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ON A CLASS OF KLEINIAN GROUPS

BY

BERNARD MASKIT

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ON A CLASS OF KLEINIAN GROUPS

Let $\hat{C}$ denote the Riemann sphere ($\hat{C} = C \cup \{\infty\}$). A Kleinian group G is a group of Mobius transformations (directly conformal self maps of $\hat{C}$) which is discontinuous somewhere. The set of points at which G is discontinuous is called the *regular set* and is denoted by $R(G)$. A connected component of $R(G)$ is called a *component* of G.

Every Mobius transformation g can be written in the form $z \to (az + b) / (cz + d)$, $ad - bc = 1$. g determines a, b, c, and d up to sign, and $tr^2(g) = (a + d)^2$ is well defined. g is called *loxodromic* if $tr^2(g)$ does not lie in the closed real segment $[-4,4]$.

A Kleinian group G is called a π^p *group* if every $g \in G$, $g \not\equiv 1$, is loxodromic, and G is isomorphic to the fundamental group of a closed orientable surface of genus p, $p \geq 2$. G then has a presentation of the form $a_1, b_1, \ldots, a_p, b_p : \pi_{i=1}^p [a_i, b_i]$.

The main result of this paper (theorem 3) is that every π^p-group has a simply connected invariant component.

Let Γ be a Fuchsian π^p-group acting on the lower half plane L. A quadratic differential on Γ is a holomorphic function φ on L, where $\varphi(\gamma(z)) (\gamma'(z))^2 = \varphi(z)$, for all $\gamma \in \Gamma$. The space of all quadratic differentials on Γ, with norm

$$\|\varphi\| = \sup_{z = x + iy \in L} |y^2 \varphi(z)|$$

is denoted by $B(\Gamma)$.

Every $\varphi \in B(\Gamma)$ induces a homomorphism x_φ of Γ into the group of all Mobius transformations as follows. Given φ, there is a unique meromorphic function w_φ on L, where the Schwartzian derivative of w_φ equals φ, and near $-i$, $w_\varphi(z) = 1/(z + i) + 0(|z + i|)$. For each $\gamma \in \Gamma$, there is a unique Mobius transformation $x_\varphi(\gamma)$ so that

$$w_\varphi \circ \gamma(z) = x_\varphi(\gamma) \circ w_\varphi(z), \text{ for all } z \in L.$$

A point $\varphi \in B(\Gamma)$ is called *regular* if x_φ is an isomorphism, and if every non-trivial element of $x_\varphi(\Gamma)$ is loxodromic. It was observed by Kra [5], that almost all $\varphi \in B(\Gamma)$ are regular.

Our main result asserts that if $\varphi \in B(\Gamma)$ is regular and $x_{\varphi}(\Gamma)$ is discontinuous, then $x_{\varphi}(\Gamma)$ has a simply connected invariant component. One might suppose that this would imply that w_{φ} is univalent in L. Theorem 5 shows that this is not necessarily true.

Throughout this paper we will use various well known facts about quadratic differentials, Fuchsian and quasi-Fuchsian groups, and quasiconformal mappings. The reader is referred to [2] for the basic definitions and proofs.

The author would like to thank Irwin Kra for several conversations during which he raised these questions.

Theorem 1: *Let G be a Kleinian group isomorphic to the fundamental group of a closed orientable surface of genus $p \geq 2$. Suppose that G has an invariant component R_0. Then R_0 is simply connected and R_0/G is a closed orientable surface of genus p.*

Proof: G is finitely generated. By Ahlfors theorem [1] $S = R_0/G$ is a closed Riemann surface from which a finite number of points have been removed.

Assume that R_0 is not simply connected. Then by the planarity theorem [6], there is a set $w_1, \ldots, w_q, q \geq 1$, of simple disjoint loops on S with the following properties. No w_i bounds either a disc or a punctured disc on S. Each w_i lifts to a loop on R_0. R_0 with group G is the highest regular covering of S for which $w_1, \ldots, w_q$ lift to loops.

Let K' be the 2-complex obtained from S by passing abstract discs $p_1, \ldots, p_q$ through $w_1, \ldots, w_q$. By lifting $p_1, \ldots, p_q$, to abstract discs sewed onto R_0, in all possible ways, we see that G is isomorphic to $\pi_1(K')$.

There are elementary homotopy equivalences to show that if $q \geq 1$, then $\pi_1(K')$ is a non-trivial free product (see [6] pgs. 352—3 and [7] pg. 228). Briefly, if w_i is dividing, contract p_i to a point; if w_i is non-dividing, contract p_i to a point, pull the two pieces of surface apart stretching the point into a 1-cell, and then pull the endpoints of the 1-cell together. The resulting 2-complex K is clearly a wedge product of homotopically non-trivial components. Hence $\pi_1(K') = \pi_1(K)$ is a non-trivial free product.

G is isomorphic to the fundamental group of a closed orientable surface, hence every subgroup of G is either free or of finite index. If G were a non-trivial free product, $G = A * B$, then A and B, being infinite groups, would both have infinite index in G. Hence A and B would both be free, and so G would be free, which it isn't.

We conclude that R_0 is simply connected. It follows at once from the classification of surfaces that S is a closed Riemann surface of genus p.

The following theorem is a special case of theorem 4 in [8].

Theorem 2: *Let G be a π^p-group with a simply connected invariant component R_0. Then either*

1) *G is degenerate;* i.e. *$R(G) = R_0$, or*
2) *G is quasi-Fuchsian;* i.e. *there is a Fuchsian group Γ, and a quasiconformal homeomorphism $w : \hat{C} \to \hat{C}$ so that $G = w \circ \Gamma \circ w^{-1}$.*

Theorem 3: *Let G be a π^p-group. Then G has a simply connected invariant component.*

Proof: Assume not. Then G has at least two components R_0 and R_1. Let H be the subgroup of G keeping R_0 invariant.

R_0/H is a connected component of $R(G)/G$. By Ahlfors theorem [1], R_0/H is a finite surface, and so H is finitely generated.

H is a subgroup of G, and so H is either free, or H is a π^q-group for some $q > p$. Is H were free, then by [7], H would be a Schottky group, contradicting the fact that H has at least two components.

H has an invariant component R_0. By theorem 1, R_0 is simply connected. By assumption R_0 is a proper subset of $R(G) \subset R(H)$ and so, by theorem 2, H is quasi-Fuchsian. There is then a Fuchsian group Δ^*, acting on the upper half plane U, and a global quasiconformal homeomorphism W, where $W(U) = R_0$, and $W \circ \Delta \circ W^{-1} = H$. Let $R_1' = W(L)$.

H is of finite index in G, hence $R(G) = R(H) = R_0 + R_1'$. R_0 and R_1' are both invariant under H, hence $[G : H] = 2$, and, using the Riemann-Hurwitz formula, $q = 2p - 1$.

Let g be some element of $G - H$. Then $g \circ h \circ g^{-1} \in H$. Let r denote reflection in the real axis. Then $W \circ r \circ W^{-1}$ commutes with every element of H. It follows that $\bar{g} = g \circ W \circ r \circ W^{-1}$ is an orientation reversing homeomorphism of R_0, where $\bar{g} \circ h \circ \bar{g}^{-1} \in H$ for every $h \in H$.

Let Γ be some Fuchsian π^p-group acting on U. Let $\psi : G \to \Gamma$ be some isomorphism, and let $\Delta = \psi(H)$. U/Δ and $S = R_0/H$ are both closed Riemann surfaces of genus q, hence they are homeomorphic. Every isomorphism of $\pi_1(S)$ is induced by a homeomorphism of S (Nielsen [9], see also Zieschang [10]). Hence there is a homeomorphism $V : U \to R_0$ so that $V^{-1} \circ h \circ V = \psi(h)$ for every $h \in H$.

Let $\gamma = \psi(g)$, and set $\gamma^* = \gamma^{-1} \circ V^{-1} \circ \bar{g} \circ V$. γ^* is an orientation reversing homeomorphism of U, and γ^* commutes with every element of Δ, which is absurd.

Theorem 4: *There exists a Kleinian group G, which is isomorphic to the fundamental group of a closed orientable surface, and which has no invariant component.*

Proof: We write the Mobius transformation $z \to (\alpha z + \beta) / (\gamma z + \delta)$ as $(\alpha, \beta; \gamma, \delta)$. Let $a = (1, -4; 0, 1)$, $b = (1, 0; 1, 1)$, $c = (-3, 4; -1, 1)$,

$d = (-i, 2i; -1, 2 + i)$, $e = (1 + 4i, 16; 1, 1-4i)$, $f = (3 + 4i, 12-16i; 1, -1-4i)$, and $g = (3i, 8-6i; 1, -2-3i)$. Let G be the group generated by $a, \ldots, g$. We remark that G is a subgroup of the Picard group.

Let C_1 and C_2 be the lines $Re\, z = 3$ and $Re\, z = -1$, respectively. Let $C_3, \ldots, C_{12}$ be the circles of radius 1 with centers at $-1, 1, 3, 2 + i, i, -1 + 4i, 1 + 4i, 3 + 4i, 2 + 3i, 3i$ respectively. Let $P_1 \ldots, P_{12}$ be the non-Euclidean planes in the upper half space, whose boundaries are the circles $C_1, \ldots, C_{12}$, respectively. Observe that $a(P_1) = P_2$, $b(P_3) = P_4$, $c(P_4) = P_5$, $d(P_6) = P_7$, $e(P_8) = P_9$, $f(P_9) = P_{10}$, and $g(P_{11}) = P_{12}$. Splitting P_4 and P_9 into two pieces each, using the plane bounded by $Re\, z = 1$, we get a polyhedron Q whose sides are pairwise identified by $a, \ldots, g$. We also observe that any two interesting sides of Q meet an angle of $\pi/2$ or π.

By Poincaré's theorem (see [3] pg. 174—8), Q is a fundamental polyhedron for G. It follows that $R(G)/G$ is the disjoint union of four 3-times punctured spheres. It also follows that $cba = d^{-1}bdc = afe = g^{-1}\, egf = 1$ is a complete set of relations for G. One sees at once that G is generated by $b, d, e,$ and g, and that these generators satisfy the one defining relation $b^{-1}\, d^{-1}\, bd\, g^{-1}\, e^{-1}\, ge = 1$.

Remark 1: G constructed above is isomorphic to a π^2-group. For every $p = 3, 4, \ldots,$ there is a Kleinian group which is isomorphic to a π^p-group, and which has no invariant component. The construction of such groups is considerably more complicated, and not worth the effort.

Theorem 5: *Let G be a Fuchsian π^p-group. Then there is a Mobius transformation a, a Fuchsian π^p-group Γ, and a* regular $\varphi \in B(\Gamma)$, so that $X_\varphi(\Gamma) = a \circ G \circ a^{-1}$, *and so that w_φ is not univalent in L.*

Proof: We normalize G so that the negative imaginary axis projects onto a simple closed curve on L/G. There is then a smallest $\varrho > 1$ so that $z \to \varrho z$ is an element of G. There is also an $\alpha > 0$ so that no two points of

$$B = \{z | 1 \leq |z| < \varrho\,, \quad 3\pi/2 - \alpha \leq arg\, z \leq 3\pi/2 + \alpha\}$$

are equivalent under G.

A fundamental set for G acting on L is a set $D \subset L$ so that the natural projection $p : L \to L/G$, when restricted to D, is a one-to-one map of D onto L/G. It is well known that there is such a fundamental set D with the following additional properties. D is bounded by a non-Euclidean polygon whose sides are pairwise identified by elements of G; these elements generate G. $B \subset D \subset \{z | 1 \leq |z| < \varrho\}$.

Let $v(\Theta)$ be a C^∞ real valued function with the following properties. v is a diffeomorphism of $[3\pi/2 - \alpha, 3\pi/2 + \alpha]$ onto $[3\pi/2 - \alpha, 7\pi/2 + \alpha)$.

$v(3\pi/2 - \alpha) = 3\pi/2 - \alpha$ $v'(3\pi/2 - \alpha) = v'(3\pi/2 + \alpha) = 1$. All the higher derivatives of v vanish at the endpoints.

We now define a function f on L. For $z \in D - B$, set $f(z) = z$. For $z = re^{i\Theta} \in B$, set $f(z) = r\, e^{iv(\Theta)}$. If $z \in L - D$, then there is a $g \in G$, and a $w \in D$, so that $g(w) = z$; set $f \circ g(w) = g \circ f(w)$. One sees at once that f is a C^∞ map of L onto $\hat{C}$, f is a local homeomorphism, and for every $g \in G$, $f \circ g = g \circ f$.

For $z \in L$, set $\mu(z) = f_{\bar{z}}/f_z$. Observe that for $g \in G$,

$$\mu(g(z))\, g'(z)/\overline{g'(z)} = \mu(z)\,.$$

By the compactness of D, there is a $k < 1$, so that $|\mu(z)| \leq k$. For $z \in U$, set $\mu(z) = \overline{\mu(z)}$. For $z \in R \cup \{\infty\}$, set $\mu(z) = 0$.

Let w be the global quasiconformal homeomorphism satisfying $w_{\bar{z}} = \mu(z)\, w_z$, $w(O) = O$, $w(1) = 1$, $w(\infty) = \infty$. Then $w(L) = L$, and for every $g \in G$, $w \circ g \circ w^{-1}$ is a Möbius transformation. Set $\Gamma = w \circ G \circ w^{-1}$. $f \circ w^{-1}$ is meromorphic in L, and is a local homeomorphism; hence φ, the Schwartzian derivative of $f \circ w^{-1}$ is holomorphic. $\varphi \in B(\Gamma)$, since $(f \circ w^{-1}) \circ (w \circ g \circ w^{-1}) = g \circ (f \circ w^{-1})$. w_φ and $f \circ w^{-1}$ have the same Schwartzian derivative and so there is a Möbius transformation a so that $w_\varphi = a \circ (f \circ w^{-1})$. It follows that $x_\varphi(\Gamma) = a \circ G \circ a^{-1}$.

Remark 2: There is a simple modification of the above construction to yield the same result starting with a quasi-Fuchsian group G. There is also an obvious modification of the construction to yield a sequence of quadratic differentials φ_n in different spaces $B(\Gamma_n)$, with the same properties.

Remark 3: Let $\varphi \in B(\Gamma)$ be as in theorem 5. Then by quasiconformal stability [4], there is a neighborhood U of φ, so that for every $\psi \in U$, $x_\psi(\Gamma)$ is quasi-Fuchsian and w_ψ is not univalent.

Remark 4: Let Γ be a Fuchsian π^p-group. It is not known whether or not there is a regular $\varphi \in B(\Gamma)$ so that x_φ is not univalent.

Remark 5: Let $\varphi \in B(\Gamma)$ be as in theorem 5. It was shown by Kra [5] that $w_\varphi(L) = \hat{C}$, and that w_φ is not a cover map.

Massachusetts Institute of Technology
Cambridge, Mass. U.S.A.

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