REMOVABILITY THEOREMS FOR QUASICONFORMAL MAPPINGS

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1. Introduction. In this paper we shall study the following removability question: Let $D$ and $D'$ be domains in the euclidean $n$-space $\mathbb{R}^n$, $n \geq 2$, let $E \subset D$ be closed in $D$, and let $f: D \rightarrow D'$ be a homeomorphism which is locally $K$-quasiconformal in $D - E$ for some $K$, which means that for every $x \in D - E$ there is a connected neighborhood $U$ of $x$ such that $f \mid U$ is a $K$-quasiconformal mapping [8, p. 20]. We ask for conditions on $E$ and on the restriction $f \mid E$ which imply the quasiconformality of $f$. A special case for $n = 2$ of this situation is considered in [5, Theorem 3] which implies that $f$ is quasiconformal if $E$ is a Jordan curve and if $f \mid E = g \mid E$ for some quasiconformal mapping $g$ of a domain $G \supset E$.

The set $E$ is called an exceptional set if $f$ is always a $\varphi$-quasiconformal mapping. One of the main results which give conditions for the exceptionality is that the set $E$ is exceptional if $E$ is of $\sigma$-finite $(n-1)$-dimensional Hausdorff measure [9, 35.1], [3, Corollary 5]; for the case $n = 2$ see also [7], [1], and [4, Satz V.3.2]. We shall give answers to the given problem in the other direction. It turns out (Theorem 1) that the condition mentioned above, namely the existence of a quasiconformal mapping $g$ of a domain $G \supset E$ such that $f \mid E = g \mid E$, implies the quasiconformality of $f$ even if no further assumptions are made on $E$. We shall also in Theorem 2 establish another form where the assumption on the restriction $f \mid E$ is weakened but $E$ is assumed to be connected or locally connected. In these results the maximal dilatation of $f$ is in general greater than $K$.

2. Notation. Throughout this paper $D$ and $D'$ are domains in $\mathbb{R}^n$ and $n \geq 2$. If $A, B \subset \mathbb{R}^n$, $d(A, B)$ denotes the euclidean distance between $A$ and $B$. For $x \in \mathbb{R}^n$ and $r > 0$ we set $B^n(x, r) = \{y \in \mathbb{R}^n \mid |y - x| < r\}$ and $S^{n-1}(x, r) = \{y \in \mathbb{R}^n \mid |y - x| = r\}$. We also use the abbreviations $B^n(r) = B^n(0, r)$ and $S^{n-1}(r) = S^{n-1}(0, r)$. If $f: D \rightarrow D'$ is a homeomorphism, if $x \in D$, and if $0 < r < d(x, \partial D)$, we set

$$L(x, f, r) = \sup_{|y - x| = r} |f(y) - f(x)|,$$

$$l(x, f, r) = \inf_{|y - x| = r} |f(y) - f(x)|.$$

The linear dilatation of $f$ at $x$ is
The $k$-dimensional Lebesgue measure is denoted by $m_k$. The $(n-1)$-dimensional measure of the unit sphere $S^{n-1}(1)$ is $\omega_{n-1}$.

3. We start with a simple distortion result for quasiconformal mappings. Let $v$ be an increasing function of the interval $[1, \infty)$ into itself and let $\varphi : A \to \mathbb{R}^n$, $A \subset \mathbb{R}^n$, be an injective mapping. We say that $\varphi$ has local $v$-bounded distortion if for every $x \in A$ there is an $s > 0$ such that $x_1, x_2 \in A$ and $0 < |x - x_2| \leq |x - x_1| < s$ imply

$$\frac{|\varphi(x_1) - \varphi(x)|}{|\varphi(x_2) - \varphi(x)|} \leq v\left(\frac{|x_1 - x|}{|x_2 - x|}\right).$$

**Lemma 1.** Let $f : D \to D'$ be a $K$-quasiconformal mapping. Then there exists an increasing function $v : [1, \infty) \to [1, \infty)$ depending only on $n$ and $K$ such that $f$ has local $v$-bounded distortion.

**Proof.** Assume $x \in D$. Choose $s > 0$ such that $fB^2(x, s) \subset B^2(f(x), t) \subset D'$ for some $t$. Let $0 < |x_2 - x| \leq |x_1 - x| < s$ and set $\alpha_i = |f(x_i) - f(x)|$, $i = 1, 2$. Assume $x_2 < \alpha_1$ and let $I''$ be the family of curves which join the boundary components of the ring $A' = \{y \mid x_2 < |y - f(x)| < x_1\}$ in $A'$. Then the $n$-modulus $M(I'')$ of $I''$ equals $\omega_{n-1}/(\log (\alpha_1/\alpha_2))^{n-1}$ [8, p. 5, 7]. For the $n$-modulus of the curve family $I = f^{-1}I'' = \{f^{-1} \circ \gamma' \mid \gamma' \in I''\}$ we get by [9, 11.7] (see also [2, Theorem 4]) the estimate

$$M(I) \geq \kappa_n\left(\frac{|x_1 - x|}{|x_2 - x|}\right)$$

where $\kappa_n : (0, \infty) \to (0, \infty)$ is a decreasing function which depends only on $n$. Since $f$ is $K$-quasiconformal, $M(I) \leq K M(I'')$. Hence

$$\frac{|f(x_1) - f(x)|}{|f(x_2) - f(x)|} \leq \exp\left(-\frac{K \omega_{n-1}}{\kappa_n}\left(\frac{1}{|x_1 - x|}\right)^{1/(n-1)}\right) = v\left(\frac{|x_1 - x|}{|x_2 - x|}\right),$$

and the lemma is proved.

The main step is the following lemma (cf. [5, Lemma 3]).

**Lemma 2.** Let $E \subset D$ be closed in $D$ and let $f : D \to D'$ be a homeomorphism which is locally $K$-quasiconformal in $D - E$ and such that $f \upharpoonright E$ has local $v$-bounded distortion for some $v$. Let $E_0$ be the set of points $x \in E$ such that for every integer $j$ there exists an integer $k \geq j$ such that $(B^2(x, 1/k) - B^2(x, 1/2k)) \cap E = \emptyset$. Then
(a) \( m_n(E_0) = 0 \).

(b) There exists a \( c < \infty \), depending only on \( n \), \( K \), and \( v \), such that \( H(x,f) < c \) if \( x \in D - E_0 \).

Proof. Since no point of \( E_0 \) is a point of outer density for \( E_0 \), \( m_n(E_0) = 0 \) [6, p. 129].

To prove (b) it suffices by [9, 34.2] to show that a uniform bound exists for \( H(x,f) \) for points \( x \in E - E_0 \). Let \( x_0 \in E - E_0 \). By performing similarity transformations we may assume that \( x_0 = f(x_0) = 0 \). There exists an integer \( k_0 \) such that \( (B^n(1/k) - B^n(1/2k)) \cap E \neq \emptyset \) for \( k \geq k_0 \). Since \( f \mid E \) has local \( v \)-bounded distortion, there exists an \( s > 0 \) such that if \( x_1, x_2 \in E \) and if \( 0 < |x_2| \leq |x_1| < s \), then

\[
\frac{|f(x_1)|}{|f(x_2)|} \leq v \left( \frac{|x_1|}{|x_2|} \right).
\]

Now let \( r \) be such that \( 0 < r < \min (d(0, \partial D), s, 1/k_0)/8 \) and such that \( B^n(L(0, f, r)) \subset D' \). We set

\[
L_r = L(0, f, r), \quad l_r = l(0, f, r),
\]

\[
A_1 = \{ x \mid r < |x| < 2r \}, \quad A_2 = \{ x \mid r/2 < |x| < r \},
\]

\[
H_1 = \{ x \mid 2r < |x| < 8r \}, \quad H_2 = \{ x \mid r/8 < |x| < r/2 \},
\]

\[
F_i = \bar{A}_i \cup \bar{H}_i, \quad i = 1, 2,
\]

\[
r_1 = \sup_{x \in E \cap F_1} |f(x)|, \quad r_2 = \inf_{x \in E \cap F_2} |f(x)|.
\]

We shall make use of the fact that the sets \( \bar{A}_1 - f^{-1}B^n(r_1) \) and \( \bar{A}_2 \cap f^{-1}B^n(r_2) \) do not meet \( E \).

Assume \( L > r_1 \) and let \( z \in fS^{n-1}(r) \) be such that \( |z| = L \). There exists \( r_1 > 1 \) such that the line segment \( J = \{ tz \mid 1 \leq t \leq r_1 \} \) is contained in \( f\bar{A}_1 \) and such that \( r_1z \in fS^{n-1}(2r) \). Assume \( r_1 < \sigma < L_r \) and let \( \Gamma \) be the family of curves which join \( f^{-1}J \) and \( f^{-1}S^{n-1}(\sigma) \) in \( \bar{A}_1 - f^{-1}B^n(\sigma) \).

Next we derive a positive lower bound for the \( n \)-modulus \( M(\Gamma) \) of \( \Gamma \). Let \( r < t < 2r \) and set \( S = S^{n-1}(t) \). Then \( S \cap f^{-1}J \neq \emptyset \). We show that also \( S \cap f^{-1}S^{n-1}(\sigma) \neq \emptyset \) holds. To prove this we first note that \( H_1 \cap E \neq \emptyset \). There is therefore a point \( u \in \bar{B}^n(r_1) \cap fH_1 \). The line segment \( \{ \lambda u \mid 0 < \lambda < 1 \} \) meets \( fS \). The assertion then follows from the fact that \( fS \) has points in both components of the complement of \( S^{n-1}(\sigma) \).

We now choose a point \( y \in S \cap f^{-1}J \). Since \( y \) does not belong to the non-empty closed set \( S \cap f^{-1}S^{n-1}(\sigma) \), there exists an open half space \( M \) such that \( y \in M \), \( M \cap S \subset S - f^{-1}S^{n-1}(\sigma) \), and \( \bar{M} \cap S \cap f^{-1}S^{n-1}(\sigma) \neq \emptyset \). Denote by \( \Gamma' \), the family of curves \( \gamma \in \Gamma \) which lie in \( \bar{M} \cap S \).
For the $n$-modulus $M_n^S(\Gamma_i)$ of $\Gamma_i$ with respect to $S$ the estimate $M_n^S(\Gamma_i) \geq b_n/t$ holds where $b_n > 0$ is a constant which depends only on $n$ [9, 10.2]. If $\varrho : R^n \to [0, \infty]$ is a Borel function such that

$$\int_{\gamma} \varrho \, ds \geq 1$$

for every rectifiable $\gamma \in \Gamma$, we have

$$\int_{s} \varrho^n \, dm_{n-1} \geq M_n^S(\Gamma_i)$$

by definition. Hence

$$\int_{\lambda} \varrho^n \, dm \geq \int_{\lambda} \varrho^n \, dm = \int_{r} \left( \int_{s} \varrho^n \, dm_{n-1} \right) \, dt$$

$$\geq \int_{r} b_n \, dt = b_n \log 2 .$$

This gives $M(\Gamma) \geq b_n \log 2 > 0$.

On the other hand, the ring $B^n(L_c) - B^n(\sigma)$ separates $J$ and $S^{n-1}(\sigma)$. Consequently, $M(\Gamma') \leq \omega_{n-1}(\log(L_c/\sigma))^{n-1}$ [8, p. 7] where $\Gamma' = \{ f \circ \gamma \mid \gamma \in \Gamma \}$. Let $D_1$ be the component of $D - E$ which contains $f^{-1}J$. Then every curve of $\Gamma$ lies in $D_1$. Since $f \mid D - E$ is locally $K$-quasiconformal, $f \mid D_1$ is $K$-quasiconformal, and we have $M(\Gamma) \leq K M(\Gamma')$. This gives $b_n \log 2 \leq K \omega_{n-1}(\log(L_c/\sigma))^{n-1}$. Hence

$$\frac{L_r}{r_1} \leq \exp \left( \frac{1}{b_n \log 2} \right) = a_n .$$

Similarly one proves $l_r/l_2 \geq a_n^{-1}$.

Let $x_i \in E \cap F_i$ be such that $|f(x_i)| = r_i$, $i = 1, 2$. Then $|x_1|/|x_2| \leq 64$. Finally we obtain the estimate

$$\frac{L_r}{t_r} \leq a_n^2 v(64) ,$$

which proves (b).

**Theorem 1.** Let $E \subset D$ be closed in $D$ and let $f : D \to D'$ be a homeomorphism which is locally $K$-quasiconformal in $D - E$ for some $K$. Suppose that there exists a quasiconformal mapping $g$ of a domain $G, E \subset G \subset D, $ such that $g \mid E = f \mid E$. Then $f$ is quasiconformal.
Proof. By Lemma 1, $f \mid E$ has local $v$-bounded distortion where $v$ depends only on $n$ and on the maximal dilatation $K(g)$ of $g$. By Lemma 2, there exists a set $E_0 \subset E$ of measure zero such that $H(x, f) < c < \infty$ for $x \in D - E_0$ where $c$ depends only on $n$, $K$, and $K(g)$. By an $n$-dimensional version of [8, Lemma 6.3], $f$ is differentiable almost everywhere. If $f$ is differentiable at $x \in D - E_0$, $|f'(x)|^n \leq c^{n-1}|J(x, f)|$ where $f'(x)$ is the derivative and $J(x, f)$ the Jacobian of $f$ at $x$. We shall show that $f$ is ACL [8, p. 15]. The quasiconformality of $f$ then follows from an $n$-dimensional version of [8, Theorem 6.11].

It suffices to prove that $f$ is ACL in $G$. To show this, let $Q$ be an open $n$-interval such that $Q \subset G$. Let $P : R^n \to R^{n-1}$ be the orthogonal projection. For each Borel set $A \subset PQ$ we set $Z_A = Q \cap P^{-1}A$ and $\psi(A) = m_n(fZ_A)$. By Lebesgue’s theorem, the set function $\psi$ has a finite derivative $\psi'(y)$ for almost every $y \in PQ$. Furthermore, $g$ is absolutely continuous on $Z_y$ and $m_1(E_0 \cap Z_y) = 0$ for almost every $y \in PQ$. Fix $y \in PQ$ such that all these three conditions are satisfied. By symmetry, it is sufficient to prove that $f$ is absolutely continuous on $Z_y$.

Let $F$ be a compact subset of $Z_y$. Since $g$ is absolutely continuous on $Z_y$, since $g \mid E_0 = f \mid E_0$, and since $m_1(E_0 \cap F) = 0$, we have $A_1(f(E_0 \cap F)) = 0$ where $A_1$ is the 1-dimensional Hausdorff measure. Hence $A_1(fF) = A_1(f(F - E_0))$. Let $k_0$ be an integer such that $0 < 1/k_0 < d(F, \partial Q)$. For each integer $k \geq k_0$ we define the set $F_k$ of points $x \in F$ such that $0 < r < 1/k$ implies $L(x, f, r) \leq c l(x, f, r)$. For every $k \geq k_0$ $F_k$ is compact and $F_k \subset F_{k+1}$. Since $H(x, f) < c$ for $x \in F - E_0$, we have

$$F - E_0 \subset \bigcup_{k = k_0}^{\infty} F_k = \hat{F}$$

and one can prove the inequality (see [9, (31.3)] and [1, p. 10])

$$A_1(f\hat{F})^n \leq c^n \psi'(y) m_1(F)^n$$ \hspace{1cm} (1)

where $\alpha < \infty$ is a constant which depends only on $n$. Consequently, also $A_1(fF)^n$ has the right hand side of (1) as an upper bound. After this a simple limiting process shows that $f$ is absolutely continuous on $Z_y$. The theorem is proved.

Theorem 2. Let $E \subset D$ be connected or locally connected and closed in $D$. Let $f : D \to D'$ be a homeomorphism which is locally $K$-quasiconformal in $D - E$ for some $K$. Suppose further that $f \mid E$ has local $v$-bounded distortion for some $v$. Then $f$ is quasiconformal.

Proof. The set $E_0 \subset E$ defined in Lemma 2 consists in this case of isolated points only, and $D - E_0$ is a domain. By (b) in Lemma 2 and by [9, 34.1] $f \mid D - E_0$ is quasiconformal. But $E_0$ is removable [9, 17.3], and the theorem is proved.
Remark. If \( \varphi \) has local \( v \)-bounded distortion for some \( v \), it does not necessarily follow that \( \varphi \) is a restriction of a quasiconformal mapping. This is shown by an \( n \)-dimensional version of the example presented in [5, p. 388]. Hence the condition on \( f \mid E \) is in this sense weaker in Theorem 2 than in Theorem 1.

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References