ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

530

REMARKS ON THE REGULARITY OF THE SOLUTIONS OF A LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

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HELSINKI 1973
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1973.530

ISBN 951-41-0086-7

Communicated 12 October 1972 by Rolf Nevanlinna

KESKUSKIRJAPAINO
HELSINKI 1973

1. Let us consider a linear partial differential equation

$$(1) \qquad L\,u = f\,,$$

where the differential operator L has constant coefficients. We assume that f belongs to the Sobolev space $H^p(\Omega)$ $(p \geqq 0)$, where Ω is a bounded open domain in the Euclidean space R^n. Let L' be the formal adjoint of L and write $M = L'\,L\,(1 - \Delta)^p$. With the help of M we construct a distribution space X^- such that $L^2(\Omega) \subset X^- \subset D'(\Omega)$. We shall show that if $u \in X^-$ is a distribution solution of (1), it can be decomposed

$$u = u_0 + \vartheta\,,$$

where $u_0 \in H_0^{p+r}(\Omega)$ $(r \geqq 0)$ and ϑ satisfies

$$(2) \qquad (M^2 + 1)\,\vartheta = 0\,.$$

Thus the part u_0 is at least as regular as the right hand side f of (1). Since the second part ϑ satisfies the homogeneous equation (2), its regularity depends only on the differential operator L. Consequently, the question about the regularity of a solution of the inhomogeneous problem (1) returns to the question about the regularity of a solution of the homogeneous problem (2). Furthermore, this result will be applied to prove that weak L^2-solutions of hypoelliptic equations can be approximated in $L^2(\Omega)$ by C^∞-functions.

2. Let Ω be a bounded open domain in the Euclidean space R^n. For a multi-index $\varrho = (\varrho_1, \ldots, \varrho_n)$ we write $D^\varrho = D_1^{\varrho_1} \ldots D_n^{\varrho_n}$ where $D_i = \partial/\partial x_i$. If $p \geqq 0$ is an integer, we denote by $H^p(\Omega)$ the Sobolev space consisting of the complex valued functions whose distribution derivates of order $\leqq p$ belong to $L^2(\Omega)$. The space $H^p(\Omega)$ is a Hilbert space with the inner product

$$(u\,,v)_p = \sum_{|\varrho| \leqq p} \frac{p!}{\varrho_1! \ldots \varrho_n!\,(p - |\varrho|)\,!} \int_\Omega D^\varrho u\,\overline{D^\varrho v}\,dx$$

$(u\,, v \in H^p(\Omega))$. Let $H_0^p(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^p(\Omega)$.

We consider a differential operator

$$L = \sum_{|\varrho| \leqq m} a_\varrho D^\varrho$$

with constant coefficients. There is an integer $r \geqq 0$ such that for each $p \geqq 0$ one has

$$\|L \varphi\|_p \geqq k_p \|\varphi\|_{p+r} \qquad (\varphi \in C_0^\infty(\Omega)), \tag{3}$$

where k_p is a positive constant (cf. [1], p. 177). The formal adjoint of L is

$$L' = \sum_{|\varrho| \leqq m} (-1)^{|\varrho|} \overline{a_\varrho} D^\varrho .$$

Let us choose an integer $p \geqq 0$. Writing

$$S = 1 - \Delta = 1 - \sum_{i=1}^{n} D_i^2 ,$$

one has for $f \in H^p(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$

$$(f, \varphi)_p = (f, S^p \varphi)_0 .$$

We consider the formally self-adjoint differential operator

$$M = L' L S^p .$$

For $\varphi \in C_0^\infty(\Omega)$ we get by (3)

$$(\varphi, M \varphi)_0 = (L \varphi, S^p L \varphi)_0 = \|L \varphi\|_p^2 \geqq k_p^2 \|\varphi\|_{p+r}^2 .$$

Let us define on $C_0^\infty(\Omega)$ an inner product

$$(\varphi, \psi)_X = (\varphi, M \psi)_0 \qquad (\varphi, \psi \in C_0^\infty(\Omega)) .$$

We denote by X the completion of $C_0^\infty(\Omega)$ with respect to the corresponding norm $\|\cdot\|_X$. Since

$$\|\varphi\|_X \geqq k_p \|\varphi\|_{p+r} \qquad (\varphi \in C_0^\infty(\Omega)) ,$$

one has $X \subset H_0^{p+r}(\Omega)$.

We define on $L^2(\Omega)$ so-called negative norm $\|\cdot\|_{X^-}$ by the formula

$$\|v\|_{X^-} = \sup_{0 \neq u \in X} \frac{|(u, v)_0|}{\|u\|_X} \qquad (v \in L^2(\Omega)) .$$

Let X^- be the completion of $L^2(\Omega)$ with respect to this norm. Take $u \in X$ and $v \in X^-$. There is a sequence $\{v_i\} \subset L^2(\Omega)$ such that $\|v_i - v\|_{X^-} \to 0$. Since

$$|(u, v_i)_0| \leqq \|u\|_X \|v_i\|_{X^-} \qquad (i = 1, 2, \ldots) ,$$

we can define the form $(\cdot\,,\cdot)_0$ on $X \times X^-$ such that

$$|(u\,,v)_0| \leqq \|u\|_X \|v\|_{X^-} \qquad (u \in X\,,\ v \in X^-)\,.$$

Each $v \in L^2(\Omega)$ determines a continuous linear functional $(\cdot\,,v)_0$ on X. Consequently, there is a continuous linear mapping $A_0 : L^2(\Omega) \to X$ such that

$$(u\,,v)_0 = (u\,,A_0\,v)_X \qquad (u \in X\,,\ v \in L^2(\Omega))\,.$$

For $v \in L^2(\Omega)$ one has

$$\|A_0\,v\|_X = \sup_{0 \neq u \in X} \frac{|(u\,,A_0\,v)_X|}{\|u\|_X} = \|v\|_{X^-}\,.$$

Thus, A_0 maps a dense subset of X^- isometrically into X. Consequently A_0 has an isometric extension $A : X^- \to X$. Since A is isometric, the range $A(X^-)$ is closed in X. If $A(X^-) \neq X$, one can find a non-zero $u \in X$ such that for each $v \in L^2(\Omega)$

$$(u\,,v)_0 = (u\,,A\,v)_X = 0\,.$$

This is true only if $u = 0$, therefore $A(X^-) = X$. Thus, the mapping $A : X^- \to X$ is an isometric isomorphism, X^- is a Hilbert space, and

$$(u\,,v)_0 = (u\,,A\,v)_X = (A^{-1}\,u\,,v)_{X^-} \qquad (u \in X\,, v \in X^-)\,.$$

This implies that the spaces X and X^- are dual with respect to the form $(\cdot\,,\cdot)_0$. If $\psi \in C_0^\infty(\Omega)$, we get

$$(\varphi\,,M\,\psi)_0 = (\varphi\,,\psi)_X = (\varphi\,,A^{-1}\,\psi)_0 \qquad (\varphi \in C_0^\infty(\Omega))\,,$$

and therefore $M\,\psi = A^{-1}\,\psi$.

Let $D'(\Omega)$ be the space of distributions in Ω. An element $u \in X$ determines a distribution

$$\varphi \to (\varphi\,,u)_0 \qquad (\varphi \in C_0^\infty(\Omega))\,.$$

We shall identify u with this distribution. Then we get $X^- \subset D'(\Omega)$. If $u\,,M\,u \in X^-$, we have

$$(\varphi\,,M\,u)_0 = (M\,\varphi\,,u)_0 \qquad (\varphi \in C_0^\infty(\Omega))\,. \tag{4}$$

Since $C_0^\infty(\Omega)$ is dense in X and $A^{-1} : X \to X^-$ is an isomorphism, the set $A^{-1}(C_0^\infty(\Omega)) = M(C_0^\infty(\Omega))$ is dense in X^-. Let us define on the domain $D(M_0) = M(C_0^\infty(\Omega))$ an operator $M_0 : D(M_0) \to X^-$ by

$$M_0\,\psi = M\,\psi \qquad (\psi \in D(M_0))\,.$$

Lemma 1. *The operator M_0 is symmetric in X^-. If $u\,,M\,u \in X^-$, one has $u \in D(M_0^*)$, where M_0^* is the adjoint of M_0 in X^-.*

Proof. For $\psi \in D(M_0) = A^{-1}(C_0^\infty(\Omega))$, one has $\varphi = A\,\psi \in C_0^\infty(\Omega)$ and we get by (4)

$$(\psi\,,\,M\,u)_{X^-} = (A^{-1}\,\varphi\,,\,M\,u)_{X^-} = (\varphi\,,\,M\,u)_0 = (M\,\varphi\,,\,u)_0 = (\psi\,,\,u)_0$$
$$= (A^{-1}\,\psi\,,\,u)_{X^-} = (M_0\,\psi\,,\,u)_{X_-}\,.$$

This implies that M_0 is symmetric in X^-, $u \in D(M_0^*)$ and $M_0^*\,u = M\,u$.

Lemma 2. *If* $f \in H^p(\Omega)$ *then* $L'\,S^p\,f \in X^-$.

Proof. Let Y be the closure of the set $L(C_0^\infty(\Omega))$ in $H^p(\Omega)$ and $Y^\perp$ the orthogonal complement of Y in $H^p(\Omega)$. Then we can decompose $f = v + f_0$ where $v \in Y$, $f_0 \in Y^\perp$. Since

$$(f_0\,,\,S^p\,L\,\varphi)_0 = (f_0\,,\,L\,\varphi)_p = 0 \qquad (\,\varphi \in C_0^\infty(\Omega)\,)\,,$$

one has $L'\,S^p\,f_0 = 0$ and therefore $L'\,S^p\,f = L'\,S^p\,v$. For $\varphi \in C_0^\infty(\Omega)$ we get

$$\|L\,\varphi\|_p^2 = (\varphi\,,\,L'\,L\,S^p\,\varphi)_0 = \|\varphi\|_X^2$$

and moreover

$$|(L'\,S^p\,v)(\varphi)| = |v(L\,S^p\,\varphi)| = |(v\,,\,L\,S^p\,\varphi)_0|$$
$$= |(v\,,\,L\,\varphi)_p| \leqq \|v\|_p\,\|L\,\varphi\|_p = \|v\|_p\,\|\varphi\|_X\,.$$

Thus the distribution $L'\,S^p\,v \in D'(\Omega)$ is bounded by the norm $\|\cdot\|_X$. Consequently, $L'\,S^p\,f = L'\,S^p\,v \in X^-$, since X^- is dual with X with respect to the form $(\cdot\,,\,\cdot)_0$.

Theorem 3. *Assume that* $f \in H^p(\Omega)$ *and let* $u \in X^-$ *be a distribution solution in* Ω *of the equation*[1]

$$L\,u = f\,.$$

Then u *can be decomposed*

$$u = u_0 - \vartheta\,,$$

where $u_0 \in X \subset H_0^{p+r}(\Omega)$ *and* $\vartheta \in X^-$ *satisfies*

$$(M^2 + 1)\,\vartheta = 0\,. \tag{5}$$

Proof. According to Lemma 2 one has

$$M\,u = L'\,S^p\,L\,u = L'\,S^p\,f \in X^-\,.$$

Thus we get by Lemma 1, $u \in D(M_0^*)$, where M_0^* is the adjoint in X of the symmetric operator M_0. Let $\bar{M}_0$ be the closure in X^- of M_0. Then u can be decomposed

[1] A distribution solution in Ω of the equation $L\,u = f$ is defined to be a distribution $u \in D'(\Omega)$ such that $(L\,u)(\varphi) := u(L'\,\varphi) = f(\varphi)$ for all $\varphi \in C_0^\infty(\Omega)$.

$$u = u_0 + \vartheta_1 + \vartheta_2 ,$$

where $u_0 \in D(\bar{M}_0)$ and the elements $\vartheta_1 , \vartheta_2 \in D(M_0^*)$ satisfy

$$M_0^* \vartheta_1 = i \vartheta_1 , \quad M_0^* \vartheta_2 = -i \vartheta_2$$

(see e.g. [2], VII.4, Theorem 3, p. 204).

For $\varphi \in C_0^\infty(\Omega)$ one has $M \varphi \in M(C_0^\infty(\Omega)) = D(M_0) = D(M_0 + i I)$ and, since $(M_0 + i I)^* \vartheta_1 = (M_0^* - i I) \vartheta_1 = 0$, we get

$$((M + i) \varphi , \vartheta_1)_0 = (M_0 + i I) M \varphi , \vartheta_1)_{X^-} = 0 .$$

This implies that $(M - i) \vartheta_1 = 0$. Similarly one has $(M + i) \vartheta_2 = 0$. Writing $\vartheta = \vartheta_1 + \vartheta_2$ we get

$$(M^2 + 1) \vartheta = (M + i)(M - i)(\vartheta_1 + \vartheta_2) = 0 .$$

Since $u_0 \in D(\bar{M}_0)$, there exists a sequence $\{\psi_i\} \subset D(M_0) = M(C_0^\infty(\Omega))$ such that

$$\|\psi_i - u_0\|_{X^-} \to 0 , \qquad \|M_0 \psi_i - \bar{M}_0 u_0\|_{X^-} \to 0 .$$

Because

$$\|\psi_i - \psi_j\|_X = \|A^{-1}(\psi_i - \psi_j)\|_{X^-} = \|M_0 (\psi_i - \psi_j)\|_{X^-} \to 0 .$$

there exists $u_0' \in X$ such that $\psi_i \to u_0'$ in X. This implies that $\psi_i \to u_0'$ also in X^-, therefore $u_0 = u'_0 \in X$.

Corollary 4. *We assume that* L *is hypoelliptic and* $f \in L^2(\Omega)$. *If* $u \in L^2(\Omega)$ *is a distribution solution of*

$$L u = f ,$$

then there exists a sequence $\{\psi_i\} \subset C^\infty(\Omega) \cap L^2(\Omega)$ *such that in* $L^2(\Omega)$, $\psi_i \to u$ *and* $L \psi_i \to f$.

Proof. Let us take $p = 0$ and apply Theorem 3. Then we get $u = u_0 + \vartheta$, where $u_0 \in X$ and ϑ satisfies (5) with $M = L' L$. Since L is hypoelliptic, $M^2 + 1$ is also hypoelliptic and therefore $\vartheta \in C^\infty(\Omega)$. For $\varphi \in C_0^\infty(\Omega)$ one has

$$\|\varphi\|_X^2 = (\varphi , L' L \varphi)_0 = \|L \varphi\|_0^2 .$$

Since $u_0 \in X$ there exists a sequence $\{\varphi_i\} \in C_0^\infty(\Omega)$ such that $\varphi_i \to u_0$ in X. Because

$$\|L \varphi_i - L \varphi_j\|_0 = \|\varphi_i - \varphi_j\|_X \to 0 ,$$

there is $v \in L^2(\Omega)$ such that $L \varphi_i \to v$ in $L^2(\Omega)$. For each $\varphi \in C_0^\infty(\Omega)$ one has

$$(L' \varphi , \varphi_i)_0 \to (L' \varphi , u_0)_0 = (L u_0)(\varphi)$$

and on the other hand

$$(L' \varphi , \varphi_i)_0 = (\varphi , L \varphi_i)_0 \to (\varphi , v)_0 = v(\varphi) .$$

This implies that $L u_0 = v$ and therefore $L \varphi_i \to L u_0$ in $L^2(\Omega)$. Thus we get for $\psi_i = \varphi_i + \vartheta \in C^\infty(\Omega) \cap L^2(\Omega)$, $\psi_i \to u_0 + \vartheta = u$ and $L \psi_i = L \varphi_i + L \vartheta \to L u_0 + L \vartheta = f$ in $L^2(\Omega)$.

I am most grateful to the referee, who has read the manuscript and suggested the following theorem, which is simpler and more general than Theorem 3 in the case $r = 0$. This theorem also implies Corollary 4.

Theorem 5. *Assume that* $f \in H^p(\Omega)$ *and let* $u \in D'(\Omega)$ *be a distribution solution in* Ω *of the equation*

$$L u = f .$$

Then u *can be decomposed*

$$u = u_0 + \vartheta ,$$

where $u_0 \in H_0^p(\Omega)$ *and* $\vartheta \in D'(\Omega)$ *satisfies*

$$L' L S^p \vartheta = 0 .$$

Proof. Let us define on $C_0^\infty(\Omega)$ an operator $L_0 : C_0^\infty(\Omega) \to H_0^p(\Omega)$ by

$$L_0 \varphi = L \varphi \qquad (\varphi \in C_0^\infty(\Omega)) .$$

It has in $H_0^p(\Omega)$ a closure $\bar{L}_0$ and according to (3) one has

$$\|\bar{L}_0 v\|_p \geqq k_p \|v\|_p \qquad (v \in D(\bar{L}_0)) .$$

This implies that the range $R(\bar{L}_0)$ is closed in $H_0^p(\Omega)$ and therefore a closed subspace of $H^p(\Omega)$. Thus, the element $f \in H^p(\Omega)$ can be decomposed $f = f_0 + f_1$, where f_0 is contained in $R(\bar{L}_0)$ and f_1 in the orthogonal complement of $R(\bar{L}_0)$ in $H^p(\Omega)$. We can find $u_0 \in D(\bar{L}_0) \subset H_0^p(\Omega)$ such that $\bar{L}_0 u_0 = f_0$. There exists a sequence $\{\varphi_n\} \subset D(L_0) = C_0^\infty(\Omega)$ such that in $H_0^p(\Omega)$ one has $\varphi_n \to u_0$ and $L \varphi_n = L_0 \varphi_n \to \bar{L}_0 u_0 = f_0$. Thus, for each $\varphi \in C_0^\infty(\Omega)$

$$(L' \varphi , \varphi_n)_0 \to (L' \varphi , u_0)_0$$

and on the other hand

$$(L' \varphi , \varphi_n)_0 = (\varphi , L \varphi_n)_0 \to (\varphi , f_0)_0 .$$

Consequently, we have

$$(\varphi , f_0)_0 = (L' \varphi , u_0)_0 \qquad (\varphi \in C_0^\infty(\Omega)) . \tag{6}$$

We identify u_0 with the distribution

$$\varphi \to (\varphi, u_0)_0 \qquad (\varphi \in C_0^\infty(\Omega))$$

and write $\vartheta = u - u_0 \in D'(\Omega)$. For each $\psi \in C_0^\infty(\Omega)$ one has $L\,S^p\,\psi \in C_0^\infty(\Omega)$ and using the relation (6) we get

$$\begin{aligned}(L'\,L\,S^p\,\vartheta)(\psi) &= (L'\,L\,S^p\,u)(\psi) - (L'\,L\,S^p\,u_0)(\psi) \\ &= (L\,u)(L\,S^p\,\psi) - u_0(L'\,L\,S^p\,\psi) \\ &= (L\,S^p\,\psi\,, f)_0 - (L'\,(L\,S^p\,\psi)\,, u_0)_0 \\ &= (L\,S^p\,\psi\,, f)_0 - (L\,S^p\,\psi\,, f_0)_0 \\ &= (L\,S^p\,\psi\,, f_1)_0 = (L_0\,\psi\,, f_1)_p = 0\,,\end{aligned}$$

since f_1 is in $H^p(\Omega)$ orthogonal to $R(\bar{L}_0) \supset R(L_0)$. Thus we have $L'\,L\,S^p\vartheta = 0$, $u_0 \in H_0^p(\Omega)$ and $u = u_0 + \vartheta$.

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References

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Printed January 1973