$ and a set $B$ in $\tau_w(y)$ such that $A \times A \subseteq W$ and $B \times B \subseteq W$. For an arbitrary element $(u, v)$ of $A \times B$ we then get $(u, x) \in A \times A \subseteq W$ and $(y, v) \in B \times B \subseteq W$ and, consequently, $(u, v) \in W^3$. We thus see that $A \times B \subseteq W^3$ and therefore

$$(A \cup B) \times (A \cup B) \subseteq W^3,$$

if $W$ is symmetric. Since $A \cup B \in \tau_w(x) \cap \tau_w(y)$, the pair $(x, y)$ therefore belongs to $\tilde{W}^3$. This indicates that
(2) \[ W \subset \tilde{W}^2 \cap (E \times E) \]
for every symmetric set \( W \) in \( \mathfrak{w} \). From the relations (1) and (2) it finally follows that \( \tilde{w}_E = \mathfrak{w} \) and \((E, \mathfrak{w})\) is thus a subspace of \((\tilde{E}, \mathfrak{w})\).

Now we show that \( E \) is dense in \((\tilde{E}, \rho)\). In order to do this we consider an arbitrary symmetric set \( W \) of \( \mathfrak{w} \) and an arbitrary point \( \tilde{x} \) of \( \tilde{E} \). Since \( q(\tilde{x}) \) is a Cauchy filter of \((E, \rho)\), there exists a set \( A \) of this filter satisfying \( A \times A \subset W \). Since \( \tilde{x} \) is not the zero filter, the set \( A \) is non-empty. Let \( a \) be a point of \( A \). From the Cauchy filter \( q(a) = \tau(w)(a) \) we choose a set \( B \) such that \( B \times B \subset W \). For an arbitrary element \((u, v)\) of \( A \times B \) we then get \((u, a) \in A \times A \subset W \) and \((a, v) \in B \times B \subset W \) and, consequently, \((u, v) \in W^2 \). From this it follows, as before, that
\[
(A \cup B) \times (A \cup B) \subset W^2.
\]
Since \( A \cup B \in q(\tilde{x}) \cap q(a) \), the pair \((\tilde{x}, a)\) thus belongs to \( \tilde{W}^2 \). Therefore we have
\[
\tilde{W}^2[\tilde{x}] \cap E \neq \emptyset,
\]
which shows that \( E \) is dense in \((\tilde{E}, \tilde{\mathfrak{w}})\).

4. The completeness of \( \tilde{E} \). In this section we first prove the following lemma, which will primarily be used in section 6, but which will also to some extent shorten the proof of the completeness of \( \tilde{E} \):

**Lemma:** If \( M \) is a dense subspace of a uniform space \( E \), then \( a \rightarrow a_M \) is a one-to-one mapping from the family of all minimal Cauchy filters of \( E \) onto the family of all minimal Cauchy filters of \( M \).

Let first \( a \) be a minimal Cauchy filter of \( E \). Then \( a_M \) is a Cauchy filter in \( M \) and there exists, consequently, a minimal Cauchy filter \( u \) of \( M \) for which \( u \subset a_M \). Let \( \tilde{u} \) be the filter of \( E \), with \( u \) as a base. If \( A \) were a set of \( a \), such that \( A \cap M \) would not belong to \( u \), then \( A \) could not belong to \( \tilde{u} \), and thus \( u \cap a \) were a Cauchy filter of \( E \) properly contained in the minimal Cauchy filter \( a \). This contradiction shows that \( u = a_M \), and thus the trace in \( M \) of every minimal Cauchy filter of \( E \) is a minimal Cauchy filter of \( M \).

Conversely, let \( u \) be a minimal Cauchy filter of \( M \). This filter \( u \) is a base of a Cauchy filter \( \tilde{u} \) in \( E \). Let \( v \) be a minimal Cauchy filter of \( E \) such that \( v \subset \tilde{u} \). Then we have \( v_M \subset \tilde{u}_M = u \) and therefore \( v_M = u \), since \( v_M \) is a Cauchy filter and \( u \) is minimal. Thus every minimal Cauchy filter of \( M \) is the trace of at least one minimal Cauchy filter of \( E \).
At last we see that if \( a \) and \( b \) are two minimal Cauchy filters of \( E \) such that \( a_M = b_M \) (\( \neq 0 \) since \( M \) is dense), then every set of \( a \) intersects every set of \( b \) and, consequently, \( a = b \). Our lemma is proved.

In order to prove that \((\tilde{E}, \tilde{\nu})\) is complete, we consider an arbitrary non-zero Cauchy filter \( \tilde{\nu} \) of \((\tilde{E}, \tilde{\nu})\). We have to show that there exists at least one point \( \tilde{x} \) of \( \tilde{E} \) such that \( \tau_{\tilde{\nu}}(\tilde{x}) \subseteq \tilde{\nu} \). Since there exists a minimal Cauchy filter \( \tilde{\nu} \) satisfying \( \tilde{\nu} \subseteq \tilde{\nu} \), we may assume that \( \tilde{\nu} \) is a minimal Cauchy filter. From our lemma it then follows that the trace \( \tilde{\nu}_E \) of \( \tilde{\nu} \) is a minimal Cauchy filter of \((E, \nu)\). Let \( \tilde{x} \) be a point of \( \tilde{E} \) such that \( \varphi(\tilde{x}) = \tilde{\nu}_E \). We show that

\[
\tilde{\nu} = \tau_{\tilde{\nu}}(\tilde{x}) .
\]

Since \( \tilde{\nu} \) and \( \tau_{\tilde{\nu}}(\tilde{x}) \) are minimal Cauchy filters, the assertion is proved, if we show that every set of \( \tilde{\nu} \) intersects every set of \( \tau_{\tilde{\nu}}(\tilde{x}) \). To see this we consider an arbitrary set \( \tilde{A} \) of \( \tilde{\nu} \) and an arbitrary set \( W \) of \( \nu \). Let \( V \) and \( U \) be symmetric sets of \( \nu \) such that \( V^2 \subseteq W \) and \( U^3 \subseteq V \). Since \( E \) is dense in \((\tilde{E}, \tilde{\nu})\), we can choose a point \( x \) from \( \tilde{U}[\tilde{x}] \cap \tilde{E} \). From the relation \((\tilde{x}, x) \in \tilde{U} \) it follows that there exists a set \( B \) in \( \varphi(\tilde{x}) \cap \varphi(x) = \tilde{\nu}_E \cap \tau_{\tilde{\nu}}(x) \) satisfying \( B \times B \subseteq U \). Since both \( \tilde{A} \cap E \) and \( B \) belong to the non-zero filter \( \tilde{\nu}_E \), their intersection \( \tilde{A} \cap B \) is non-empty. If \( y \) is a point of \( \tilde{A} \cap B \), then by (2)

\[
(x, y) \in B \times B \subseteq U \subseteq U^3 \subseteq V .
\]

From this and from \((\tilde{x}, x) \in \tilde{U} \subseteq \tilde{V} \) it then follows that

\[
(\tilde{x}, y) \in \tilde{V} = \tilde{\nu}_E \subseteq \tilde{V} ,
\]

i.e. that \( y \in \tilde{W}[\tilde{x}] \). This shows that the intersection \( \tilde{W}[\tilde{x}] \cap \tilde{A} \) is non-empty and thus the equation (3) holds. From this equation it finally follows that \( \tilde{\nu} \) converges to \( \tilde{x} \), and thus \((\tilde{E}, \tilde{\nu})\) is complete.

5. The postulate \( T_0^r \). Let \((\tilde{x}, \tilde{y})\) be an element of \( \tilde{E} \times \tilde{E} \) belonging to every set \( \tilde{W} \). For every set \( W \) of \( \nu \) it then exists a set \( M \) of \( \varphi(\tilde{x}) \cap \varphi(\tilde{y}) \) satisfying \( M \times M \subseteq W \). This shows that the family

\[
c = \{ A \cup B : A \in \varphi(\tilde{x}) , B \in \varphi(\tilde{y}) \}
\]

is a Cauchy filter in \((E, \nu)\). Since it is included in both \( \varphi(\tilde{x}) \) and \( \varphi(\tilde{y}) \), we have thus \( \varphi(\tilde{x}) = c = \varphi(\tilde{y}) \). Therefore we see that if \((\tilde{x}, \tilde{y})\) is a pair of elements of \( \tilde{E} \) for which \( \varphi(\tilde{x}) \neq \varphi(\tilde{y}) \), then there always exists a set \( \tilde{W} \) of
\[ \tilde{W} \text{ such that } (\tilde{x}, \tilde{y}) \in \tilde{W}. \] The set \( \tilde{W}[\tilde{x}] \) is then a neighborhood of \( \tilde{x} \) that does not contain \( \tilde{y} \). This indicates that every point of \( \tilde{E} - E \) is \( T_1 \)-separated from every other point of \( \tilde{E} \).

6. **The uniqueness of the completion.** Let now \( (\tilde{E}, \tilde{w}) \) and \( (\hat{E}, \hat{w}) \) be two completions of \( (E, w) \) satisfying the conditions of our theorem. Let \( \tilde{x} \) be an arbitrary point of \( \tilde{E} \). Then the trace \( \tau_{\tilde{w}}(\tilde{x})_E \) of \( \tau_{\tilde{w}}(\tilde{x}) \) is, according to the lemma of section 4, a minimal Cauchy filter of \( (E, w) \). After this, the same lemma shows that \( \tau_{\tilde{w}}(\tilde{x})_E \) is the trace in \( E \) of a minimal Cauchy filter \( \hat{u} \) of \( (\hat{E}, \hat{w}) \). Since \( (\hat{E}, \hat{w}) \) is complete, there exists a point \( \hat{x} \) of \( \hat{E} \) such that \( \hat{u} = \tau_{\hat{w}}(\hat{x}) \). Thus we see that with every point \( \tilde{x} \) of \( \tilde{E} \) we can associate a point \( \hat{x} \) of \( \hat{E} \) in such a way that
\[
(4) \quad \tau_{\hat{w}}(\hat{x})_E = \tau_{\tilde{w}}(\tilde{x})_E.
\]
If, in particular, \( \tilde{x} \) lies in \( E \), we may obviously suppose that \( \hat{x} = \tilde{x} \).

Now we consider the mapping \( f : (\tilde{E}, \tilde{w}) \to (\hat{E}, \hat{w}) \) defined by
\[
f(\tilde{x}) = \hat{x}.
\]
If \( \tilde{x} \) is in \( \tilde{E} - E \), then \( \hat{x} \) is in \( \hat{E} - E \). In fact, if \( \hat{x} \) were in \( E \), then
\[
\tau_{\tilde{w}}(\tilde{x})_E = \tau_{\hat{w}}(\hat{x})_E = \tau_{\tilde{w}}(\tilde{x})_E
\]
and, consequently, \( \tau_{\tilde{w}}(\tilde{x}) = \tau_{\hat{w}}(\hat{x}) \). This is a contradiction, however, since the points of \( \tilde{E} - E \) are \( T_1 \)-separated from other points of \( (\tilde{E}, \tilde{w}) \). If \( \hat{x} \) is an arbitrary point of \( \hat{E} - E \), then \( \tau_{\hat{w}}(\hat{x})_E \) is a minimal Cauchy filter of \( (E, w) \) and, as a consequence, there exists a point \( \tilde{x} \) of \( \tilde{E} \) such that
\[
\tau_{\tilde{w}}(\tilde{x})_E = \tau_{\hat{w}}(\hat{x})_E.
\]
By (4) we then have \( \tau_{\tilde{w}}(f(\tilde{x})) = \tau_{\tilde{w}}(\hat{x}) \), i.e. \( f(\tilde{x}) = \hat{x} \). Since \( f \) is the identity in \( E, \tilde{x} \) belongs to \( \tilde{E} - E \). This shows that \( f \) is a mapping from \( \tilde{E} - E \) onto \( \tilde{E} - E \). Finally we observe that if \( f(\tilde{x}) = f(\tilde{y}) \), then
\[
\tau_{\tilde{w}}(\tilde{x})_E = \tau_{\tilde{w}}(f(\tilde{x}))_E = \tau_{\hat{w}}(f(\tilde{y}))_E = \tau_{\tilde{w}}(\tilde{y})_E
\]
and, consequently, \( \tau_{\tilde{w}}(\tilde{x}) = \tau_{\tilde{w}}(\tilde{y}) \). From this it follows, if \( \tilde{x} \) and \( \tilde{y} \) belong to \( \tilde{E} - E \), that \( \tilde{x} = \tilde{y} \). This indicates that \( f \) is a one-to-one mapping from \( \tilde{E} - E \) onto \( \hat{E} - E \). Since it coincides in \( E \) with the identity mapping, it is thus a one-to-one map from \( \tilde{E} \) onto \( \hat{E} \).

Next we prove that \( f \) is a uniform isomorphism. Because of the symmetry it suffices to show that \( f \) is uniformly continuous. Let \( \hat{W} \) be an
arbitrary symmetric set of \( \tilde{w} \). Since the restriction \( f|E : (E, w) \to (\tilde{E}, \tilde{w}) \) is uniformly continuous, there exists a set \( \tilde{W} \) in \( \tilde{w} \) such that

\[
\tilde{W}^3 \cap (E \times E) \subseteq \tilde{W}.
\]

We consider an arbitrary point \((\tilde{x}, \tilde{y})\) of \( \tilde{W} \). From (4) it follows that there exists a symmetric set \( \tilde{V} \) of \( \tilde{w} \) such that \( \tilde{V} \subseteq \tilde{W} \) and

\[
\tilde{V}[\tilde{x}] \cap E \subseteq \tilde{W}[f(\tilde{x})] \cap E, \quad \tilde{V}[\tilde{y}] \cap E \subseteq \tilde{W}[f(\tilde{y})] \cap E.
\]

Since \( E \) is dense in \( \tilde{E} \), we can choose a point \( x \) from \( \tilde{V}[\tilde{x}] \cap E \) and a point \( y \) from \( \tilde{V}[\tilde{y}] \cap E \). From the relations \((x, \tilde{x}), (\tilde{y}, y) \in \tilde{V} \subseteq \tilde{W} \) and \((\tilde{x}, \tilde{y}) \in \tilde{W} \) it follows that \((x, y) \in \tilde{W}^3 \) and therefore, by (5), that \((x, y) \in \tilde{W} \).

On the other hand, by (6) \( x \in \tilde{W}[f(\tilde{x})] \) and \( y \in \tilde{W}[f(\tilde{y})] \), i.e. \( (f(\tilde{x}), x) \in \tilde{W} \) and \((y, f(\tilde{y})) \in \tilde{W} \) and, consequently, \((f(\tilde{x}), f(\tilde{y})) \in \tilde{W}^3 \). Thus we see that from \((\tilde{x}, \tilde{y}) \in \tilde{W} \) it always follows that \((f(\tilde{x}), f(\tilde{y})) \in \tilde{W}^3 \). This proves that \( f \) is a uniform isomorphism, as we wished to show.

We have seen that the identity mapping \( i : (E, w) \to (E, w) \) can be extended to a uniform isomorphism \( f : (\tilde{E}, \tilde{w}) \to (\tilde{E}, \tilde{w}) \). It remains to be shown that this extension is unique. In order to see this, we suppose that \( g : (\tilde{E}, \tilde{w}) \to (\tilde{E}, \tilde{w}) \) is another such extension and that \( f(\tilde{x}) \neq g(\tilde{x}) \). Then the points \( f(\tilde{x}) \) and \( g(\tilde{x}) \) belong to \( \tilde{E} - E \) and, as a consequence, \( \tau^w_0(f(\tilde{x})) \neq \tau^w_0(g(\tilde{x})) \). From this it follows, since the neighborhood filters are minimal Cauchy filters, that there exist disjoint sets \( \hat{A} \in \tau^w_0(f(\tilde{x})) \) and \( \hat{B} \in \tau^w_0(g(\tilde{x})) \). On the other hand, we see that, since \( f \) and \( g \) are continuous, there exists a neighborhood \( \hat{U} \) of \( \tilde{x} \) such that \( f(\hat{U}) \subseteq \hat{A} \) and \( g(\hat{U}) \subseteq \hat{B} \). Since \( E \) is dense in \( \tilde{E} \), we can choose a point \( x \) from \( \hat{U} \cap E \). Then the point \( x = f(x) = g(x) \) belongs to \( f(\hat{U}) \cap g(\hat{U}) \) and, consequently, to \( \hat{A} \cap \hat{B} \). This contradiction shows that \( f \) is unique.

7. Proof of the corollary. Now we suppose that \((E, w)\) is a uniform \( T_0 \)-space and show that then \((\tilde{E}, \tilde{w})\) is a \( T_0 \)-space, too. We have seen above that every point of \( \tilde{E} - E \) is \( T_0 \)-separated from every other point of \( \tilde{E} \). Thus we only have to consider two points \( x \) and \( y \), both belonging to \( E \). If they are not \( T_0 \)-separated, then \( \tau^w_0(x) = \tau^w_0(y) \) and, consequently, \( \tau^w_0(x)_E = \tau^w_0(y)_E \). On the other hand, since both \( \tau^w_0(x)_E \) and \( \tau^w_0(x) \) are minimal Cauchy filters in \((E, w)\) and since every set of the former filter intersects every set of the latter filter \((x)\), we see that \( \tau^w_0(x)_E = \tau^w_0(x) \) and, likewise, \( \tau^w_0(y)_E = \tau^w_0(y) \). Thus \( \tau^w_0(x) = \tau^w_0(y) \), from which it finally follows, since \((E, w)\) is a \( T_0 \)-space, that \( x = y \). This completes the proof of the corollary.
References


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