ANNALES ACADEMIAE SCIENTIARUM FENNICAE

Series A

I. MATHEMATICA

542

ON PAIR ALGEBRAS AND STATE-INFORMATION IN AUTOMATA

BY

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HELSINKI 1973
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https://doi.org/10.5186/aasfm.1973.542

ISBN 951—41—0110—3

Communicated 8 January 1973 by ARTO SALOMAA

KESKUSKIRJAPAINO
HELSINKI 1973

1. Introduction

In structure theory of sequential machines state-information is usually represented by a partition or cover on the state-set. When discussion is limited to realizations by one-to-one assignments, the partition model appears quite satisfactory both intuitively and mathematically. In their studies summarized in [2], HARTMANIS and STEARNS employ successfully the lattice of partitions with the substitution property and the notion of partition pairs to describe decompositions and loop-free realizations as well as information-flow in networks of machines.

In order to be able to deal with more general forms of realizations, partitions were replaced by covers (called set systems in [2]), where blocks are allowed to overlap. While many of the central results still hold and some very general results concerning decomposability have been proved (cf. for ex. ZEIGER [4]) using this formalism, the interpretation of covers as state-information pose some serious difficulties. LIU [3] has suggested that cover-information could equivalently be represented by a symmetric, reflexive binary relation. However, it is evident that this simplification of the formalism must involve some loss of detail because the correspondence is not one-to-one. This problem gave the original impetus to this paper. However, it became evident that our questions can be stated and answered in a much more general form. Therefore we first study general lattice morphisms and their effects on pair algebras. The relationship between the two information-representations is then easily described applying the general results.

In the use of terminology we mostly follow BIRKHOFF [1]. The discussion has been limited to finite lattices, and the applications therefore to finite automata. Also, some of the results have not been stated in their most general form. Thus, it would sometimes suffice to assume a join or meet morphism instead of a lattice morphism. On the other hand, the results apply with trivial modifications to input-state and the other pair algebras of an automaton not discussed here.

2. Lattice morphisms

In the following K and L, possibly with subscripts, will always denote finite lattices. The meet and join of two lattice elements x and y is written as xy and $x + y$, respectively. Also, $\inf\{x \mid P(x)\}$ and sup $\{x \mid P(x)\}$ denote the infimum and supremum of the elements x having property P. The least element is 0 and the greatest element 1. These notations are used for any lattice, because the lattice will always be known from the context.

Let $h : K \to L$ be a lattice epimorphism. Then $h^{-1}(u)$ is a sublattice of K, for every $u \in L$. The greatest element of this sub-lattice is denoted by $1_h(u)$ and the least element by $0_h(u)$. When h is understood, we omit the subscripts. The equivalence relation on K induced by h is denoted by E_h. We may view 0_h and 1_h as mappings for which the following lemma is easily verified.

Lemma 2.1. For any lattice epimorphism $h : K \to L$,

(1) $0_h : L \to K$ is a join monomorphism, and

(2) $1_h : L \to K$ is a meet monomorphism.

A lattice K is called *pseudo-complemented* if, for every element $x \in K$, the set of elements $x_1 \in K$ such that $x\,x_1 = 0$ contains a greatest element. This element

$$x^* := \sup\{x_1 \in K \mid xx_1 = 0\}$$

is then called the *pseudo-complement* of x. In our applications x^* will represent the smallest amount of information which combined with x gives perfect state information.

Proposition 2.2. Let K be a pseudo-complemented lattice and $h : K \to L$ a lattice epimorphism such that $h^{-1}(0) = \{0\}$. Then L is pseudo-complemented and, for any x, $y \in K$,

(1) $$h(x)^* = h(x^*)\,,$$

(2) $$x\,E_h y \text{ implies } x^*\,E_h y^*\,, \text{ and}$$

(3) $$x^* = 1(h(x)^*)\,.$$

Proof: Let $u \in L$ and $x \in K$ such that $h(x) = u$. Then $u\,h(x^*) = h(xx^*) = 0$. Suppose $uv = 0$, for some $v = h(y) \in L$. Then $0 = h(x)h(y) = h(xy)$ implies $xy = 0$. Therefore, $y \leqq x^*$, $v \leq h(x^*)$ and $h(x^*) = h(x)^*$. Statement (2) follows immediately from (1). By the assumption $h^{-1}(0) = \{0\}$, $h(x1(h(x)^*)) = h(x)^*\,h(x) = 0$ implies $1(h(x)^*) \leqq x^*$. On the other hand, $h(x^*) = h(x)^*$ implies $x^* \leqq 1(h(x^*)) = 1(h(x)^*)$. Thus we have verified (3).

3. Pair algebras

A subset $\Gamma \subseteq K_1 \times K_2$, where K_1 and K_2 are finite latticcs is said to form a *pair algebra* [2] if

(A) for any $x_1, x_2 \in K_1$ and $y_1, y_2 \in K_2$, $(x_1, y_1), (x_2, y_2) \in \Gamma$ implies $(x_1 + x_2, y_1 + y_2), (x_1 \cdot x_2, y_1 \cdot y_2) \in \Gamma$, and

(B) $(0, y) \in \Gamma$ and $(x, 1) \in \Gamma$, for every $x \in K_1$ and $y \in K_2$.

We relate pair algebras to the more familiar concept of a subdirect product. A subset $\Gamma \subseteq K_1 \times K_2$ is a subdirect product of K_1 and K_2 if it satisfies conditon (A) and

(C) $\mathrm{pr}_1(\Gamma) = K_1$ and $\mathrm{pr}_2(\Gamma) = K_2$, where pr_1 and pr_2 are the projections to K_1 and K_2.

Proposition 3.1: A subset $\Gamma \subseteq K_1 \times K_2$ is a pair algebra iff it is a subdirect product and $(0, 1) \in \Gamma$.

Proof: Every pair algebra is a subdirect product because condition (B) implies condition (C). Suppose Γ is a subdirect product and $(0, 1) \in \Gamma$. For any $x \in K_1$, there exists by (C) a pair $(x, y) \in \Gamma$. Using (A) we have $(x, 1) = (x + 0, y + 1) \in \Gamma$. Dually, $(0, y) \in \Gamma$, for any $y \in K_2$.

It is often convenient to consider a pair algebra as a lattice with the naturally defined componentwise operations and partial ordering.

Lemma 3.2. For any lattice epimorphisms $h_1 : K_1 \to L_1$ and $h_2 : K_2 \to L_2$,

(1) $h := h_1 \times h_2$ is a lattice epimorphism,

(2) $h(\Gamma)$ is a pair algebra in $L_1 \times L_2$, for any pair algebra $\Gamma \subseteq K_1 \times K_2$, and

(3) $h^{-1}(\Delta)$ is a pair algebra in $K_1 \times K_2$, for any pair algebra $\Delta \subseteq L_1 \times L_2$.

The lemma follows from well known properties of direct products and morphisms. We observe that $h(0, 1) = (0, 1)$.

For any pair algebra $\Gamma \subseteq K_1 \times K_2$, two mappings $\mathrm{m}_\Gamma : K_1 \to K_2$ andi $M_\Gamma : K_2 \to K_1$ can be defined as folllows. For any $x \in K_1$ and $y \in K_2$, $m_\Gamma(x) := \inf \{y_1 \in K_2 \mid (x, y_1) \in \Gamma\}$, and $M_\Gamma(y) := \sup \{x_1 \in K_1 \mid (x_1, y) \in \Gamma\}$.

A number of properties of these mappings can be found in [2]. In the applications where lattice elements represent state information in an automaton, $m_\Gamma(x)$ is the maximum next-state information that can be computed from the present-state information x. Similarly, $M_\Gamma(y)$ is the minimum present-state information required for the next-state information y.

Proposition 3.3. Let $h_1 : K_1 \to L_1$, $h_2 : K_2 \to L_2$ be lattice epimorphisms and $\Gamma \subseteq K_1 \times K_2$, $\Delta \subseteq L_1 \times L_2$ pair algebras such that $h_1 \times h_2(\Gamma) = \Delta$. Then

$$m_\Delta(u) = h_2(m_\Gamma(0(u))) \text{, for any } u \in L_1 , \tag{1}$$

$$M_\Delta(v) = h_1(M_\Gamma(1(v))) \text{, for any } v \in L_2 , \tag{2}$$

$$h_2((m_\Gamma(x)) \geqq m_\Delta(h_1(x)) \text{, for any } x \in K_1 \text{, and} \tag{3}$$

$$h_1(M_\Gamma(y)) \leqq M_\Delta(h_2(y)) \text{, for any } y \in K_2 . \tag{4}$$

Proof: Because the first two statements are dual to each other (in a generalized sense), we prove (1) only. We claim that $(u , v) \in \Delta$ iff there exists a $y \in K_2$ such that $h_2(y) = v$ and $(0(u) , y) \in \Gamma$. Suppose y satisfies these conditions. Then

$$(u , v) = (h_1(0(u)) , h_2(y)) \in h_1 \times h_2(\Gamma) = \Delta .$$

Conversely, let $(u , v) \in \Delta$. Then there exists a pair $(x , y) \in \Gamma$ such that $h_1 \times h_2(x , y) = (u , v)$. But $h_1(x) = u$ implies $0(u) \leq x$ and $(0(u) , y) \in \Gamma$. Also $h_2(y) = v$. Using this result we have

$$\begin{aligned} h_2(m_\Gamma(0(u))) &= \inf \{h_2(y) \mid (0(u) , y) \in \Gamma\} \\ &= \inf \{v \mid (u , v) \in \Delta\} \\ &= m_\Delta(u) . \end{aligned}$$

Statement (3) follows from (1) because $x \geq 0(h_1(x))$ and both h_2 and m_Γ are isotone mappings. Similarly, (4) follows from (2).

Corollary 3.3.1. With the assumptions of Proposition 3.3, we have

$$0(m_\Delta(u)) \leqq m_\Gamma(0(u)) \text{, for any } u \in L_1 \text{, and} \tag{1}$$

$$1(M_\Delta(v)) \geqq M_\Gamma(1(v)) \text{, for any } v \in L_2 . \tag{2}$$

We shall now show that these results can be strengthened when we make some additional assumptions concerning h_1 and h_2. In particular, we can compute m_Δ and M_Δ in the pair algebra Γ using any element from the corresponding E_{h_1}- or E_{h_2}- class.

Proposition 3.4. Let $h_1 : K_1 \to L_1$, $h_2 : K_2 \to L_2$ be lattice epimorphisms and $\Gamma \subseteq K_1 \times K_2$, $\Delta \subseteq L_1 \times L_2$ pair algebras such that $h_1 \times h_2(\Gamma) = \Delta$ and, for any $x \in K_1$, $v \in L_2$,

(a) $$(x, 1(v)) \in \Gamma \text{ if } (h_1(x), v) \in \Delta .$$

Then

(1) $$m_\Delta(h_1(x)) = h_2(m_\Gamma(x)) , \text{ for any } x \in K_1 ,$$

(2) $$1(M_\Delta(v)) = M_\Gamma(1(v)) , \text{ for any } v \in L_2 ,$$

(3) $$1(m_\Delta(u)) \geqq m_\Gamma(1(u)) , \text{ for any } u \in L_1 , \text{ and}$$

(4) $$x_1 E_{h_1} x_2 \text{ implies } m_\Gamma(x_1) E_{h_2} m_\Gamma(x_2) , \text{ for any } x_1 , x_2 \in K_1 .$$

Proof: Statement (1) can be proved using (a) as follows. If $(x, y) \in \Gamma$, then $(h_1(x), h_2(y)) \in \Delta$. Conversely, let $(h_1(x), v) \in \Delta$. Then $v = h_2(y)$ and $(x, y) \in \Gamma$ if we choose $y = 1(v)$. Hence,

$$\begin{aligned} h_2(m_\Gamma(x)) &= \inf\{h_2(y) \mid (x, y) \in \Gamma\} \\ &= \inf\{v \mid (h_1(x), v) \in \Delta\} \\ &= m_\Delta(h_1(x)) . \end{aligned}$$

Consider now any $v \in L_2$ and let $x = 1(M_\Delta(v))$. Then $(M_\Delta(v), v) \in \Delta$ implies $(1(M_\Delta(v)), 1(v)) \in \Gamma$ by condition (a). Therefore, $1(M_\Delta(v)) \leq M_\Gamma(1(v))$. Combining this with Corollary 3.3.1 we get (2). Let $v \in L_1$. Writing $x = 1(u)$ in (1), we get $m_\Delta(u) = h_2(m_\Gamma(1(u)))$. Therefore, $m_\Gamma(1(u)) \leq 1(m_\Delta(u))$. Statement (4) follows easily from (1).

We note that statement (4) in Proposition 3.4 means that the m_Γ-operator can be computed in the image algebra »modulo E_{h_2}» assuming that the argument is known »modulo E_{h_1}». Assumption (a) is equivalent to the condition that $(1(u), 1(v)) \in \Gamma$ whenever $(u, v) \in \Delta$.

The following results can be obtained similarly as Proposition 3.4.

Proposition 3.5. Let $h_1 : K_1 \to L_1$, $h_2 : K_2 \to L_2$ be lattice epimorphisms and $\Gamma \subseteq K_1 \times K_2$, $\Delta \subseteq L_1 \times L_2$ pair algebras such that $h_1 \times h_2(\Gamma) = \Delta$ and, for any $u \in L_1$ and $y \in K_2$,

(b) $$(0(u), y) \in \Gamma \text{ if } (u, h_2(y)) \in \Delta .$$

Then

(1) $$M_\Delta(h_2(y)) = h_1(M_\Gamma(y)) , \text{ for any } y \in K_2 ,$$

(2) $$0(m_\Delta(u)) = m_\Gamma(0(u)) , \text{ for any } u \in L_1 ,$$

(3) $$0(M_\Delta(v)) \leqq M_\Gamma(0(v)) , \text{ for any } v \in L_2 , \text{ and}$$

(4) $$y_1 E_{h_2} y_2 \text{ implies } M_\Gamma(y_1) E_{h_1} M_\Gamma(y_2) , \text{ for any } y_1 , y_2 \in K_2 .$$

Again, we note that under condition (b) M_Γ can be computed in Δ »modulo E_{h_1}». The condition (b) means that $(0(u), 0(v)) \in \Gamma$ whenever $(u, v) \in \Delta$.

We conclude this section by the following result which can easily be proved.

Proposition 3.6. With the notations and combined assumptions of Propositions 3.4. and 3.5., $\Gamma \cap (h_1^{-1}(u) \times h_2^{-1}(v))$ is a pair algebra in $h_1^{-1}(u) \times h_2^{-1}(v)$, for any $u \in L_1$, $v \in L_2$.

4. Covers and relations

Let Q be a finite set. A collection α of subsets of Q is called a cover (or set system in [2]) if $\bigcup \{A \mid A \in \alpha\} = Q$ and $A_1 \supseteq A_2$ implies $A_1 = A_2$, for any $A_1, A_2 \in \alpha$. The set of all covers of Q is denoted by C_Q. For any $\alpha, \beta \in C_Q$, we write

$$\alpha \leq \beta \text{ iff } (\forall A \in \alpha)(\exists B \in \beta)\, A \subseteq B .$$

Then C_Q becomes a distributive lattice, cf. [2], where

$$\alpha \cdot \beta = \max \{A \cap B \mid A \in \alpha , B \in \beta\}$$

and

$$\alpha + \beta = \max \{\alpha \cup \beta\} .$$

The least cover 0 consists of singletons only, and the greatest cover is $1 = \{Q\}$.

Let R_Q denote the set of all symmetric, reflexive relations on Q. If we order the relations by the usual inclusion relation, R_Q becomes a Boolean lattice, where $\varrho \cup \sigma$ and $\varrho \cap \sigma$ are the join and meet of $\varrho, \sigma \in R_Q$.

The greatest element is $Q \times Q$ and the least element the equality relation ε.

Following Liu [3] we assign to every α in C_Q the relation $r(\alpha) := \{(q, q') \mid (\exists A \in \alpha)\, q, q' \in A\}$.
Clearly, we get a mapping from C_Q to R_Q, but it is not one-to-one, if Q has more than two elements.

Proposition 4.1. For any set Q, $r : C_Q \to R_Q$ is a lattice epimorphism. For any $\varrho \in R_Q$, the sublattice $r^{-1}(\varrho)$ has the least element

$$0_r(\varrho) = \max \{\{q , q'\} \mid (q , q') \in \varrho\} ,$$

and the greatest element

$$1_r(\varrho) = \max\{C \mid C \times C \subseteq \varrho\}$$

In particular, $r^{-1}(\varepsilon) = \{0\}$.

We omit the straightforward proof.

Proposition 4.2. For any set Q,

(1) $$0_r : R_Q \to C_Q \text{ is a lattice monomorphism, and}$$

(2) $$1_r(\varrho \cap \sigma) = 1_r(\varrho) \cdot 1_r(\sigma)\,, \text{ for any } \varrho, \sigma \in R_Q\,.$$

This proposition is a special case of Lemma 2.1. except for the claim that 0_r is a meet morphism. This again follows from the following observation. Let C_{Q2} be the set of all covers of Q with no blocks containing more than two elements. Clearly, C_{Q2} is a sublattice of C_Q and r restricted to C_{Q2} is an isomorphism with 0_r as its inverse.

Because C_Q and R_Q are finite distributive lattices, they certainly are pseudo-complemented. Using Proposition 2.2. and the last part of Proposition 4.1., we get the following result.

Proposition 4.3. For any $\alpha, \beta \in C_Q$,

(1) $$\alpha\beta = 0 \text{ iff } r(\alpha) \cap r(\beta) = \varepsilon\,,$$

(2) $$r(\alpha^*) = r(\alpha)^*\,, \text{ and}$$

(3) $$\alpha^* = 1(r(\alpha)^*)\,.$$

In order to interprete these results, we assume that Q is the state set of an automaton (Q, X, T), where X is the set of input symbols and $T : Q \times X \to Q$ the transition function. We will call this automaton simply T. For any $q \in Q$ and $x \in X$, we use the notation $qx^T := T(q, x)$.

Every cover $\alpha \in C_Q$ represents now some information about the state of T. Similarly, each relation $\varrho \in R_Q$ can be viewed as some state-information (cf. [3]). It follows from Proposition 4.1. that the relation representation of some state-information determines its cover representation with the accuracy of a sublattice of C_Q only. On the other hand, this correspondence between the two representations is preserved in the combination of information (meet) and the combination of ignorance (join). The pseudo-complement α^* represents the minimum information which combined with α gives perfect state information. Proposition 4.3 tells us that this can essentially be computed in R_Q using $r(\alpha)$. Also, the question whether the cover information α and β jointly give perfect state information can be resolved considering the relational equivalents.

(We have ignored here the difficulties involved in the interpretation of $\alpha\beta$ as the combined information, cf. [2]). In general, the combined information $\alpha\beta$ can be computed in R_Q »modulo E_r» only.

5. Relation pairs and cover pairs

From here on Q will always be the state set of the automaton T. The pair $(\alpha, \beta) \in C_Q \times C_Q$ is called a *cover pair* (system pair in [2]) if, for every $A \in \alpha$ and $x \in X$ there exists a block $B \in \beta$ such that $Ax^T := \{q \mid (\exists\, q' \in A)\, q'\, x^T = q\} \subseteq B$. The set of cover pairs CP_T of T forms a pair algebra, cf. [2].

We call $(\varrho, \sigma) \in R_Q \times R_Q$ a *relation pair* if, for any $q, q' \in Q$ and $x \in X$, $(q, q') \in \varrho$ implies $(qx^T, q'\, x^T) \in \sigma$. The set of all relation pairs of T is denoted by RP_T.

Proposition 5.1. For any automaton T, RP_T is a pair algebra in $R_Q \times R_Q$.

We omit the simple proof. The m- and M-operators can now be introduced as usual. They are denoted by m_C and M_C in CP_T, and by m_R and M_R in RP_T. For any $\varrho \in R_Q$, we can evidently compute $m_R(\varrho)$ and $M_R(\varrho)$ as follows:

$$m_R(\varrho) = \{(qx^T, q'\, x^T) \mid (q, q') \in \varrho\,,\ x \in X\} \cup \varepsilon$$

and

$$M_R(\varrho) = \{(q, q') \mid (qx^T, q'\, x^T) \in \varrho\,,\ x \in X\}\,.$$

Proposition 5.2. For any automaton T and $\varrho, \sigma \in R_Q$,

(1) $$r \times r(CP_T) = RP_T\,,$$

(2) $$(\varrho, \sigma) \in RP_T \text{ iff } (0(\varrho), 0(\sigma)) \in CP_T\,, \text{ and}$$

(3) $$(\varrho, \sigma) \in RP_T \text{ iff } (1(\varrho), 1(\sigma)) \in CP_T\,.$$

Proof: We first prove that $(\alpha, \beta) \in CP_T$ implies $(r(\alpha), r(\beta)) \in RP_T$. This gives one half of all of the statements. Thus, let $(q, q') \in r(\alpha)$ and $x \in X$. Then $q, q' \in A$, for some $A \in \alpha$ such that $Ax^T \subseteq B$ where $B \in \beta$. This implies $(qx^T, q'x^T) \in r(\beta)$ as required. To prove the second part of (2), we have to show that $(\varrho, \sigma) \in RP_T$ implies $(0(\varrho), 0(\sigma)) \in CP_T$. Now every block of $0(\varrho)$ is of the form $\{q, q'\}$, where $(q, q') \in \varrho$ ($q = q'$ possible). For any $x \in X$, $(q\, x^T, q'\, x^T) \in \sigma$, and hence $\{q\, x^T, q'\, x^T\} \subseteq B$, for some $B \in 0(\sigma)$. Statement (2) implies $RP_T \subseteq r \times r\,(CP_T)$, and thus we have the second half of (1). Finally, let $(\varrho, \sigma) \in RP_T$, and $A \in 1(\varrho)$, $x \in X$. Then $A \times A \subseteq \varrho$, and $(Ax^T) \times (Ax^T) \subseteq \sigma$. Hence, $Ax^T \subseteq B$ for some $B \in 1(\varrho)$. This proves statement (3).

It follows from Proposition 5.2. that all results of Section 3 are applicable here.

A relation $\varrho \in R_Q$ is called an SP relation (SP for substitution property) if $(\varrho, \varrho) \in RP_T$. Similarly, SP covers are defined.

Corollary 5.2.1. For any automaton T and $\varrho \in R_Q$, the following three conditions are equivalent:

(1) ϱ is an SP relation,

(2) $0(\varrho)$ is an SP cover, and

(3) $1(\varrho)$ is an SP cover.

This result cannot be extended to apply to any $\alpha \in h^{-1}(\varrho)$.

The next corollary follows from statements (1) and (3) of Proposition 3.4, and statement (2) of Proposition 3.5.

Corollary 5.2.2. For any automaton T, $\alpha, \beta \in C_Q$ and $\varrho \in R_Q$,

(1) $$m_R(r(\alpha)) = r(m_C(\alpha)),$$

(2) $$1(m_R(\varrho)) \geqq m_C(1(\varrho)),$$

(3) $$0(m_R(\varrho)) = m_C(0(\varrho)), \text{ and}$$

(4) $$\alpha E_r \beta \text{ implies } m_C(\alpha) E_r m_C(\beta).$$

Similarly, we get the following results.

Corollary 5.2.3. For any automaton T, $\alpha, \beta \in C_Q$ and $\varrho \in R_Q$,

(1) $$M_R(r(\alpha)) = r(M_C(\alpha)),$$

(2) $$0(M_R(\varrho)) \leqq M_C(0(\varrho)),$$

(3) $$1(M_R(\varrho)) = M_C(1(\varrho)), \text{ and}$$

(4) $$\alpha E_r \beta \text{ implies } M_C(\alpha)\, E_r M_C(\beta).$$

6. Concluding remarks

As a conclusion we comment on some aspects of the previous results which may be of significance to the structure theory of sequential machines.

We have seen that the algebra of symmetric, reflexive relations is an epimorphic image of the algebra of covers. Thus relation information provides a representation of cover information, although not a faithful one. Using Corollaries 5.2.2 and 5.2.3 we can compute $m_R(\varrho)$ and $M_R(\varrho)$ in the cover algebra using any $\alpha \in r^{-1}(\varrho)$ as the argument:

$$m_R(\varrho) = r(m_C(\alpha))$$

and

$$M_R(\varrho) = r(M_C(\alpha)) \ .$$

On the other hand, m_C and M_C can be computed »modulo E_r» using the relation representation even if the cover is known »modulo E_r» only. Thus the possible loss of information resulting when we transfer from the cover representation to the relation representation will not spread during computations.

From Corollary 5.2.1 we see that, for any serial decomposition which could be found using covers, there exists a serial decomposition which can be detected from the lattice of SP relations. Using Proposition 4.3 too, we get the same result for parallel decompositions.

These observations suggest that the cover representation could sometimes be replaced by a narrower formalism based on symmetric, reflexive relations without any essential loss power.

Acknowledgement

This work has been supported by Grant A7911 of the National Research Council of Canada.

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Printed August 1973