CONFORMAL CAPACITY AND QUASIREGULAR MAPPINGS

BY

URI SREBRO

HELSINKI 1973
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1973.529
Communicated 10 November 1972 by Jussi Väisälä
1. Introduction and terminology

1.1 By a condenser in $\mathbb{R}^n$ we mean a triple $E = (D; C_0, C_1)$ where $D$ is a domain in $\mathbb{R}^n$, and $C_0, C_1$ are disjoint compact sets in $D$, the closure of $D$ in $\mathbb{R}^n = \mathbb{R}^n \cup \{ \infty \}$. The capacity (namely the conformal capacity or the $n$-capacity) is defined by

$$\text{cap } E = \inf_{u \in W(E)} \int_D |\nabla u|^n dm,$$

where $W(E)$ is the set of all non-negative, continuous and ACL functions $u : D \to \mathbb{R}^1$ such that $u(x) \to j$ as $x \to x_j$ for all $x_j \in C_j$, $j = 0, 1$. If $C_0 = \emptyset$ or $C_1 = \emptyset$ we set $\text{cap } E = 0$. Note that $\text{cap } (D ; C_0, C_1) = \text{cap } (D ; C_1, C_0)$.

1.2. Let $f : D \to \mathbb{R}^n$ be a non-constant quasiregular mapping (see [2] for terminology) and $E = (D; C_0, C_1)$ a condenser. The cluster set of $f$ on a set $A \subseteq D$ is denoted by $C(f, A)$, i.e. $C(f, A)$ is the set of all points $y \in \mathbb{R}^n$ such that $f(x) \to y$ as $x \to a$ for $a \in A$. We will show that

(a) $C(f, C_0) \cap C(f, C_1) = \emptyset$ $\Rightarrow$ $\text{cap } E \leq N(f, D)K_0(f) \text{cap } fE$, where $fE = (fD ; C(f, C_0), C(f, C_1))$, and that

(b) $C_0, C_1 \subseteq D$ and $C(f, \partial D) \subseteq \partial fD \Rightarrow \text{cap } fE \leq K_1(f) \text{cap } E$, where $fE = (fD ; fC_0 \setminus f(D \setminus C_0), fC_1)$.

Here $K_0(f)$ and $K_1(f)$ are the outer and inner dilatations of $f$ in $D$, and $N(f, D) = \sup \text{cardinality } f^{-1}(y)$ over all $y \in \mathbb{R}^n$, see [2].

We conclude with an application of (b) to the boundary behavior of quasiregular mappings.

1.3. The notation and terminology will usually be as in [2]. Quasiconformal is abbreviated by qc, quasiregular by qr. The $L^n(D)$ norm of $|\nabla u| = \left( \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i} \right)^n \right)^{1/n}$ will be denoted by $\|\nabla u\|_D$ or by $\|\nabla u\|$.

---

Supported by a research grant from the Finnish Government.
2. Preliminary results on the capacity of condensers

2.1. Let \( E = (D; C_0, C_1) \) be a condenser in \( \mathbb{R}^n \). Let \( W_0(E) \) denote the set of all non-negative, continuous and ACL functions \( u : D \to \mathbb{R}^1 \) such that \( C_0 \cap \text{spt} \ u = \phi \) and \( C_1 \cap \text{spt} (1 - u) = \phi \). Finally, let \( W_0^\infty(E) = W_0(E) \cap \mathcal{C}^\infty_0(D) \). Clearly \( W_0^\infty(E) \subseteq W_0(E) \subseteq W(E) \).

2.2. Lemma. \( \text{cap} \ E = \inf \{ \| \nabla u \|^n : u \in W_0(E) \} \).

Proof. We may assume that \( \text{cap} \ E < \infty \), since otherwise there is nothing to prove. For \( u \in W(E) \) with \( \| \nabla u \| < \infty \) and \( 0 < \delta < \frac{1}{2} \), we define \( u_\delta : D \to \mathbb{R}^1 \) by setting \( u_\delta(x) = 0 \) iff \( 0 \leq u(x) \leq \delta \), \( u_\delta(x) = 1 \) iff \( 1 - \delta \leq u(x) \leq 1 \) and \( u_\delta(x) = \frac{u(x) - \delta}{1 - 2\delta} \) otherwise. It is not hard to verify that \( u_\delta \in W_0(E) \) and that \( \| \nabla u_\delta \| \leq \frac{\| \nabla u \|}{1 - 2\delta} \). Now take the infimum of \( \| \nabla u \|^n \) over all \( u \in W(E) \); then let \( \delta \to 0 \) and the result follows.

2.3. Lemma. \( \text{cap} \ E = \inf \{ \| \nabla u \|^n : u \in W_0^\infty(E) \} \).

Proof. We may assume that \( \text{cap} \ E < \infty \). Given \( \epsilon > 0 \) and \( u \in W_0(E) \) with \( \| \nabla u \| < \infty \), let
\[
\delta = \min \{ \text{dist} (C_0, \text{spt} u), \text{dist} (C_1, \text{spt} (1 - u)) \}.
\]
We now continue with the techniques of [3]. Define \( D_0 = D_{-1} = \phi \) and
\[
D_i = \left\{ x \in D : |x| < i \quad \text{and} \quad \text{dist} (x, \partial D) > \frac{1}{i} \right\}, \quad i = 1, 2, \ldots .
\]
Choose a partition of unity \( \sum_{i=1}^\infty \psi_i \equiv 1 \) on \( D \) such that
\[
\text{spt} \ \psi_i \subseteq D_{i+1} \setminus D_{i-1}, \quad i = 1, 2, \ldots ,
\]
and let \( \varphi_i : \mathbb{R}^n \to \mathbb{R}^1, \ i = 1, 2, \ldots , \) be non-negative \( C^\infty \) functions such that
\[
\text{spt} \ \varphi_i \subseteq B_n \left( \frac{1}{(i + 1)(i + 2)} \right) \cap B_n \left( \frac{\delta}{2} \right) \text{ and } \int \varphi_i(x) dm(x) = 1 \quad \text{and}
\]
\[
\| \nabla (\varphi_i \ast \psi_i u) - \nabla (\psi_i u) \| < \frac{\epsilon}{2i}.
\]
\[
\text{spt} \ \varphi_i \ast \psi_i u \subseteq D_{i+2} \setminus D_{i-2}; \quad \text{hence the series}
\]
$v = \sum_{i=1}^{\infty} q_i \ast \psi_i u$

converges and defines a function in $W_0^\omega(E)$. Finally for $k = 1, 2, \ldots$ we have:

$$
\|\nabla u - \nabla v\|_{D_k} = \sum_{i=1}^{k+1} \|\nabla (q_i \ast \psi_i u) - \nabla (\psi_i u)\|_{D_k} \leq \sum_{i=1}^{k+1} \|\nabla (q_i \ast \psi_i u) - \nabla (\psi_i u)\| < \varepsilon
$$

Letting $k \to \infty$, we conclude by Lebesgue monotone convergence theorem and Minkowski's inequality that $\|\nabla v\| \leq \|\nabla u\| + \varepsilon$. Now take the infimum of $\|\nabla u\|$ over $W_0(E)$, let $\varepsilon \to 0$ and the result follows by 2.2 and the inclusion $W_0^\omega(E) \subset W_0(E)$.

3. Capacity inequalities

3.1. Theorem. Let $f : D \to R^n$ be non-constant and qr, and let $E = (D; C_0, C_1)$ be a condenser. If $C(f, C_0) \cap C(f, C_1) = \phi$, then

$$
\text{cap } E \leq N(f, D)K_0(f) \text{ cap } f E,
$$

where $f E = (fD; C(f, C_0), C(f, C_1))$.

Proof. We may assume that $N(f, D) < \infty$ and that $\text{cap } f E < \infty$. (Actually, the capacity of a condenser is always finite.) Given $v \in W_0^\omega(fE)$ with $\|\nabla v\|_{fD} < \infty$, we define $u = v \circ f$. Clearly $u$ is non-negative and continuous in $D$. Let $U' = fD \setminus \text{spt } v$ and $U = f^{-1}(U')$. Then $\text{spt } u = D \setminus U$; and since $C(f, C_0) \cap \text{spt } v = \phi$ it follows, by the definition of a cluster set and the nature of $U$ and $U'$, that $C_0 \cap \text{spt } u = \phi$. In the same way $C_1 \cap \text{spt } (1 - u) = \phi$. Finally $f$ is ACL and differentiable a.e. in $D$, cf. [2, 2.26], and $v \in C^\infty(fD)$, hence $u$ is ACL and

$$
|\nabla u|^n \leq |(\nabla u) \circ f|^n |f'|^n \leq K_0(f)(\nabla v) \circ f,^n J(f) \text{ a.e. in } D,
$$

where $J(f)$ denotes the Jacobian of $f$ in $D$. Consequently $u \in W_0(E)$ and

$$
\text{cap } E \leq \|\nabla u\|^n \leq K_0(f) \int_{D} |(\nabla v) \circ f, ^n J(f) |d m \leq K_0(f) N(f, D)\|\nabla v\|_D^n.
$$

Now take the infimum of $\|\nabla v\|_D$ over all $v \in W_0^\omega(fD)$ and the result follows by virtue of 2.3.

3.2. Theorem. Let $f : D \to R^n$ be qr and $E = (D; C_0, C_1)$ a condenser with $C_0, C_1 \subset D$ and $C(f, \partial D) \subset \partial f D$. Then
cap \( \tilde{f}E \leq K_1(f) \) cap \( E \),

where \( fE = (\tilde{f}D; fC_0 \setminus f(D \setminus C_0), fC_1) \).

**Proof.** We may assume that \( \text{cap } E < \infty \). Given \( u \in W_\infty^\alpha(E) \) with \( \| \nabla u \| < \infty \) we define \( v : fD \to R^n \) by \( v(y) = \sup \{ u(x) : x \in f^{-1}(y) \} \). Clearly \( v(y) \geq 0 \) for all \( y \in fD \), \( v(y) = 0 \) for all \( y \in fC_0 \setminus f(D \setminus C_0) \) and \( v(y) = 1 \) for all \( y \in fC_1 \). \( f \) is \( qr \), hence \( f \) is open and discrete and since \( C(f, \partial D) \subset \partial fD \), it follows that \( f \) is closed. Thus, by [1, 3.3], \( N(f, D) < \infty \). Consequently \( v \) is continuous in \( D' \). Indeed, given \( \varepsilon > 0 \) and \( y \in D' \) with \( f^{-1}(y) = \{ x_1, \ldots, x_k \} \), choose neighborhoods \( U_i \subset D \) of \( x_i \), \( i = 1, \ldots, k \), such that \( |u(x) - u(x_i)| < \varepsilon \) for all \( x \in U_i, i = 1, \ldots, k \). Then \( A = \bigcap_{i=1}^k fU_i \) is open in \( D' \), \( B = f(D \setminus \bigcup_{i=1}^k U_i) \) is closed rel. \( D' \), and so \( V = A \setminus B \) is a neighborhood of \( y \) and \( |v(y') - v(y)| < \varepsilon \) for all \( y' \in V \).

Applying the arguments of [2, 7.8—7.13] we conclude that \( v \) is ACL in \( D' \). Thus \( v \in W(\tilde{f}E) \). Then the arguments and computations of [2, 7.15—7.17] yield \( \| \nabla v \|_{C^0} \leq K_1(f) \| \nabla u \|^{\alpha} \). Take the infimum of \( \| \nabla u \|^{\alpha} \) over all \( u \in W_\infty^\alpha(D) \) and the result follows by 2.3.

4. An application

4.1. A domain \( D \subset R^n \) is said to be quasiconformally accessible at a point \( b \in \partial D \) iff for every neighborhood \( U \) of \( b \) and for every continuum \( C_0 \subset D \setminus U \), there is a positive number \( \delta \), and a neighborhood \( V \) of \( b \) with \( \tilde{V} \subset U \) such that \( \text{cap } (D; C_0, C_1) > \delta \) for every continuum \( C_1 \) in \( D \) which meets both \( \partial V \) and \( \partial U \). cf. [4, 1.7].

4.2. **Theorem.** Let \( f : D \to R^n \) be \( qr \) with \( C(f, \partial D) \subset \partial fD \). If \( D \) is locally connected at a point \( b \in \partial D \) and \( D' = fD \) is qc accessible at all point \( y \in C(f, b) \), then \( C(f, b) = \{ y \} \).

**Proof.** For qc mappings see [4, 2.4]. Suppose that \( C(f, b) \) contains more than one point. Then there are sequences \( \{ x_i \} \) and \( \{ x'_i \} \) in \( D \) with \( x_i \to b, x'_i \to b, f(x_i) \to y \) and \( f(x'_i) \to y' \) with \( y \neq y' \). Let \( C'_0 \) be any continuum in \( D' \). \( C(f, \partial D) \subset \partial fD \) implies that \( C'_0 = f^{-1}C'_0 \) is compact in \( D \). Choose a neighborhood \( U \) of \( y \) such that \( y' \notin \tilde{U} \) and \( C'_0 \cap \tilde{U} = \phi, y \in \partial D' \) and \( D' \) is qc accessible at \( y; \) hence there is \( \delta > 0 \) and a neighborhood \( V \) of \( y \) with the properties stated in 4.1. \( D \) is locally connected at \( b \), therefore \( x_i \) and \( x'_i \) may be joined by an arc \( \gamma_i \) in \( D \) whose diameter tends to 0 as \( i \to \infty \). \( f(x'_i) \in V \) and \( f(x'_i) \in D' \setminus U \) for all \( i \) sufficiently large, hence \( \text{cap } (D'; C'_0, f\gamma_i) > \delta \).
while $\text{cap}(D; C_0, \gamma_i) \to 0$ as $i \to \infty$, violating Theorem 3.1. Thus $C(f, b) = \{y\}$.

4.3. **Corollary** Let $D$ be a Jordan domain in $\mathbb{R}^n$ and $f: D \to \mathbb{R}^n$ qr with $fD \subset B^n$ and $C(f, \partial D) \subset \partial B^n$. Then $f$ has a continuous extension on $D$. 
References


Technion, Haifa, Israel
and
University of Helsinki, Helsinki, Finland

Printed February 1973