ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

524

ON AN INEQUALITY CONNECTED WITH THE COEFFICIENT CONJECTURE FOR FUNCTIONS OF BOUNDED BOUNDARY ROTATION

BY

D. AHARONOV and S. FRIEDLAND

HELSINKI 1972
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1973.524

ISBN 951-41-0065-4

Communicated 13 October 1972

KESKUSKIRJAPAINO
HELSINKI 1973

1. Introduction

For $k \geq 2$ let V_k denote the class of locally univalent analytic functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \tag{1.1}$$

that map $|z| < 1$ conformally onto a domain whose boundary rotation is at most $k\pi$. (See [5] for the definition and basic properties of the class V_k).

The function

$$f_k(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right] = \sum_{n=1}^{\infty} A_n z^n \tag{1.2}$$

belongs to V_k and the coefficient conjecture for the class V_k is that for a function (1.1) in V_k,

$$|a_n| \leq A_n \ (n > 1) \tag{1.3}$$

This conjecture was proved for $n = 2$ by Pick (see [5]) for $n = 3$ by Lehto [5] and for $n = 4$ in [9], [6], [1] and [3].

In support of the conjecture Noonan has shown [7] that for a given function (1.1) in V_k, $\lim\limits_{n\to\infty} \dfrac{|a_n|}{A_n}$ exists and in less then 1 unless $f(z) = e^{-i\varphi} f_k(e^{i\varphi} z)$. Recently Brannan, Clunie and Kirwan [2] established the conjecture (1.3) for $n \leq 14$ and for all n in case that function (1.1) has real coefficients or if $k \geq 4$. This was done by a remarkable extension of the classical Herglotz formula. With the aid of this generalized formula they showed that the conjecture would follow for any $n > 1$ and any $k \geq 2$ if the following inequality

$$\left(\frac{1+xz}{1-z}\right)^{\alpha} \ll \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad \alpha \geq 1\,, \quad |x| = 1 \tag{1.4}$$

holds. By $\sum\limits_{n=1}^{\infty} \alpha_n z^n \ll \sum\limits_{n=1}^{\infty} \beta_n z^n$ we mean $|\alpha_n| \leq |\beta_n|$ for $n = 1, 2, \ldots$.

The aim of this paper is to prove the inequality (1.4) and thus to establish the coefficient conjecture. In fact, as was shown in [2], the inequality (1.4) implies the coefficient conjecture (1.3) for the larger class of close-to-convex functions of order $\beta(\beta = \frac{k}{2} - 1 \geq 0)$. This class was introduced by Pommerenke [8]. (See also Goodman [4] for further properties of this class).

2. A stronger inequality

In order to show (1.4) it is more convenient to consider the inequality

$$\frac{(1+xz)^\alpha}{1-z} \ll \frac{(1+z)^\alpha}{1-z}\,, \quad \alpha \geq 1\,, \quad |x| = 1\,. \tag{2.1}$$

It is clear that (2.1) implies (1.4) since

$$\left(\frac{1+xz}{1-z}\right)^\alpha = \frac{(1+xz)^\alpha}{1-z}\,\frac{1}{(1-z)^{\alpha-1}} \ll \frac{(1+z)^\alpha}{1-z}\,\frac{1}{(1-z)^{\alpha-1}} = \left(\frac{1+z}{1-z}\right)^\alpha .$$

Obviously it is enough to consider the case $1 < \alpha < 2$. Indeed $(1-xz)^p \ll (1+z)^p$ for any natural p and thus

$$\frac{(1+xz)^\alpha}{1-z}(1-xz)^p \ll \frac{(1+z)^\alpha}{1-z}(1+z)^p$$

We note that

$$(1+xz)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k z^k \tag{2.2}$$

Now

$$\frac{(1+xz)^\alpha}{1-z} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k z^k \sum_{m=0}^{\infty} z^m = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \binom{\alpha}{k} x^k\Big) z^n .$$

Thus (2.1) is equivalent to

$$\Big|\sum_{k=0}^{n} \binom{\alpha}{k} x^k\Big| \leq \sum_{k=0}^{n} \binom{\alpha}{k}\,, \quad \alpha \geq 1\,, \quad |x| = 1\,, \quad n = 1, 2, \ldots \tag{2.3}$$

For $n = 1, 2$ the inequality evidently holds. Let $x = e^{i\phi}$ and consider separately the inequality (2.3) at first for $0 \leq \phi \leq \frac{\pi}{n}$ and after for $\frac{\pi}{n} \leq \phi \leq \pi$.

3. The case $0 \leq \phi \leq \frac{\pi}{n}$

The inequality (2.3) can be written as

$$(3.1)\qquad [\sum_{k=0}^{n} \tbinom{\alpha}{k} \cos k\phi]^2 + [\sum_{1}^{n} \tbinom{\alpha}{k} \sin k\phi]^2 \le [\sum_{k=0}^{n} \tbinom{\alpha}{k}\,]^2$$

or

$$[\sum_{k=1}^{n} \tbinom{\alpha}{k} \sin k\phi]^2 \le [\sum_{k=0}^{n} \tbinom{\alpha}{k} (1 - \cos k\phi)\,] [\sum_{k=0}^{n} \tbinom{\alpha}{k} (1 + \cos k\phi)\,]$$

$$= 4 \left[\sum_{k=0}^{n} \tbinom{\alpha}{k} \sin^2 \frac{k\phi}{2}\right] \left[\sum_{k=0}^{n} \tbinom{\alpha}{k} - \sum_{k=0}^{n} \tbinom{\alpha}{k} \sin^2 \frac{k\phi}{2}\right].$$

Noting that $\dfrac{\sin u}{u}$ is decreasing for $0 \le u \le \pi$ we obtain

$$|\tbinom{\alpha}{k} \sin k\phi| \ge |\tbinom{\alpha}{k+1} \sin (k+1)\phi|$$

for $1 \le k \le n-1$, $0 \le \phi \le \dfrac{\pi}{n}$ and $1 < \alpha < 2$.

This implies

$$\sum_{k=1}^{n} \tbinom{\alpha}{k} \sin k\phi = \alpha \sin\phi - \frac{\alpha(\alpha-1)}{2} \sin 2\phi - \frac{\alpha(\alpha-1)(2-\alpha)}{6} \sin 3\phi + \dots$$

$$\le \alpha \sin\phi + \frac{\alpha(\alpha-1)}{2} \sin 2\phi\,.$$

Thus (3.1) is true if

$$(3.2)\qquad [\alpha \sin\phi + \alpha(\alpha-1) \sin\phi \cos\phi]^2 \le 4 \left[\sum_{k=1}^{n} \tbinom{\alpha}{k} \sin^2 \frac{k\phi}{2}\right] \left[\sum_{k=0}^{n} \tbinom{\alpha}{k} - \sum_{k=0}^{n} \tbinom{\alpha}{k} \sin^2 \frac{k\phi}{2}\right] = 4\, w[\sum_{k=0}^{n} \tbinom{\alpha}{k} - w]$$

In fact we show the stronger inequality

$$(3.3)\qquad \alpha^2 \sin^2\phi\, [1 + 2(\alpha-1) + (\alpha-1)^2 \cos^2\phi] \le 4w[\sum_{k=0}^{n} \tbinom{\alpha}{k} - w]$$

Using again the monotonicity of $\dfrac{\sin u}{u}$ we have

$$\left|\binom{\alpha}{k}\right| \sin^2 \frac{k\phi}{2} \ge |\tbinom{\alpha}{k+1}| \sin^2 \frac{(k+1)}{2}\phi\,, \quad 2 \le k \le n-1\,, \quad 0 \le \phi \le \frac{\pi}{n}\,,$$

as

$$\frac{k-\alpha}{k+1}\left(\frac{k+1}{k}\right)^2 < 1 \quad \text{for } 1 < \alpha < 2\,.$$

For $n \geq 3$ and $\frac{n\phi}{2} \leq \frac{\pi}{2}$ we obtain

$$w = \sum_{k=0}^{n} \binom{\alpha}{k} \sin^2 \frac{k\phi}{2} = \alpha \sin^2 \frac{\phi}{2} + \frac{\alpha(\alpha-1)}{2} \sin^2\phi - \frac{\alpha(\alpha-1)(2-\alpha)}{6} \sin^2 \frac{3\phi}{2} + \ldots$$

$$\leq \alpha \sin^2 \frac{\phi}{2} + \frac{\alpha(\alpha-1)}{2} \sin^2\phi \leq \alpha \sin^2 \frac{\pi}{6} + \frac{\alpha(\alpha-1)}{2} \sin^2 \frac{\pi}{3} = \frac{\alpha}{4} + \frac{\alpha(\alpha-1)}{2} \frac{3}{4} =$$

$$= \frac{\alpha}{4} + \frac{3}{8}(\alpha - 1) < \frac{\alpha}{4} + \frac{3}{8}\alpha = \frac{5}{8}\alpha, \quad (1 < \alpha < 2).$$

Now, the function $Q(w) = w\,[\sum_{k=0}^{n} \binom{\alpha}{k} - w]$ is increasing in the domain $0 \leq w \leq \frac{5}{8}\alpha$.

Indeed

$$Q'(w) = \sum_{k=0}^{n} \binom{\alpha}{k} - 2w > \sum_{k=0}^{n} \binom{\alpha}{k} - \frac{5}{4}\alpha > 1 + \alpha - \frac{5}{4}\alpha > 0,$$

for

$$1 < \alpha < 2.$$

We claim that

$$w \geq \frac{\alpha}{4} \sin^2\phi \left[1 + \frac{(\alpha-1)(3\alpha-2)}{2}\right]. \tag{3.4}$$

Indeed

$$w \geq \alpha \sin^2 \frac{\phi}{2} + \frac{\alpha(\alpha-1)}{2} \sin^2\phi - \frac{\alpha(\alpha-1)(2-\alpha)}{6} \sin^2 \frac{3\phi}{2}$$

$$= \sin^2\phi \left[\alpha \frac{\sin^2 \frac{\phi}{2}}{\sin^2\phi} + \frac{\alpha(\alpha-1)}{2} - \frac{\alpha(\alpha-1)(2-\alpha)}{6} \frac{\sin^2 \frac{3\phi}{2}}{\sin^2\phi}\right]$$

$$\geq \sin^2\phi \left[\frac{\alpha}{4} + \frac{\alpha(\alpha-1)}{2} - \frac{\alpha(\alpha-1)(2-\alpha)}{6} \frac{9}{4}\right]$$

$$= \frac{\alpha}{4} \sin^2\phi \left[1 + \frac{\alpha-1}{2}(3\alpha-2)\right].$$

Recalling that $Q(w)$ increases for $0 \leq w \leq \frac{5}{8}\alpha$ it is enough to show the inequality (3.3) in case that w attains its lower bound in (3.4):

$$(3.5)\qquad \alpha[1 + 2(\alpha - 1) + (\alpha - 1)^2 \cos^2\phi] \leq \left[1 + \frac{\alpha - 1}{2}(3\alpha - 2)\right]$$

$$\times \left[\sum_{k=0}^{n} \binom{\alpha}{k} - \frac{\alpha}{4}\sin^2\phi\left(1 + \frac{(\alpha - 1)(3\alpha - 2)}{2}\right)\right]$$

Again, as

$$\sum_{k=0}^{n} \binom{\alpha}{k} \geq 1 + \alpha + \frac{\alpha(\alpha - 1)}{2}\left[1 - \frac{2 - \alpha}{3}\right] = (1 + \alpha)\left[1 + \frac{\alpha(\alpha - 1)}{6}\right]$$

(3.5) is true if

$$(3.6)\qquad \alpha\,[1 + 2(\alpha - 1) + (\alpha - 1)^2 \cos^2\phi] \leq \left[1 + \frac{\alpha - 1}{2}(3\alpha - 2)\right]\left[(1 + \alpha)\right.$$

$$\left.\left(1 + \frac{\alpha(\alpha - 1)}{6}\right) - \frac{\alpha}{4}\sin^2\phi\left(1 + \frac{(\alpha - 1)(3\alpha - 2)}{2}\right)\right].$$

As $\cos^2\phi = 1 - \sin^2\phi$, the above inequality is linear in $\sin^2\phi$ (or in $\cos^2\phi$). Thus it is enough to check the end points. But $0 \leq \sin\phi \leq \sin\frac{\pi}{n} \leq \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

Therefore (3.6) would follow if we show the following two inequalities:

$$(3.7)\qquad \alpha^3 = \alpha\,[1 + 2(\alpha - 1) + (\alpha - 1)^2] < \left[1 + \frac{\alpha - 1}{2}(3\alpha - 2)\right]$$

$$(1 + \alpha)\left(1 + \frac{\alpha(\alpha - 1)}{6}\right).$$

$$(3.8)\qquad \alpha\left[1 + 2(\alpha - 1) + \frac{(\alpha - 1)^2}{4}\right] < \left[1 + \frac{(\alpha - 1)}{2}(3\alpha - 2)\right]\left[(1 + \alpha)\right.$$

$$\left.\left(1 + \frac{\alpha(\alpha - 1)}{6}\right) - \frac{3}{16}\alpha\left(1 + \frac{(\alpha - 1)(3\alpha - 2)}{2}\right)\right]$$

For the proof of (3.7) put $\alpha = 1 + \beta$. Then (3.7) is equivalent to the inequality

$$(1 + \beta)^3 < \left[1 + \frac{\beta}{2}(3\beta + 1)\right](2 + \beta)\left[1 + \frac{(\beta + 1)\beta}{6}\right]$$

which can be easily obtained by simple calculations. To establish (3.8) we first note that

$$-\left[1 + \frac{\alpha - 1}{2}(3\alpha - 2)\right] > -(1 + \alpha)\,,\quad \text{for}\quad 1 < \alpha < 2\,.$$

Thus (3.8) would follow if

$$(3.9) \qquad \alpha\left[1-2(\alpha-1)+\frac{(\alpha-1)^2}{4}\right] < \left[1+\frac{\alpha-1}{2}(3\alpha-2)\right](1+\alpha)$$
$$\left[1+\frac{\alpha(\alpha-1)}{6}-\frac{3\alpha}{16}\right]$$

The function $1-\frac{3\alpha}{16}+\frac{\alpha(\alpha-1)}{6}=1+\frac{8\alpha^2-17\alpha}{48}$ attains its minimum at the point $\alpha=\frac{17}{16}$ and so $1+\frac{8\alpha^2-17\alpha}{48}\geq 1-\frac{289}{32\cdot 48}>\frac{4}{5}$.

Finally it is enough to show

$$(3.10) \qquad \alpha\left[1-2(\alpha-1)+\frac{(\alpha-1)^2}{4}\right] < \frac{4}{5}\left[1+\frac{\alpha-1}{2}(3\alpha-2)\right](1+\alpha)\,.$$

Put again $\alpha=\beta+1$ and then (3.10) holds if we show that the polynomial $12-28\beta+11\beta^2+19\beta^3$ is positive for $0<\beta<1$. Clearly this polynomial is positive for $\beta>0$.

4. Two lemmas

Let

$$(4.1) \qquad \varepsilon_n(x)=(1+x)^\alpha-\sum_{k=0}^{n}\binom{\alpha}{k}x^k=\sum_{k=n}^{\infty}\binom{\alpha}{k+1}x^{k+1}$$

We bring now an integral representation of $\varepsilon_n(x)$:

Lemma 1.

$$(4.2) \qquad \varepsilon_n(x)=\frac{\sin\pi(\alpha-1)}{\pi}(-1)^{n+1}x^{n+1}\int_0^1\frac{(1-r)^\alpha r^{n-\alpha}}{1+xr}\,dr$$

where $|x|\leq 1$, $1<\alpha<2$ and $n\geq 3$.

Proof. Denote by $B(a,b)$ the Beta function:

$$(4.3) \qquad B(a\,,b)=\int_0^1 r^{a-1}(1-r)^{b-1}\,dr,\quad a>0\,,b>0\,.$$

We recall the following well known properties of the Beta function:

$$
(4.4)\qquad
\begin{aligned}
&B(a\,,b) = B(b\,,a)\,, \quad B(a-1\,,b) = \frac{a}{a+b}\,B(a\,,b)\,,\\
&B(a\,,1-a) = \frac{\pi}{\sin a} \quad \text{for} \quad 0 < a < 1\,.
\end{aligned}
$$

Now for $1 < \alpha < 2$ we have

$$
B(n-\alpha+1\,,\alpha+1) = \frac{(n-\alpha)}{(n+1)}\,\frac{(n-\alpha-1)}{n}\cdots\frac{2-\alpha}{3}\,B(2-\alpha\,,\alpha+1)
$$

$$
= \frac{(n-\alpha)}{(n+1)}\,\frac{(n-\alpha-1)}{n}\cdots\frac{2-\alpha}{3}\,\frac{\alpha(\alpha-1)}{2\cdot 1}\,B(2-\alpha\,,\alpha-1)
$$

On the other hand

$$
(4.5)\quad \binom{\alpha}{n+1} = \frac{\alpha(\alpha-1)\ldots(\alpha-n)}{(n+1)\,!} = (-1)^{n+1}\,\frac{\alpha(\alpha-1)\,(2-\alpha)\ldots(n-\alpha)}{(n+1)\,!}\,.
$$

Combining the two last equalities we obtain the representation of the binomial coefficients $\binom{\alpha}{n+1}$ with help of the Beta function:

$$
(4.6)\qquad \binom{\alpha}{n+1} = (-1)^{n+1}\,\frac{\sin\pi(\alpha-1)}{\pi}\,B(n-\alpha+1\,,\alpha+1)\,.
$$

For $|x| < 1$ we have

$$
\varepsilon_n(x) = \sum_{k=n}^{\infty}\binom{\alpha}{k+1}x^{k+1} = \frac{\sin\pi(\alpha-1)}{\pi}\sum_{k=n}^{\infty}\int_0^1 r^{k-\alpha}(1-r)^{\alpha}(-x)^{k+1}\,dr
$$

$$
= \frac{\sin\pi(\alpha-1)}{\pi}\,(-x)^{n+1}\int_0^1 \frac{r^{n-\alpha}(1-r)^{\alpha}}{1-xr}\,dr\,, \quad 1 < \alpha < 2\,, \quad 3 \le n\,.
$$

This proves the equality (4.2) for $|x| < 1$. The case $|x| = 1$ follows from continuity argument as $\sum_{k=n}^{\infty}\left|\binom{\alpha}{k+1}\right| < \infty$.

Remark. A more »natural» proof of (4.1) is obtained by using the Cauchy integral theorem for analytic functions. Clearly

$$
-\varepsilon_n(x) = \int_{|z|=t}\frac{(1+xz)^{\alpha}-(1+x)^{\alpha}}{z^{n+1}(1-z)}\,, \quad 0 < t < 1\,.
$$

Replace then the curve $|z| = t$ by $|z| = R$, $R > 1$ and a radial slit emanating from $z = -\frac{1}{x}$. Realizing that $\frac{(1+xz)^\alpha - (1+x)^\alpha}{z^{n+1}(1-z)}$ is regular at $z = 1$ and letting $R \to \infty$ one gets an alternative proof of lemma 1.

Lemma 2. Let $\varepsilon_n(x)$ be defined as above, then

$$|\varepsilon_n(x)| < \frac{\sin \pi(\alpha - 1)}{\pi} \frac{1}{n}, \quad \text{for} \quad Re(x) < 0\,. \tag{4.7}$$

$$|\varepsilon_n(x)| \leq \frac{\sin \pi(\alpha - 1)}{\pi} \frac{1}{n^2}, \quad \text{for} \quad Re(x) \geq 0\,. \tag{4.8}$$

where $|x| = 1$ and $n \geq 3$.

Prrof. Using (4.2) one gets

$$|\varepsilon_n(x)| \leq \frac{\sin \pi(\alpha - 1)}{\pi} \int_0^1 \frac{(1-r)^\alpha r^{n-\alpha}}{1-r} dr$$

Let $|x| \leq 1$. Since $(1-r)^\delta \leq 1 - \delta r$ in the range $0 \leq r \leq 1$, $0 \leq \delta \leq 1$, for $\delta = \alpha - 1$ we obtain:

$$|\varepsilon_n(x)| \leq \frac{\sin \pi(\alpha - 1)}{\pi} \int_0^1 (1-r)^\delta r^{n-\alpha} dr \leq \frac{\sin \pi(\alpha - 1)}{\pi} \int_0^1 (1 - \delta r) r^{n-\alpha} dr$$

$$= \frac{\sin \pi(\alpha - 1)}{\pi} \left[\frac{1}{n - \alpha + 1} - \frac{\delta}{n - \alpha + 2}\right] < \frac{\sin \pi(\alpha - 1)}{\pi} \frac{1}{n}.$$

This established (4.7). To show (4.8) we note that
$\frac{1}{|1 + xr|} \leq 1$ for $Re(x) \geq 0$. Thus

$$|\varepsilon_n(x)| \leq \frac{\sin \pi(\alpha - 1)}{\pi} \int_0^1 (1-r)^\alpha r^{n-\alpha} dr \leq \frac{\sin \pi(\alpha - 1)}{\pi} \int_0^1 (1-r) r^{n-\alpha}(1 - \delta r) dr$$

$$= \frac{\sin \pi((\alpha - 1)}{\pi} \left[\frac{1}{n - \alpha + 1} - \frac{\alpha}{n - \alpha + 2} + \frac{\delta}{n - \alpha + 3}\right] \leq$$

$$\frac{\sin \pi(\alpha - 1)}{\pi} \frac{1}{n^2}$$

(The last part is established as follows:

$$\frac{1}{n-\alpha+1}-\frac{\alpha}{n-\alpha+2}+\frac{\delta}{n-\alpha+3}=\frac{1}{n-\alpha+1}-\frac{1}{n-\alpha+2}-$$

$$\delta\left(\frac{1}{n-\alpha+3}-\frac{1}{n-\alpha+2}\right)=\frac{1}{(n-\alpha+1)(n-\alpha+2)}-$$

$$\frac{\delta}{(n-\alpha+3)(n-\alpha+2)}=\frac{1}{n-\alpha+2}\left[\frac{1}{n-\alpha+1}-\frac{\delta}{n-\alpha+3}\right].$$

Since $\frac{1}{n-\alpha+2}<\frac{1}{n}$ it is enough to show $\frac{1}{n-\alpha+1}-$ $\frac{\delta}{n-\alpha+3}\leq\frac{1}{n}$, which is equivalent to $\frac{1}{n-\alpha+1}-\frac{1}{n}\leq\frac{\alpha-1}{n-\alpha+3}$ and is obviously true for $n\geq 3$ and $1<\alpha<2)$.

5. The case $\frac{\pi}{n}\leq\phi\leq\pi$

Using the definition of $\varepsilon_n(x)$ (4.1) we have

$$\left|\sum_{k=0}^{n}\binom{\alpha}{k}x^k\right|\leq|1+x|^\alpha+|\varepsilon_n(x)|.$$

Om the other hand

$$(1+1)^\alpha\leq\left|\sum_{k=0}^{n}\binom{\alpha}{k}\right|+|\varepsilon_n(1)|$$

Thus the inequality $\left|\sum_{k=0}^{n}\binom{\alpha}{k}x^k\right|\leq\left|\sum_{k=0}^{n}\binom{\alpha}{k}\right|$ would follow if

$$|1+x|^\alpha+|\varepsilon_n(x)|\leq 2^\alpha-|\varepsilon_n(1)|,\quad x=e^{i\phi},\ \frac{\pi}{n}\leq\phi\leq\pi.$$

Put $\frac{\phi}{2}=\theta$ then

$$|1+x|=|e^{i\phi}+1|=2\cos\theta$$

Therefore it is enough to show

$$|\varepsilon_n(x)|+|\varepsilon_n(1)|<2^\alpha(1-\cos\theta) \tag{5.1}$$

for $1<\alpha<2$ and $\frac{\pi}{2n}\leq\theta\leq\frac{\pi}{2}$ as $1-(\cos\theta)^\alpha\leq 1-\cos\theta$.

Assume first $Re(x) < 0$. By Lemma 2 we have:

$$|\varepsilon_n(x)| + |\varepsilon_n(1)| \leq \frac{\sin \pi(\alpha - 1)}{\pi}\left(\frac{1}{n} + \frac{1}{n^2}\right)$$

$$\leq \frac{\sin \pi(\alpha - 1)}{\pi}\left(\frac{1}{9} + \frac{1}{3}\right) \leq \frac{4}{9\pi}$$

Now the condition $Re(e^{i\phi}) < 0 \left(\text{i.e. } \frac{\pi}{2} \leq \phi \leq \pi\right)$ implies that $\cos\theta \leq \frac{\sqrt{2}}{2}$.

But

$$\frac{4}{9\pi} < 2\left(1 - \frac{\sqrt{2}}{2}\right) < 2^{\alpha}\,(1 - \cos\theta) \quad \text{for} \quad \alpha > 1\,.$$

Assume now $Re(x) \geq 0$. In this case Lemma 2 implies

$$|\varepsilon_n(x)| + |\varepsilon_n(1)| \leq \frac{2}{\pi}\,\frac{\sin \pi(\alpha - 1)}{n^2}$$

Finally we show

$$\frac{2}{\pi}\sin\frac{\pi(\alpha - 1)}{n^2} < 2^{\alpha}(1 - \cos\theta) = 2^{\alpha+1}\sin^2\frac{\theta}{2}\,,\ \frac{\pi}{2n} \leq \theta \leq \frac{\pi}{4}\,.$$

It is enough to consider the above inequality for $\frac{\theta}{2} = \frac{\pi}{4n}$:

$$\frac{2}{\pi}\,\frac{\sin \pi(\alpha - 1)}{n^2} < 2^{\alpha+1}\sin^2\frac{\pi}{4n}\,. \tag{5.2}$$

Clearly $\sin^2\frac{\pi}{4n} \geq \left(\frac{\sin\frac{\pi}{12}}{\frac{\pi}{12}}\right)^2\left(\frac{\pi}{4n}\right)^2$ for $n \geq 3$.

So (5.2) is reduced to

$$\frac{2}{\pi}\,\frac{\sin \pi(\alpha - 1)}{n^2} < 2^{\alpha+1}\left(\frac{\sin\frac{\pi}{12}}{\frac{\pi}{12}}\right)^2\frac{\pi^2}{16n^2} \tag{5.3}$$

which is equivalent to an obvious inequality

$$\sin \pi(x - 1) < 2^{\alpha} \frac{\pi^3}{16} \left(\frac{\sin \frac{\pi}{12}}{\frac{\pi}{12}} \right)^2 .$$

This completes the proof of the inequality (2.1).

In conclusion we remark that we showed in fact that:

$$\left| \sum_{k=0}^{n} \binom{\alpha}{k} x^k \right| < \sum_{k=0}^{n} \binom{\alpha}{k}$$

for $|x| = 1$ and $x \neq 1$.

Department of Mathematics
Technion City, Haifa
Israel

References

[1] BRANNAN, D. A.: On functions of bounded boundary rotation II, Bull. London Math. Soc. 1 (1969).

[2] —»— CLUNIE, J. G. and KIRWAN, W. E.: On the coefficient problem for functions of bounded boundary rotation, Ann. Acad. Sci. Fenn. A I 523 (1973).

[3] COONCE, H. B.: On functions of bounded boundary rotation, Trans. Amer. Math. Soc. (to appear).

[4] GOODMAN, A. W.: On close-to-convex function of higher order (to appear).

[5] LEHTO, O.: On the distortion of conformal mappings with bounded boundary rotation. Ann. Acad. Sci. Fenn. A I 124 (1952).

[6] LONKA, H. and TAMMI, O.: On the case of step-functions in extremal problems of the class with bounded boundary rotations, Ann. Acad. Sci. Fenn. A I 418 (1968).

[7] NOONAN, J.: Asymptotic behaviour of functions with bounded boundary rotations, Trans. Amer. Math. Soc., 164 (1972), 397–410.

[8] POMMERENKE, CH.: On close-to-convex analytic functions, Trans. Amer. Math. Soc., 114 (1965), 176–186.

[9] SCHIFFER, M. and TAMMI, O.: On the fourth coefficient of univalent functions with bounded boundary rotation, Ann. Acad. Sci. Fenn. A I 396 (1967).

Printed February 1973