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ON THE TENSORPRODUCT AND PRODUCT HOM(f, g) OF COMPACT OPERATORS IN LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

BY

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HEIKKI APIOLA

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INTRODUCTION

In his paper [12] K. Vala proved that for normed spaces E_1, E_2, E_3, E_4 and non-zero linear operators $f: E_1 \to E_2$ and $g: E_3 \to E_4$ the following statement is true: The mapping which to each linear operator $u: E_2 \to E_3$ assigns the composed operator $g\,u\,f: E_1 \to E_4$ is precompact if and only if f and g are both precompact operators.

In the present paper the corresponding statement will be proved allowing the spaces E_k to be arbitrary locally convex Hausdorff topological vector spaces. Using the concept of an $\mathfrak{S}$-precompact operator with respect to a family $\mathfrak{S}$ of bounded subsets of the domain space the generalization appears in the following form: The mapping described above, which will be called $\mathrm{Hom}(f,g)$, maps the equicontinuous subsets of $L(E_2, E_3)$ into precompact subsets of $L(E_1, E_4)$ with respect to an $\mathfrak{S}_1$-topology, if and only if f is $\mathfrak{S}_1$-precompact and g is precompact (theorem 3.3.). An analogous result for compact instead of precompact operators will be proved, too (theorem 3.7.).

These theorems are general enough to be applied in several different ways to give quite a number of results concerning compact or precompact sets or mappings of locally convex spaces. On one hand the mapping $\mathrm{Hom}(f,g)$ can be regarded as a generalized transpose of f, as the mapping $\mathrm{Hom}(f, id)$ equals the transpose of f if the mapping id is the identity transformation of the scalar field. One of the consequences of the general theorems mentioned is thus what we call the Schauder theorem for locally convex spaces (theorem 3.11.).

Another application possibility arises from the interpretation of the bilinear mapping Hom as a tensorproduct mapping. This method, together with the standard tools of the duality theory, give us, among other things, the result that the tensorproduct of two precompact operators is precompact with respect to the ε-tensorproduct spaces (theorem 4.5.). (It should be emphasized that an operator is called precompact if it sends all bounded sets into precompact sets.)

A third point of view is to consider the mapping $\mathrm{Hom}(id_E, id_F)$ which equals the identity transformation of the space $L(E,F)$ of linear operators. One of the results obtained this way is a very general form of the theorem of Alaoglu-Bourbaki (theorem 5.2.).

1. Preliminaries

All the vector spaces under consideration will be supposed to be defined over the field $\mathbf{K}$ of real or complex numbers. If A is a set and F is a topological vector space, the symbols $\mathfrak{F}(A,F)$ or F^A will be used for the space of all mappings from A into F. If E is a topological vector space over the same field as F, we denote by $L(E,F)$ the subspace of $\mathfrak{F}(E,F) = F^E$ consisting of continuous linear mappings from E into F. The space $L(E,\mathbf{K})$, the dual of E, will be denoted by E'.

The topology of uniform convergence on $\mathfrak{F}(A,F)$ will be defined by means of the sets $f + N(A,V)$, which form a fundamental system of neighbourhoods of $f \in \mathfrak{F}(A,F)$ as V runs through a fundamental system of neighbourhoods of the origin in F. Here $N(A,V)$ stands for the set of functions mapping A into V. This topology is a uniform topology but not usually a vector space topology.

The topology of uniform convergence on the sets S of a family $\mathfrak{S}$ of subsets of A (or briefly, the $\mathfrak{S}$ — topology) is defined as the initial topology with respect to the mappings

$$f \to f \mid S : \mathfrak{F}(A,F) \to \mathfrak{F}(S,F)\,, \quad S \in \mathfrak{S}\,,$$

where each $\mathfrak{F}(S,F)$ is equipped with the topology of uniform convergence. A subbasis of neighbourhoods of the origin is formed by the sets $N(S,V) = \{f \mid f(S) \subset V\}$ as S runs through $\mathfrak{S}$ and V through a fundamental system of neighbourhoods of the origin in F. If the family $\mathfrak{S}$ is directed by set-theoretic inclusion, the sets $N(S,V)$ form a basis. The notation $\mathfrak{F}_{\mathfrak{S}}(A,F)$ will be used for the space $\mathfrak{F}(A,F)$ equipped with the $\mathfrak{S}$-topology. The topology induced on a subspace G of $\mathfrak{F}_{\mathfrak{S}}(A,F)$ is a vector space topology, if and only if $f(S)$ is bounded in F for each $f \in G$ and $S \in \mathfrak{S}$.

If E is a topological vector space and $\mathfrak{S}$ a family of bounded subsets of E, the subspace $L_{\mathfrak{S}}(E,F)$ of $\mathfrak{F}_{\mathfrak{S}}(E,F)$ is a topological vector space, which is locally convex if F is. Moreover $L_{\mathfrak{S}}(E,F)$ is Hausdorff if F is Hausdorff and $\mathfrak{S}$ is a covering of E. In the following table we shall list the $\mathfrak{S}$-topologies of special importance later:

the topology of	the set $\mathfrak{S}$	notation of space
— pointwise (or simple) convergence	finite subsets of E	$L_s(E,F)$
— precompact convergence	precompact subsets of E	$L_c(E,F)$
— bounded convergence	bounded subsets of E	$L_b(E,F)$
— equicontinuous convergence	equicontinuous subsets of E'	$L_e(E',F)$

The topology of pointwise (resp. bounded) convergence will also be called the weak (resp. strong) topology, especially if $F = \mathbf{K}$. In the case of a dual pair (F, G) the first three topologies on F are denoted by $\sigma(F, G)$, $\lambda(F, G)$ and $\beta(F, G)$ respectively. For dual pairs we shall also need the Mackey topology $\tau(F, G)$, which is the topology of uniform convergence on the balanced, convex, $\sigma(G,F)$-compact subsets of G. It is the finest topology compatible with the duality between F and G.

Bounded and equicontinuous subsets of $L(E, F)$ are characterized as follows:

A subset H of $L_{\mathfrak{S}}(E,F)$ is *bounded* if and only if one of the following equivalent conditions is valid:

(b1) $\bigcap_{u \in H} u^{-1}(V)$ absorbs each $S \in \mathfrak{S}$ for every neighbourhood V in F,

(b2) $\bigcup_{u \in H} u(S)$ is bounded in F for every $S \in \mathfrak{S}$.

A subset H of $L(E, F)$ is *equicontinuous*, if and only if one of the following equivalent conditions is valid:

(e1) $\bigcap_{u \in H} u^{-1}(V)$ is a neighbourhood in E for each neighbourhood V in F,

(e2) For each neighbourhood V in F there exists a neighbourhood U in E such that $\bigcup_{u \in H} u(U) \subset V$.

(The word »neighbourhood» alone without any reference to a point in a topological vector space is to be understood as a neighbourhood of the origin.)

An equicontinuous subset of $L(E,F)$ is bounded for any $\mathfrak{S}$-topology, because the neighbourhoods of the origin absorb bounded sets and thus (e1) implies (b1).

If E is a locally convex space and $\mathfrak{S}$ is a covering of E by bounded sets, E will be called *$\mathfrak{S}$-barrelled* if every barrel that absorbs all sets $S \in \mathfrak{S}$ is a neighbourhood of the origin. If E is $\mathfrak{S}$-barrelled with respect to the family $\mathfrak{S}$ of finite (resp. bounded) subsets of E, then E is barrelled (resp. infrabarrelled) in the terminology of [6]. A characteristic property of $\mathfrak{S}$-barrelled spaces is the following theorem of Banach-Steinhaus type:

Theorem 1.1. *If E is $\mathfrak{S}$-barrelled and F is an arbitrary locally convex space, then every bounded subset H of $L_{\mathfrak{S}}(E, F)$ is equicontinuous.*

Proof. Let V be a balanced, convex and closed neighbourhood in F. The set

$$T = \bigcap_{u \in H} u^{-1}(V)$$

is balanced, convex and closed and absorbs each $S \in \mathfrak{S}$. Thus T is a neighbourhood in E, which implies that H is equicontinuous. $\square$

A mapping f from a set A into a topological vector space F is called *precompact* (resp. *compact*) if the image $f(A)$ is precompact (resp. relatively compact) in F. If $\mathfrak{S}$ is a family of subsets of A, f is called $\mathfrak{S}$-*precompact* (resp. $\mathfrak{S}$-*compact*) if $f(S)$ is precompact (resp. relatively compact) in F for each S in $\mathfrak{S}$.

Let E be a topological vector space and $\mathfrak{S}$ a family of bounded subsets of E. A continuous linear mapping $f: E \to F$ is called an $\mathfrak{S}$-*precompact operator* (resp. $\mathfrak{S}$-*compact operator*) if f is an $\mathfrak{S}$-precompact (resp. $\mathfrak{S}$-compact) mapping. If $\mathfrak{S}$ is the family of all bounded subsets of E, an $\mathfrak{S}$-precompact (resp. $\mathfrak{S}$-compact) operator is called precompact (resp. compact). (The continuity requirement included in the definition of a precompact (resp. compact) operator does not in general follow from the property of mapping bounded sets into precompact (resp. compact) sets unless E is bornological.)

Remark. In the terminology of Grothendieck [4] a precompact (resp. compact) operator means a linear mapping such that $f(U)$ is precompact (resp. relatively compact) for some neighbourhood U. This type of f is a precompact (resp. compact) operator in the sense defined above, as is readily seen. Of course both definitions coincide if E is a normed space.

We denote by $T_{\mathfrak{S}}(E,F)$ the set of $\mathfrak{S}$-precompact operators from E into F. The set of precompact operators will be denoted by $T(E,F)$. The symbol $A(E,F)$ stands for the set of operators of finite rank from E into F. Let us collect some basic properties of $\mathfrak{S}$-precompact operators:

Theorem 1.2. *If E and F are topological vector spaces and $\mathfrak{S}$ is a family of bounded subsets of E, the following conditions hold*:

(i) $T(E,E)$ *is an ideal of* $L(E,E)$

(ii) $T_{\mathfrak{S}}(E,F)$ *is closed in* $L_{\mathfrak{S}}(E,F)$

(iii) $A(E,F) \subset T_{\mathfrak{S}}(E,F)$.

Proof. Parts (i) and (iii) are immediate and the proof of (ii) is similar to that of [9] ch. III statement 9.3, p. 110. $\square$

From parts (ii) and (iii) it follows that the closure of $A(E,F)$ in $L_{\mathfrak{S}}(E,F)$ is contained in $T_{\mathfrak{S}}(E,F)$. Whether this inclusion is in fact an equality is one form of the »problème d' approximation» of A. Grothendieck ([4]

part I, »proposition» 37, p. 170). A negative answer to this question has recently been given by Per Enflo, whose counterexample shows that even in the case of reflexive and separable Banach spaces E and F the inclusion may be strict.

2. Compact sets in function spaces

We begin this section with two general results concerning uniform spaces. We use the terminology and notations of Bourbaki [1], chapter II, in connection with uniformity. First we give an elementary proof of a well-known result (c.f. [1], II. 4. 2 proposition 3, p. 203).

Lemma 2.1. *Let X be a set, $(Y_i)_{i\in I}$ a family of uniform spaces, and for each $i \in I$ let f_i be a mapping of X into Y_i. A set $A \subset X$ is precompact for the initial uniformity with respect to the mappings $(f_i)_{i\in I}$, if and only if $f_i(A)$ is precompact in Y_i for each $i \in I$.*

Proof. 1° If A is precompact, then $f_i(A)$ is precompact, since f_i is uniformly continuous for all $i \in I$.

2° Suppose $f_i(A)$ is precompact for all $i \in I$. Let U be an arbitrary entourage in X. Then there exists a finite subset $K \subset I$ and for each $k \in K$ an entourage V_k in Y_k such that

$$\bigcap_{k\in K} g_k^{-1}(V_k) \subset U ,$$

where $g_k(x,y) = (f_k(x), f_k(y))$. As the sets $f_k(A)$ are precompact, there is for each $k \in K$ a finite covering of A by sets $A^k_{i_k}$, $i_k \in I_k$ such that

$$(f_k(x) , f_k(y)) \in V_k \text{ as soon as } x , y \in A^k_{i_k}, i_k \in I_k .$$

We will show that the set of intersections

$$\{ \bigcap_{k\in K} A^k_{i_k} \mid i_k \in I_k \text{ for all } k \in K\}$$

is a finite covering of A by U-small sets.

If $x \in A$, then for each $k \in K$ there is an index $i_k \in I_k$ such that $x \in A^k_{i_k}$, hence

$$x \in \bigcap_{k\in K} A^k_{i_k} .$$

To prove that each such intersection is U-small, let

$$x, y \in \bigcap_{k\in K} A^k_{i_k} , \; i_k \in I_k .$$

Then $g_k(x, y) = (f_k(x), f_k(y)) \in V_k$ for all $k \in K$, which implies that

$$(x, y) \in \bigcap_{k \in K} g_k^{-1}(V_k) \subset U$$

and the proof is complete. $\square$

The following definition is due to Vala (c.f. [12], p. 4):

Definition 2.2. *A set H of mappings from a set A into a uniform space Y is said to have equal variation if for every entourage V in Y there is a finite covering $(A_i)_{i \in I}$ of A such that $(f(x), f(y)) \in V$ for all $f \in H$, whenever $x, y \in A_i$, $i \in I$.*

We shall now »translate» the form of Ascoli's theorem proved by K. Ylinen (c.f. [14], theorem 2.1) into the language of uniform spaces.

Theorem 2.3. *Let A and B be arbitrary sets, Y a uniform space and Φ a mapping from $A \times B$ into Y. Consider the following four conditions:*

I *The range of the mapping $\Phi(\cdot, y) : A \to Y$ is precompact for all $y \in B$.*

II *The range of the mapping $\Phi(x, \cdot) : B \to Y$ is precompact for all $x \in A$.*

III *The set $\{\Phi(\cdot, y) \mid y \in B\}$ has equal variation.*

IV *The set $\{\Phi(x, \cdot) \mid x \in A\}$ has equal variation.*

The following statements hold:

III *implies* I, IV *implies* II,

II *and* III *together imply* IV,

I *and* IV *together imply* III.

Proof. The first two statements are trivial. By symmetry it suffices to prove one of the remaining two.

Suppose II and III are valid. To prove condition IV, choose an arbitrary entourage V of Y and a symmetric entourage U such that $U \circ U \circ U \subset V$. By condition III there exists a finite covering $(A_i)_{i \in I}$ of A such that

$$u, v \in A_i, i \in I \text{ implies}$$
$$(\Phi(u, y), \Phi(v, y)) \in U \text{ for all } y \in B.$$

For each $i \in I$ choose $x_i \in A_i$. By condition II and lemma 2.1 the set B is precompact for the initial uniformity defined by the mappings $\Phi(x, \cdot)$, $x \in A$. Thus there is a finite covering $(B_j)_{j \in J}$ of B by W-small sets, where

$$W = \bigcap_{i \in I} g_i^{-1}(U)\,,\; g_i(w\,,z) = (\Phi\,(x_i\,,w)\,,\; \Phi(x_i\,,z))\,.$$

We claim that $(B_j)_{j \in J}$ is the required covering corresponding to the given entourage V. So let $w, z \in B_j, j \in J$, and let $x \in A$. Then $(\Phi(x_i, w), \Phi(x_i, z)) \in U$ for all $i \in I$, and $x \in A_k$ for some $k \in I$. Thus we have:

$$(\Phi(x, w)\,,\; \Phi(x_k, w)) \in U\,,$$

$$(\Phi(x_k, w)\,,\; \Phi(x_k, z)) \in U\,,$$

$$(\Phi(x_k, z)\,,\; \Phi(x, z)) \in U\,,$$

which implies that

$$(\Phi(x, w)\,,\; \Phi(x, z)) \in U \circ U \circ U \subset V\,,$$

i.e. condition IV holds. $\square$

As a consequence we get the following generalization of theorem 1 in [12].

Theorem 2.4. *Let A be a set and F a topological vector space. A set H of precompact mappings from A into F is precompact in $\mathfrak{F}\,(A,F)$ for the topology of uniform convergence, if and only if the following conditions are satisfied:*

(1) *The set $H(x) = \{f(x) \mid f \in H\}$ is precompact for all $x \in A$,*

(2) *H has equal variation.*

Proof. Define a mapping Φ: $A \times H \to F$ by $\Phi(x.f) = f(x)$. If B is replaced by H and Y by F in theorem 2.3, conditions I—IV take the following form:

I Each $f \in H$ is precompact.

II $H(x)$ is precompact for all $x \in A$.

III H has equal variation

IV H is precompact in $\mathfrak{F}\,(A, F)$ for the topology of uniform convergence.

As condition I is valid by assumption, it follows that II and III together are equivalent to IV. $\square$

As a consequence of the preceding theorem we get the following result:

Theorem 2.5. *Let H be a set of $\mathfrak{S}$-precompact mappings from A into F, where $\mathfrak{S}$ is a family of subsets of the given set A. H is precompact for the $\mathfrak{S}$-topology, if and only if the following conditions are satisfied:*

(1) *$H(x)$ is precompact for all $x \in \bigcup \mathfrak{S}$,*

(2) The set $H \mid S = \{f \mid S \mid f \in H\}$ has equal variation for all $S \in \mathfrak{S}$.

Proof. As the $\mathfrak{S}$-topology is the initial topology with respect to the mappings $u \to u|S$, $S \in \mathfrak{S}$, it follows from lemma 2.1 that the set H is precompact in $\mathfrak{F}_{\mathfrak{S}}(A, F)$, if and only if $H|S$ is precompact in $\mathfrak{F}(S, F)$ for all $S \in \mathfrak{S}$, which by theorem 2.4 is equivalent to conditions (1) and (2). $\square$

From now on we shall restrict ourselves to topological vector spaces and continuous linear mappings between them. For the purpose of subsequent reference we formulate as a theorem the following fact, which is a direct consequence of the above theorem.

Theorem 2.6. *Let E and F be topological vector spaces and $\mathfrak{S}$ a covering of E by bounded sets. A subset $H \subset T_{\mathfrak{S}}(E, F)$ is precompact for the $\mathfrak{S}$-topology, if and only if:*

(1) *$H(x)$ is precompact for all $x \in E$, and*

(2) *$H|S$ has equal variation for all $S \in \mathfrak{S}$.*

Next we examine conditions under which a set of operators is relatively compact. We assume that E and F are locally convex Hausdorff spaces and $\mathfrak{S}$ is a covering of E by bounded sets. If a set $H \subset L_{\mathfrak{S}}(E, F)$ is relatively compact, it is precompact, and thus condition (2) above is valid. Condition (1) must be replaced by the stronger condition:

(1′) $H(x)$ is relatively compact for all $x \in E$,

which results from the continuity of the mapping: $f \to f(x)$ from $L_{\mathfrak{S}}(E, F)$ into F for all $x \in E$. These conditions are also sufficient for the relative compactness of H if it is equicontinuous, as will presently be proved.

Theorem 2.7. *An equicontinuous subset $H \subset T_{\mathfrak{S}}(E, F)$ is relatively compact in $L_{\mathfrak{S}}(E, F)$, if (and only if)*

(1′) *$H(x)$ is relatively compact for all $x \in E$,*

(2) *$H|S$ has equal variation for all $S \in \mathfrak{S}$.*

Proof. Supposing conditions (1′) and (2) it suffices, in view of theorem 2.6, to show that the closure $\bar{H}$ is a complete subset of $L_{\mathfrak{S}}(E, F)$.

Let $\mathfrak{F}$ be a Cauchy filter on $\bar{H}$. Then for any $x \in E$ the set

$$\mathfrak{F}(x) = \{\Phi(x) \mid \Phi \in \mathfrak{F}\}$$

is a basis of a Cauchy filter on $\bar{H}(x) \subset \overline{H(x)}$, hence converges to an element $u_1(x)$ in $\overline{H(x)}$, as the latter set is complete by condition (1′). In this way we obtain an element u_1 of the function space F^E, which belongs to the pointwise closure of $\bar{H}$. Since $\bar{H}$ is equicontinuous, its pointwise closure is included in $L(E, F)$ (c.f. [2] proposition 4, ch. III, § 3, N° 5, p. 23) and thus $u_1 \in L(E, F)$. To show that $\mathfrak{F}$ converges to u_1 for the $\mathfrak{S}$-topology, let $S \in \mathfrak{S}$ and let V be an arbitrary closed balanced neighbourhood in F. As $\mathfrak{F}$ is a Cauchy filter for the $\mathfrak{S}$-topology, there exists $\Phi \in \mathfrak{F}$ such that

$$u(x) - v(x) \in V \quad \text{for all} \quad u \in \Phi\,,\ v \in \Phi\,,\ x \in S\,.$$

Since $u_1(x)$ is a cluster point of the filter basis $\mathfrak{F}(x)$ for any x, and V is closed and balanced, it follows that

$$u(x) - u_1(x) \in V \quad \text{for all} \quad u \in \Phi \quad \text{and} \quad x \in S\,,$$

in other words, $\mathfrak{F}$ converges to u_1 for the $\mathfrak{S}$-topology. $\square$

3. Product Hom(f,g) of precompact operators

Let E_1, E_2, E_3 and E_4 be topological vector spaces over the same field. For $f \in L(E_1, E_2)$ and $g \in L(E_3, E_4)$ denote by Hom(f, g) the mapping: $u \to g\,u\,f$ from $L(E_2, E_3)$ into $L(E_1, E_4)$. Thus Hom$(f,g)(u)$ is the composed mapping:

$$E_1 \xrightarrow{f} E_2 \xrightarrow{u} E_3 \xrightarrow{g} E_4\,.$$

From now on we shall assume that the spaces $E, F, E_1, F_1,$ etc., are locally convex Hausdorff spaces. We shall also make the convention that whenever an $\mathfrak{S}$-topology is considered, $\mathfrak{S}$ will be assumed to be a covering of the space in question by bounded sets and directed by inclusion. To begin with we shall state some auxiliary results.

Lemma 3.1. *Let* $g \in L(E_3, E_4)$, $f \in L(E_1, E_2)$ *with* $f(S_1) \in \mathfrak{S}_2$ *for all* $S_1 \in \mathfrak{S}_1$, *where* $\mathfrak{S}_i$ *is a family defining an* $\mathfrak{S}_i$*-topology for* E_i, $i = 1, 2$. *Then*

$$\mathrm{Hom}(f,g) : L_{\mathfrak{S}_2}(E_2, E_3) \to L_{\mathfrak{S}_1}(E_1, E_4)$$

is a continuous linear mapping.

Proof. Given a neighbourhood $N(S_1, V_4)$ in $L_{\mathfrak{S}_1}(E_1, E_4)$, choose a neighbourhood V_3 in E_3 such that $g(V_3) \subset V_4$. The set $S_2 = f(S_1)$ belongs to $\mathfrak{S}_2$ by assumption. Now, if $u \in N(S_2, V_2)$, then $\mathrm{Hom}(f, g)(u) = g\,u\,f \in N(S_1, V_4)$, and the continuity is established. $\square$

It follows from the above lemma that if the spaces $L(E_2, E_3)$ and $L(E_1, E_4)$ are both equipped with the topology of pointwise, compact, precompact or bounded convergence, then $\mathrm{Hom}(f, g)$ is continuous for continuous f and g.

We shall state another lemma whose proof is, if possible, even more straightforward than that of lemma 3.1, and will therefore be omitted.

Lemma 3.2. *If $f \in L(E_1, E_2)$ and $g \in L(E_3, E_4)$, the mapping $\mathrm{Hom}(f, g)$ maps equicontinuous sets of $L(E_2, E_3)$ into equicontinuous sets of $L(E_1, E_4)$.*

We now turn to our main theorem.

Theorem 3.3. *Suppose $f \in L(E_1, E_2)$, $g \in L(E_3, E_4)$, $f \neq 0$, $g \neq 0$. Then the mapping $\mathrm{Hom}(f, g)$ maps the equicontinuous subsets of $L(E_2, E_3)$ into precompact subsets of $L_{\mathfrak{S}_1}(E_1, E_4)$, if and only if f is $\mathfrak{S}_1$-precompact and g is precompact.*

Proof. 1° Suppose first that f is $\mathfrak{S}_1$-precompact and g is precompact. Let $B \subset L(E_2, E_3)$ be equicontinuous. We have to show that the set $H = \mathrm{Hom}(f, g)\,(B)$ is a precompact subset of $L_{\mathfrak{S}_1}(E_1, E_4)$. The mappings $g\,u\,f$ are $\mathfrak{S}_1$-precompact for all u in $L(E_2, E_3)$, because g and u are continuous and f is $\mathfrak{S}_1$-precompact. Thus $H \subset T_{\mathfrak{S}_1}(E_1, E_4)$ and it remains for us to check the validity of conditions (1) and (2) in theorem 2.6.

(1): Let $x \in E_1$. As B is equicontinuous, it is pointwise bounded, and thus the set $\{(u\,f)\,(x) \mid u \in B\}$ is bounded in E_3. Now, g is supposed to be precompact, whence

$$H(x) = \{(g\,u\,f)\,(x) \mid u \in B\}$$

is precompact.

(2): Let $S_1 \in \mathfrak{S}_1$. We have to show that $H \mid S_1$ has equal variation. Let V_4 be a neighbourhood in E_4. As g is continuous, there is a neighbourhood $V_3 \subset E_3$ such that $g(V_3) \subset V_4$. The equicontinuity of B implies that the set

$$V_2 = \bigcap_{u \in B} u^{-1}(V_3)$$

is a neighbourhood in E_2. As f is $\mathfrak{S}_1$-precompact, there is a finite covering $\{A_i\}_{i \in I}$ of S_1 such that

$$f(x) - f(y) \in V_2, \quad \text{as soon as} \quad x, y \in A_i\,,\ i \in I\,.$$

If $x, y \in A_i$, $i \in I$, then, by construction of V_2,

$$(u\,f)\,(x) - (u\,f)\,(y) \in V_3 \quad \text{for all} \quad u \in B\,,$$

which implies that

$$(g\,u\,f)\,(x) - (g\,u\,f)\,(y) \in V_4 \quad \text{for all} \quad u \in B\,.$$

Thus $H|S_1$ has equal variation, and we have completed part 1°.

2° Suppose conversely that Hom(f,g) maps equicontinuous sets into precompact sets. To prove the precompactness of g, let B_3 be a bounded subset of E_3. We have to establish the precompactness of the image $g(B_3)$.

As E_2 is Hausdorff and $f \neq 0$, there is an element $x_1 \in E_1$ and a continuous seminorm p_2 of E_2 such that for $y = f(x_1)$ we have $p_2(y) = \alpha > 0$. According to the Hahn-Banach theorem ([8], p. 29, corollary 2), we can define $h \in E_2'$ as follows:

$$\begin{cases} h(y) = \dfrac{1}{\alpha}\, p_2(y) = 1 \\[2ex] |h(x_2)| \leq \dfrac{1}{\alpha}\, p_2(x_2) \quad \text{for all} \quad x_2 \in E_2 \,. \end{cases}$$

For each $z \in B_3$ define a linear mapping $u_z \in L(E_2, E_3)$ by $u_z(x_2) = h(x_2)z$. Next we prove that the set

$$K = \{u_z \mid z \in B_3\} \subset L(E_2, E_3)$$

is equicontinuous. Therefore let V_3 be a neighbourhood in E_3. As B_3 is bounded, there is $\lambda > 0$ such that $\mu B_3 \subset V_3$ for $|\mu| \leq \lambda$. The set

$$V_2 = \{x_2 \in E_2 \mid p_2(x_2) \leq \alpha\lambda\}$$

is a neighbourhood in E_2. For each $x_2 \in V_2$ and $z \in B_3$ we have:

$$u_z(x_2) = h(x_2)\, z \in h(x_2)\, B_3 \subset V_3 \,,$$

since

$$|h(x_2)| \leq \frac{1}{\alpha}\, p_2(x_2) \leq \lambda \,.$$

Having thus proved the equicontinuity of K we know by assumption that the set $H = \text{Hom}(f, g)\,(K)$ is precompact in $L_{\mathfrak{S}_1}(E_1, E_4)$. Moreover its elements, having one-dimensional ranges, are $\mathfrak{S}_1$-precompact. By theorem 2.6 we conclude that the set

$$H(x_1) = \{(g\, u_z f)\,(x_1) \mid z \in B_3\} = \{(g\, u_z)\,(y) \mid z \in B_3\} = \{g(z) \mid z \in B_3\} = g(B_3)$$

is precompact, and thus g is precompact.

Next we have to prove the $\mathfrak{S}_1$-precompactness of f. Choose $z \in E_3$ and continuous seminorms p_3 of E_3 and p_4 of E_4 such that $p_3(z) > 0$ and $p_4(g(z)) > 0$. Denote

$$\beta = p_4(g(z)) \,/\, p_3(z) \,.$$

Let $S_1 \in \mathfrak{S}_1$ and let V_2 be an arbitrary neighbourhood in E_2. Then

V_2 includes a set of the form $\{x_2 \mid p_2(x_2) \leq \varepsilon\}$ for some continuous seminorm p_2 and some $\varepsilon > 0$.

For each pair $(x, y) \in S_1 \times S_1$ define $h_{xy} \in E_2'$ as follows:

$$\begin{cases} h_{xy}(f(x) - f(y)) = \dfrac{p_2(f(x) - f(y))}{p_3(z)} \\ |h_{xy}(x_2)| \leq \dfrac{p_2(x_2)}{p_3(z)} \text{ for all } x_2 \in E_2 \end{cases}$$

(This definition is again justified by the Hahn-Banach theorem.) Then define $u_{xy} \in L(E_2, E_3)$ by

$$u_{xy}(x_2) = h_{xy}(x_2)z .$$

The set $K = \{u_{xy} \mid x, y \in S_1\}$ is directly seen to be an equicontinuous subset of $L(E_2, E_3)$, which implies by assumption that $H = \mathrm{Hom}(f, g)\,(K)$ is precompact. Moreover the elements of H, being operators of finite rank, are $\mathfrak{S}_1$-precompact. By condition (2) of theorem 2.6 the set $H|S_1$ has equal variation. Thus there is a finite covering of S_1 by sets A_i, $i \in I$ such that

$p_4((g\,u\,f)\,(x) - (g\,u\,f)\,(y)) \leq \beta\varepsilon$ for all $u \in K$, whenever $x, y \in A_i$, $i \in I$.

Now let $x, y \in A_i$, $i \in I$. As

$$u_{xy}(f(x) - f(y)) = \frac{p_2(f(x) - f(y))\, z}{p_3(z)} ,$$

we have:

$$\beta\,\varepsilon \geq p_4\,(g\,u_{xy}\,(f(x) - f(y))) = p_2\,(f(x) - f(y))\,\frac{p_4\,(g(z))}{p_3(z)} =$$
$$= \beta\,p_2\,(f(x) - f(y)) .$$

Hence $p_2\,(f(x) - f(y)) \leq \varepsilon$, which implies that $f(x) - f(y) \in V_2$. We have thus been able to construct for each neighbourhood V_2 in E_2 a finite covering $\{A_i\}_{i \in I}$ of S_1 such that $f(x) - f(y) \in V_2$ as soon as $x, y \in A_i$, $i \in I$. Since S_1 was an arbitrary element of $\mathfrak{S}_1$, the $\mathfrak{S}_1$-precompactness of f has been established. $\square$

Remark. It suffices for the »only if» part of the above theorem to assume that the restriction of $\mathrm{Hom}(f, g)$ to operators of finite rank maps equicontinuous sets into precompact sets. This is the case because the equicontinuous sets, both denoted by K in part 2° of the proof, consist of operators with one-dimensional ranges.

In the above theorem the topology of $L(E_1, E_4)$ and the condition for f are in a simple relationship with each other. In the next two theorems we consider the »strongest» and the »weakest» condition for f, and the corresponding topologies. The proof of the following theorem is immediate.

Theorem 3.4. *Let f and g be as in theorem 3.3. The mapping* $\mathrm{Hom}(f, g)$ *maps the equicontinuous sets of $L(E_2, E_3)$ into precompact sets of $L_b(E_1, E_4)$, if and only if f and g are precompact operators.*

If the topology of bounded convergence is replaced by that of precompact convergence, the »weakest» condition for f, which in fact means no condition at all, is obtained.

Theorem 3.5. *Let f and g be as in theorem 3.3. The mapping* $\mathrm{Hom}(f, g)$ *maps equicontinuous sets of $L(E_2, E_3)$ into precompact sets of $L_c(E_1, E_4)$, if and only if g is a precompact operator.*

Proof. Since f is supposed to be continuous, it is $\mathfrak{S}_1$-precompact with respect to the family $\mathfrak{S}_1$ of all precompact subsets of E_1. $\square$

Next we study conditions under which $\mathrm{Hom}(f, g)$ is a precompact operator from $L_{\mathfrak{S}_2}(E_2, E_3)$ into $L_{\mathfrak{S}_1}(E_1, E_4)$. Necessary conditions are, according to theorem 3.3, that f be $\mathfrak{S}_1$-precompact and g be precompact, since the equicontinuous sets are bounded in $L_{\mathfrak{S}_2}(E_2, E_3)$ for any $\mathfrak{S}_2$. For these conditions to be sufficient, an extra assumption on E_2 will be needed. As we have included continuity in the definition of a precompact operator, we also have to add some assumptions about the family $\mathfrak{S}_2$ (unless the $\mathfrak{S}_1$-topology is that of pointwise convergence).

Theorem 3.6. *Let $f \in L(E_1, E_2)$, $g \in L(E_3, E_4)$, $f \neq 0$, $g \neq 0$. Suppose further that E_2 is an $\mathfrak{S}_2$-barrelled space, where $\mathfrak{S}_2$ is a family of bounded subsets of E_2 containing all precompact sets. Then*

$$\mathrm{Hom}(f, g) : L_{\mathfrak{S}_2}(E_2, E_3) \to L_{\mathfrak{S}_1}(E_1, E_4)$$

is a precompact operator, if (and only if) f is $\mathfrak{S}_1$-precompact and g is precompact.

Proof. First of all, $\mathrm{Hom}(f, g)$ is continuous by lemma 3.1, as $f(S_1) \in \mathfrak{S}_2$ for all $S_1 \in \mathfrak{S}_1$. The bounded subsets of $L_{\mathfrak{S}_2}(E_2, E_3)$ are equicontinuous in view of the $\mathfrak{S}_2$-barrelledness of E_2 (c.f. theorem 1.1), and by theorem 3.3 are thus mapped into precompact subsets of $L_{\mathfrak{S}_1}(E_1, E_4)$. $\square$

If in the above theorem we take for $\mathfrak{S}_i$ the family of all bounded subsets of E_i, $i = 1,2$ and assume E_2 to be infrabarrelled, we see that for non-zero f and g

$$\mathrm{Hom}(f, g) : L_b\,(E_2, E_3) \to L_b\,(E_1, E_4)$$

is a precompact operator, if and only if the same is true of f and g. The corresponding result for normed spaces proved by Vala in [12] follows as a special case from this observation, because normed spaces are infrabarrelled.

Remark. If the space $L(E_1, E_4)$ is equipped with the topology of pointwise convergence, no barrelledness assumption will be needed for the precompactness of $\mathrm{Hom}(f, g)$ provided that g is precompact. Indeed, condition (1) in the proof of the first part of theorem 3.3 makes use only of the pointwise boundedness of the given set $B \subset L(E_2, E_3)$, and condition (2) is trivially satisfied for the family $\mathfrak{S}_1$ of finite subsets of E_1. Thus

$$\mathrm{Hom}(f, g) : L_s(E_2, E_3) \to L_s(E_1, E_4)$$

is precompact, if and only if g is precompact. (The condition concerning $\mathfrak{S}_2$ in theorem 3.6 is immaterial here, since the continuity of $\mathrm{Hom}(f, g)$ in this case follows immediately from lemma 3.1.)

Instead of precompactness we can also study the relative compactness of a set of functions using theorem 2.7 instead of 2.6 in the proof of theorem 3.3. Suppose first that the product $\mathrm{Hom}(f,g)$ of two non-zero operators f and g maps the equicontinuous subsets of $L(E_2, E_3)$ into compact subsets of $L_{\mathfrak{S}_1}(E_1, E_4)$. In part 2° of the proof of theorem 3.3 we first show that g is precompact by proving that the set $H = \mathrm{Hom}(f, g)\,(K)$ is precompact, where K is an equicontinuous set constructed in the proof. This time our assumption tells us that H is relatively compact, which, according to the remark preceding theorem 2.7, implies that $H(x_1)$ is relatively compact. Now, $H(x_1) = g(B_3)$, where B_3 is a preassigned bounded set in E_3, hence g is a compact operator. As for f, we cannot conclude more than in theorem 3.3. In fact the compactness of g together with the $\mathfrak{S}_1$-precompactness of f suffice to make $\mathrm{Hom}(f, g)$ map equicontinuous sets into compact sets.

Theorem 3.7. *Let* $f \in L(E_1, E_2)$, $g \in L(E_3, E_4)$, $f \neq 0$, $g \neq 0$. *Then* $\mathrm{Hom}(f, g)$ *maps the equicontinuous subsets of* $L(E_2, E_3)$ *onto relatively compact subsets of* $L_{\mathfrak{S}_1}(E_1, E_4)$, *if and only if* f *is* $\mathfrak{S}_1$*-precompact and* g *is compact.*

Proof. Suppose f is $\mathfrak{S}_1$-precompact and g is compact. If $B \subset L(E_2, E_3)$ is equicontinuous, the set $H = \mathrm{Hom}(f,g)\,(B)$ is also equicontinuous by lemma 3.2. The compactness of g implies that the set $H(x)$ in condition (1) of the first part of the proof of theorem 3.3 is relatively compact. Thus condition (1′) of theorem 2.7 is valid. Since condition (2) remains unaltered, the conclusion follows from theorem 2.7. The proof is thus complete, the »only if» part being treated before the statement of the theorem. $\square$

Remark 3.8. *Theorems 3.4, 3.5 and 3.6 remain valid, if the word »precompact» is replaced by »compact», where it refers to either* g *or* $\mathrm{Hom}(f, g)$.

Let us state explicitly one result of this type:

Theorem 3.9. *Let* $f \in L(E_1, E_2), g \in L(E_3, E_4), f \neq 0, g \neq 0$ *and suppose that* E_2 *is infrabarrelled. Then*

$$\mathrm{Hom}(f, g) : L_b\,(E_2, E_3) \to L_b\,(E_1, E_4)$$

is a compact operator, if and only if f *is precompact and* g *is compact.*

The above theorem is especially valid for normed spaces, because they are infrabarrelled.

Next we turn our attention to a situation, in which the spaces E_3 and E_4 above equal the scalar field $\mathbf{K}$ and g is the identity transformation of $\mathbf{K}$. If $f \in L(E,F)$, the mapping $\mathrm{Hom}(f, id)\,(u)$ for $u \in F'$ is then the composed mapping:

$$E \xrightarrow{f} F \xrightarrow{u} \mathbf{K} \xrightarrow{id} \mathbf{K}\,.$$

Thus $\mathrm{Hom}(f, id)$ is the mapping: $u \to u\,f$ from F' into E', in other words: $\mathrm{Hom}(f, id) = {}^tf =$ the transpose of f. Consequently we get theorems of »Schauder-type», i.e. theorems on the compactness of the transpose of a compact operator, as special cases of the theory presented above.

Theorem 3.10. *Let* $f \in L(E, F)$ *and equip the dual* E' *with an* $\mathfrak{S}$-*topology. Then* f *is* $\mathfrak{S}$-*precompact, if and only if the transpose* ${}^tf$ *maps the equicontinuous sets of* F' *into (pre)compact sets of* $E'_{\mathfrak{S}}$.

Proof. As the identity of $\mathbf{K}$ is compact, it follows from theorem 3.7 that ${}^tf$ maps the equicontinuous sets of F' into compact sets of $E'_{\mathfrak{S}}$, provided that f is $\mathfrak{S}$-precompact. As for the converse, it suffices, by theorem 3.3, for the $\mathfrak{S}$-precompactness of f to assume that ${}^tf$ maps the equicontinuous sets of F' into precompact sets of $E'_{\mathfrak{S}}$. $\square$

For another proof of the above theorem see [8] ch. VIII, lemma 6, p. 152.

The following theorem, which is a direct consequence of theorem 3.10, could be called the Schauder theorem for locally convex spaces.

Theorem 3.11. *An operator $f \in L(E,F)$ is precompact, if and only if its transpose maps the equicontinuous subsets of F' into compact subsets of E'_b.*

Corollary. *If in the above theorem F is supposed to be infrabarrelled, then f is precompact, if and only if its transpose is compact with respect to the strong topologies.*

The classical Schauder theorem for normed spaces immediately follows from this corollary.

Remark 3.12. It follows from the remark after theorem 3.6 that the transpose of an operator $f \in L(E,F)$ is precompact if the duals are equipped with the weak topologies $\sigma(F',F)$ and $\sigma(E',E)$.

To show that the extra assumption made about F in the above corollary and about E_2 in theorem 3.6 cannot in general be avoided, we give a simple example of a precompact operator whose transpose is not precompact for the strong topologies.

Example 3.13. Let E be an infinite-dimensional normed space and F the same space equipped with the weak topology $\sigma(E,E')$. The mapping id: $E_s \to E_s$ is precompact, as follows from the above remark if we interpret this mapping as the transpose of $id : E'_s \to E'_s$. Thus $id : E \to F$ is precompact.

On the other hand

$$^t id = id : F'_b \to E'_b$$

is not precompact, for otherwise there would exist a precompact neighbourhood in the normed space $E'_b = F'_b$, which would contradict the assumption on the infinite-dimensionality of E (c.f. [6] theorem 2.10.3, p. 147). (This also proves that F cannot be infrabarrelled.)

4. Tensorproduct of precompact operators

The mapping Hom, which to each pair (f,g) in $L(E_1,E_2) \times L(E_3,E_4)$ assigns the element $\mathrm{Hom}(f,g)$ in $L(L_b(E_2,E_3),\ L_b(E_1,E_4))$ defined in the previous section, is a bilinear mapping. It is not difficult to see that the subspace of $L(L_b(E_2,E_3),\ L_b(E_1,E_4))$ spanned by the image of the bilinear mapping Hom can be interpreted as a tensorproduct of $L(E_1, E_2)$ and $L(E_3, E_4)$. However, it is perhaps more natural to interpret the mapping $(f, g) \to \mathrm{Hom}({}^tf, g)$ as a tensorproduct of f and g, because this interpretation coincides with the conventional concept of the tensorproduct of two linear mappings f and g, as will be shown in the proof of theorem 4.5. This interpretation makes it possible to use the techniques developed in the preceding section for solving the problem of the precompactness of the tensorproduct of precompact operators.

First we need some auxiliary results. The symbols E, F, E_1, F_1, etc., stand for locally convex Hausdorff spaces as before.

Lemma 4.1. *The space $A(E'_\tau, F)$ of continuous linear mappings of finite rank from E' equipped with the Mackey topology $\tau(E', E)$ into F is a tensorproduct of E and F with respect to the bilinear mapping: $(x,y) \to (x' \to \langle x, x' \rangle y)$.*

Proof. The bilinear mapping mentioned above induces a linear mapping from $E \otimes F$ into $A(E'_\tau, F)$, which to each $x \otimes y \in E \otimes F$ assigns the continuous linear mapping: $x' \to \langle x, x' \rangle y$. To prove that the range of this mapping is all of $A(E'_\tau, F)$ we have to be able to represent every element $u \in A(E'_\tau, F)$ in the form

$$u(x') = \sum_{j=1}^{n} \langle x_j , x' \rangle y_j , \quad x' \in E' .$$

Given $u \in A(E'_\tau, F)$, choose a basis $\{y_1, \ldots, y_n\}$ of the range of u. Then u has the representation

$$u(x') = \sum_{i=1}^{n} \alpha_i(x')\, y_i , \quad x' \in E' .$$

By the Hahn-Banach theorem there exist elements $y'_1, \ldots, y'_n$ in F' such that $\langle y_i, y'_j \rangle = \delta_{ij}$. Thus

$$\alpha_j(x') = \langle u\,(x')\,, y'_j \rangle = \langle x', {}^tu(y'_j) \rangle ,$$

where the transpose ${}^tu$ is a mapping from F' into the dual of E'_τ, which can be identified with E, since $\tau(E', E)$ is compatible with the duality between E' and E. Setting $x_j = {}^tu(y'_j) \in E$ we have the desired representation.

The proof of the injectiveness of the linear mapping from $E \otimes F$ into $A(E'_\tau, F)$ is quite straightforward, the only non-trivial argument being the fact that E' separates the points of E. The details are omitted. $\square$

Remark 4.2. Instead of $A(E'_\tau, F)$ we could have taken $A(F'_\tau, E)$ and the bilinear mapping $(x, y) \to (y' \to < y, y' > x)$.

Remark 4.3. Any topology compatible with the duality between E' and E could have been used instead of $\tau(E', E)$, which is the finest of them.

The following fact from the theory of duality will be needed:

Lemma 4.4. *If T is a barrel in E' for the topology $\tau(E', E)$, it is a neighbourhood for the strong topology $\beta(E', E)$.*

Proof. Let T be a barrel in E'_τ. As T is a convex set, closed for the topology $\tau(E', E)$, it is also closed for the weak topology $\sigma(E', E)$. Thus T is a balanced, convex, $\sigma(E', E)$-closed subset of E', which implies by the theorem of bipolars ([6] theorem 3.3.1, p. 192) that T equals its bipolar $T^{\circ\circ}$. Since T is absorbing, its polar T° is weakly bounded, thus $T = (T^\circ)^\circ$ is a neighbourhood for $\beta(E', E)$. $\square$

We are now going to prove that the tensorproduct of two non-zero operators f and g is precompact for the ε-topologies of the tensorproduct spaces in question, if and only if f and g are both precompact. We recall that the ε-topology for the tensorproduct $E \otimes F$ is the topology induced by $L_e(E'_\tau, F)$ on $E \otimes F$ regarded as a subspace of the former (c.f. [10] exposé n° 7, II or [11] p. 434 and 429). Recall that the subscript e denotes the topology of uniform convergence on the equicontinuous subsets of E'.

Theorem 4.5. *The tensorproduct*

$$f_1 \otimes f_2 : E_1 \otimes_\varepsilon E_2 \to F_1 \otimes_\varepsilon F_2$$

of operators $f_i \in L(E_i, F_i)$, $f_i \neq 0$, $i = 1,2$ is precompact, if and only if f_1 and f_2 are both precompact.

Proof. 1° In view of lemma 4.1 we are allowed to make the identifications: $E_1 \otimes_\varepsilon E_2 = A_e(E'_{1\tau}, E_2)$ and $F_1 \otimes_\varepsilon F_2 = A_e(F'_{1\tau}, F_2)$, where the tensor $x_1 \otimes x_2 \in E_1 \otimes E_2$ is to be identified with the mapping: $x' \to < x_1, x' > x_2$ and $y_1 \otimes y_2$ with: $y' \to < y_1, y' > y_2$. Thus we have

$$(f_1 \otimes f_2)\,(x_1 \otimes x_2) = f_1(x_1) \otimes f_2(x_2) = (y' \to < f_1\,(x_1)\,,\, y' > f_2\,(x_2))$$
$$= (y' \to f_2\,(< x_1\,,\, {}^t\!f_1\,(y') > x_2)) = f_2 \circ (x_1 \otimes x_2) \circ {}^t\!f_1\,,$$

which implies by linearity that

$$(f_1 \otimes f_2)\,(u) = \mathrm{Hom}({}^t\!f_1\,, f_2)\,(u)\,,\; u \in A(E'_{1\tau}, E_2)\,.$$

The mapping ${}^t f_1 = \mathrm{Hom}(f_1, id_{\mathbf{K}})$ maps the equicontinuous sets of F_1' into equicontinuous sets of E_1' by lemma 3.2, which by lemma 3.1 implies that $f_1 \otimes f_2$ is continuous for the ε-topologies.

Now, f_1 is a precompact operator, hence its transpose ${}^t f_1$ maps the equicontinuous sets of F_1' into precompact sets of E_{1b}' according to theorem 3.11. Moreover, ${}^t f_1 : F_{1b}' \to E_{1b}'$ is continuous by lemma 3.1. As f_2 is assumed to be precompact, it follows from theorem 3.3 that $\mathrm{Hom}({}^t f_1, f_2)$ maps the equicontinuous subsets of $L(E_{1b}', E_2)$ into precompact subsets of $L_e(F_{1b}', F_2)$.

Next we prove that the bounded subsets of $L_e(E_{1\tau}', E_2)$ are equicontinuous in $L(E_{1b}', E_2)$. So let B be a bounded subset of $L_e(E_{1\tau}', E_2)$ and let V be a balanced, convex and closed neighbourhood in E_2. Then the set

$$V' = \bigcap_{u \in B} u^{-1}(V)$$

is balanced, convex and closed. Moreover it absorbs each equicontinuous set of E_1' (c.f. section 1, condition (b1)); hence it is in particular absorbing. Thus V' is a barrel in $E_{1\tau}'$, which implies by lemma 4.4 that it is a neighbourhood in E_{1b}'. This in turn implies that B is an equicontinuous subset of $L(E_{1b}', E_2)$ (section 1, condition (e1)).

We have thus proved that $\mathrm{Hom}({}^t f_1, f_2)$ is a precompact operator from $L_e(E_{1\tau}', E_2)$ into $L_e(F_{1b}', F_2)$, which implies that its restriction to $A_e(E_{1\tau}', E_2)$, i.e. the operator $f_1 \otimes f_2$, is precompact.

2° As we saw above, the mapping $(f_1 \otimes f_2)(u)$ for $u \in A(E_{1\tau}', E_2)$ is composed according to the diagram

$$F_1' \xrightarrow{{}^t f_1} E_1' \xrightarrow{u} E_2 \xrightarrow{f_2} F_2 .$$

It follows from [6], propositions 3.12.3 and 3.12.5, that ${}^t f_1 \in L(F_{1\tau}', E_{1\tau}')$.

The mapping $f_1 \otimes f_2 = \mathrm{Hom}({}^t f_1, f_2)$ maps the bounded sets of $A_e(E_{1\tau}', E_2)$ into precompact sets of $L_e(F_{1\tau}', F_2)$ by assumption. Thus in particular the equicontinuous subsets of the former are mapped into precompact subsets of the latter. Hence it follows from the remark after theorem 3.3 that f_2 is precompact.

To prove the precompactness of f_1 identify $E_1 \otimes_\varepsilon E_2$ with $A_e(E_{2\tau}', E_1)$ and $F_1 \otimes_\varepsilon F_2$ with $A_e(F_{2\tau}', F_1)$ (c.f. remark 4.2.), which means that $f_1 \otimes f_2$ is to be identified with $\mathrm{Hom}({}^t f_2, f_1)$. Then proceed exactly as above. $\square$

Using theorem 3.7 instead of 3.3 in the latter part of the proof of the foregoing theorem, it is seen that the compactness of both f_1 and f_2 is necessary for $f_1 \otimes f_2$ to be compact for the ε-topologies.

If E and F are complete, then $L(E'_\tau, F)$ is also complete according to [11], prop. 42.3, p. 430. Thus the completed ε-tensorproduct $E \hat{\otimes}_\varepsilon F$ can be identified with the closure of $A(E'_\tau, F)$ in $L(E'_\tau, F)$. Making use of this identification we get the following result:

Theorem 4.6. *If* E_1, E_2, F_1 and F_2 *are complete and if* $f_i \in L(E_i, F_i)$, $i = 1,2$ *are compact operators, then the extended tensorproduct*

$$f_1 \hat{\otimes} f_2 : E_1 \hat{\otimes}_\varepsilon E_2 \to F_1 \hat{\otimes}_\varepsilon F_2$$

is compact.

Proof. If we make the above identifications, the mapping $f_1 \hat{\otimes} f_2$ equals the restriction of $\mathrm{Hom}({}^tf_1, f_2)$ to the closure of $A_e(E'_{1\tau}, E_2)$ in $L_e(E'_{1\tau}, E_2)$. Proceeding exactly as in part 1° of the proof of theorem 4.5 we conclude that the restriction of $\mathrm{Hom}({}^tf_1, f_2)$ to $E_1 \hat{\otimes}_\varepsilon E_2$ maps the bounded sets into precompact sets of $F_1 \hat{\otimes}_\varepsilon F_2$, and thus into compact sets because of the completeness of the latter space. $\square$

Next we study conditions under which the tensorproduct of two linear mappings is precompact when the tensorproduct spaces are equipped with the projective or π-topology, i.e. the finest locally convex topology for which the canonical bilinear mapping: $(x,y) \to x \otimes y$ is continuous. Recall that the ε-topology is always coarser than the π-topology, but if E is nuclear (c.f. [4] part II § 2, n° 1, p. 34) these topologies coincide on $E \otimes F$ for any locally convex space F. These observations, together with theorem 4.5 and the well-known fact that the tensorproduct of continuous linear mappings is continuous for the π-topologies, give us the following result.

Theorem 4.7. *If the operators* $f_i \in L(E_i, F_i)$, $i = 1,2$ *are precompact and* F_1 *or* F_2 *is nuclear, the tensorproduct*

$$f_1 \otimes f_2 : E_1 \otimes_\pi E_2 \to F_1 \otimes_\pi F_2$$

is a precompact operator.

The conclusion of theorem 4.7 can also be made if instead of F_1 (or F_2) the spaces E_1 and E_2 are assumed to satisfy suitable conditions. The proof of the following lemma is quite straightforward and does not depend at all on the previous theorems.

Lemma 4.8. *The tensorproduct* $f_1 \otimes f_2$ *of two precompact operators* $f_i \in L(E_i, F_i)$, $i = 1,2$ *is precompact for the* π*-topologies if every bounded* *subset of* $E_1 \otimes_\pi E_2$ *is contained in the balanced convex hull of a set* $A_1 \otimes A_2 = \{x_1 \otimes x_2 \mid x_1 \in A_1, x_2 \in A_2\}$, *where* A_i *is bounded in* $E_i, i = 1{,}2$.

Proof. If A_i is bounded in E_i, the set $B_i = f_i(A_i) \subset F_i$, $i = 1{,}2$ is precompact by assumption. Now

$$(f_1 \otimes f_2)(A_1 \otimes A_2) = f_1(A_1) \otimes f_2(A_2) = B_1 \otimes B_2 .$$

The mapping $(y_1, y_2) \to y_1 \otimes y_2$ is continuous from $F_1 \times F_2$ into $F_1 \otimes_\pi F_2$, hence in particular hypocontinuous, which implies that its restriction to $B_1 \times B_2$ is uniformly continuous ([6] proposition 4.7.3, p. 360). Thus $B_1 \otimes B_2$ is precompact and the conclusion follows from the assumption on the bounded sets of $E_1 \otimes_\pi E_2$. $\square$

The condition on bounded sets of $E_1 \otimes_\pi E_2$ in the above lemma is called the »Problème des Topologies» by A. Grothendieck (c.f. [4] part I, p. 33). It is not in general satisfied, as can be seen by taking for E_1 the product R^{N} and for E_2 the locally convex direct sum $R^{(\mathrm{N})}$ (c.f. [4] part I, p. 34). On the other hand it follows from [4] part I, proposition 5, p. 43, that the condition is satisfied for spaces of type (DF), which means a locally convex space with a countable fundamental system of bounded sets and with the property that every strongly bounded subset of the dual, which is a countable union of equicontinuous sets, is equicontinuous (c.f. [3] definition 1, p. 63). It follows that every infrabarrelled space with a fundamental sequence of bounded sets is of type (DF). Normed spaces in particular are of type (DF). Further, it follows from [3] théorème 1, p. 61 that the strong dual of a metrisable space is a (DF)-space. It should also be noted that there are no metrisable (DF)-spaces other than normed spaces, for every metrisable space with a fundamental sequence of bounded sets is normable ([7] § 29.1.(2), p. 396).

As a consequence of lemma 4.8 we get the following:

Theorem 4.9. *If* E_1 *and* E_2 *are of type* (DF), *the tensorproduct*

$$f_1 \otimes f_2 : E_1 \otimes_\pi E_2 \to F_1 \otimes_\pi F_2$$

of precompact operators $f_i : E_i \to F_i, i = 1{,}2$ *is a precompact operator.*

In fact every bounded set of the completed π-tensorproduct of two (DF)-spaces E_1 and E_2 is contained in the balanced closed convex hull of a set $A_1 \otimes A_2$, where A_k is bounded in E_k, $k = 1{,}2$, according to [4] part I, proposition 5, p. 43. Thus we have the following:

Corollary 4.10. *If* E_1 and E_2 *are of type* (DF), *the extended tensor-product*

$$f_1 \hat{\otimes} f_2 \; : \; E_1 \hat{\otimes}_\pi E_2 \to F_1 \hat{\otimes}_\pi F_2$$

of precompact operators $f_i : E_i \to F_i$, $i = 1{,}2$ *is a compact operator.*

Next we are going to prove that the converse of theorem 4.9 holds without any extra assumptions on the spaces. For normed spaces it follows directly from a result of J. R. Holub (c.f. [5] proposition 3.1, p. 4). We are going to present the proof of Holub modified to locally convex spaces.

Theorem 4.11. *If* $f_i \in L(E_i, F_i)$, $f_i \neq 0$, $i = 1{,}2$ *and if*

$$f_1 \otimes f_2 \; : \; E_1 \otimes_\pi E_2 \to F_1 \otimes_\pi F_2$$

is precompact, then f_1 *and* f_2 *are both precompact operators.*

Proof. Choose $x_0 \in E_2$ and a continuous seminorm q_2 in F_2 such that

$$q_2 \left(f_2(x_0)\right) = 1 ,$$

and an element $y_2' \in F_2'$ such that

$$< f_2 (x_0) \, , y_2' > \; = q_2 \left(f_2(x_0)\right) = 1 \, .$$

Define a continuous linear mapping

$$P : E_1 \to E_1 \otimes_\pi E_2 \quad \text{by} \quad Px = x \otimes x_0 \, .$$

The mapping $(y_1, y_2) \to \; < y_2, y_2' > y_1$ is a continuous bilinear mapping from $F_1 \times F_2$ into F_1. The corresponding linear mapping

$$Q : F_1 \otimes_\pi F_2 \to F_1 \, ; \, y_1 \otimes y_2 \to \; < y_2, y_2' > y_1$$

is continuous according to the definition of the π-topology. Now,

$$\begin{aligned} (Q \circ (f_1 \otimes f_2) \circ P) \, (x) &= Q \, (f_1 \, (x) \otimes f_2 \, (x_0)) = \\ &= \; < f_2 \, (x_0), y_2' > f_1(x) = f_1(x) \quad \text{for} \quad x \in E_1 \, , \end{aligned}$$

i.e. $f_1 = Q \circ (f_1 \otimes f_2) \circ P$, where Q and P are continuous and $f_1 \otimes f_2$ is precompact. Thus f_1 is precompact. The precompactness of f_2 follows from reasons of symmetry. □

It is also readily seen from the above proof that the compactness of $f_1 \otimes f_2$ for the π-topologies implies the compactness of f_1 and f_2.

Remark 4.12. If we compare the above proof with that of the first half of part 2° of theorem 3.3, we see that the method is basically the same in both cases. Indeed, if we denote the mapping $z \to u_z$ by P and the mapping $u \to u(x_1)$ by Q in the proof of theorem 3.3, then g will be composed according to the diagram

$$E_3 \xrightarrow{P} L(E_2, E_3) \xrightarrow{\mathrm{Hom}(f,g)} L_{\mathfrak{S}_1}(E_1, E_4) \xrightarrow{Q} E_4$$

In fact, in the proof of theorem 3.3 we have shown that P sends bounded sets into equicontinuous sets and used this together with the continuity of Q to prove the precompactness of g.

5. Applications to Montel-type spaces

In this section we shall consider some classes of locally convex spaces with the following property in common: every bounded subset is precompact. We shall study some permanence properties of these spaces using the results of sections 3 and 4.

A locally convex Hausdorff space will be called *pre-Schwartz* if its bounded subsets are precompact. This class of spaces includes the following important subclasses: *semi-Montel*, *Montel*, *Schwartz* and *nuclear* spaces. Recall that a semi-Montel space is one whose bounded subsets are relatively compact and that a Montel space is an infrabarrelled semi-Montel space. In fact a Montel space, being quasicomplete and infrabarrelled, is barrelled.

We make the convention that in this section all the spaces under consideration will contain non-zero vectors. The results concerning the spaces of linear operators are consequences of the following special case of theorems 3.3 and 3.7.

Theorem 5.1. *If E and F are locally convex Hausdorff spaces and $\mathfrak{S}$ is a covering of E by bounded sets, then the identity*

$$id : L(E, F) \to L_{\mathfrak{S}}(E, F)$$

maps equicontinuous sets into precompact (resp. compact) sets, if and only if $id_E : E \to E$ is *$\mathfrak{S}$-precompact and $id_F : F \to F$ is precompact (resp. compact).*

Proof. Set $E_1 = E_2 = E$, $E_3 = E_4 = F$, $f = id_E$, $g = id_F$ in theorem 3.3 (resp. 3.7). $\square$

An immediate consequence is the following generalization of the Alaoglu-Bourbaki theorem (c.f. [6] theorem 3.4.1, p. 201).

Theorem 5.2. *The equicontinuous subsets of $L(E,F)$ are relatively com-*

pact for an $\mathfrak{S}$-topology, if and only if $\mathfrak{S}$ is a family of precompact subsets of E and F is a semi-Montel space.

The »strengthened» version of the Alaoglu-Bourbaki theorem (c.f. [6] ex. 3.10.7 (a), p. 242), which states that the equicontinuous subsets of the dual of a locally convex Hausdorff space are relatively compact for the topology of precompact convergence, follows immediately from this theorem, since the scalar field is a semi-Montel space.

Remark. The »if» part of theorem 5.2. can also be deduced by means of [2] ch. III, § 3, corollary to proposition 4 and proposition 5, p. 23. Indeed, if F is semi-Montel and H is an equicontinuous subset of $L(E,F)$, then H is pointwise bounded in particular. Thus the set $H(x)$ is bounded and hence relatively compact in F for each $x \in E$. This implies, by the corollary mentioned above, that H is relatively compact in $L_s(E,F)$. The rest follows from proposition 5, which states that on the equicontinuous subsets of $L(E,F)$ the topology of pointwise convergence coincides with that of precompact convergence.

It should be noted that we have not made use of Tihonov's theorem in the proof of theorem 5.2, nor have we used proposition 5 mentioned above.

If the space E in the preceding theorem is assumed to be pre-Schwartz, then the topology of precompact convergence coincides with that of bounded convergence. Thus we have the following result:

Corollary 5.3. *If E is a pre-Schwartz space, then the equicontinuous subsets of $L_b(E,F)$ are relatively compact, if and only if F is a semi-Montel space.*

Next we shall study conditions under which $L_{\mathfrak{S}}(E,F)$ is a semi-Montel space. Necessary conditions are, in view of theorem 5.2, that F is semi-Montel and $\mathfrak{S}$ is a family of precompact subsets of E. In particular, for $L_b(E,F)$ to be semi-Montel it is necessary for E to be pre-Schwartz and F to be semi-Montel. These conditions are also sufficient in the presence of a suitable barrelledness assumption on E, as will be shown in the following theorem (see also [4] part I, § 4, n° 1, corollaire 1, p. 99).

Theorem 5.4. *If F is semi-Montel and E is $\mathfrak{S}$-barrelled with respect to a family $\mathfrak{S}$ of precompact subsets of E, then $L_{\mathfrak{S}}(E,F)$ is semi-Montel. In particular, if E is Montel and F is semi-Montel, then $L_b(E,F)$ is semi-Montel.*

Proof. The bounded subsets of $L_{\mathfrak{S}}(E,F)$ are equicontinuous because of the $\mathfrak{S}$-barrelledness of E (c.f. theorem 1.1), hence the conclusion follows from theorem 5.2. $\square$

Remark. The condition for E to ensure that $L_b(E,F)$ is semi-Montel can of course be weakened into the form: E is pre-Schwartz and infrabarrelled.

Remark. The space $L_s(E,F)$ is pre-Schwartz for arbitrary E if F is pre-Schwartz (see the remark after theorem 3.6).

To show that in general the necessary conditions mentioned before theorem 5.4 are not alone sufficient for $L_b(E,F)$ to be semi-Montel, we can again use example 3.13. Thus, if E is an infinite-dimensional normed space, the space E_s is a pre-Schwartz space, as its identity transformation is precompact. If we choose $F = \mathbf{K}$, then F is certainly a Montel space, but $L_b(E_s, F) = E'_b$ is not even a pre-Schwartz space, as $id : E'_b \to E'_b$ is not precompact.

From theorem 5.4 it follows that the dual of a barrelled space equipped with the topology of precompact convergence (or any coarser $\mathfrak{S}$-topology) is semi-Montel and the dual of a Montel space equipped with any $\mathfrak{S}$-topology is semi-Montel. In fact the strong dual E'_b of a Montel space E is even a Montel space, because Montel spaces are reflexive, which implies that the Mackey topology and the strong topology of the dual coincide, and thus E'_b is barrelled by lemma 4.4. Thus we get the following theorem, the first part of which is the content of proposition 3.9.9. in [6].

Theorem 5.5. *If E is a Montel space, its strong dual is a Montel space. Conversely, if the strong dual of a locally convex Hausdorff space E is semi-Montel, then E is pre-Schwartz.*

Finally we turn our attention to tensorproducts. Since the identity

$$id : E \otimes F \to E \otimes F$$

equals $id_E \otimes id_F$, we have the following results:

Theorem 5.6. *The tensorproduct $E \otimes_\varepsilon F$ is a pre-Schwartz space, if and only if the same is true of both E and F.*

Proof. Theorem 4.5. $\square$

Theorem 5.7. *If E and F are complete semi-Montel spaces, their completed ε-tensorproduct $E \hat{\otimes}_\varepsilon F$ is semi-Montel.*

Proof. Theorem 4.6. $\square$

Remark. If we add the extra assumption that E should be quasinormable (c.f. [3] III.1. définition 4, p. 106), then the preceding theorem can also be deduced by means of [4] part I, § 4, n° 2, corollaire, p. 118, which states that $L_e(E'_\tau, F)$ is semi-Montel if E is a quasicomplete Schwartz space and F is a semi-Montel space. (For the definition of Schwartz spaces see [3] III.4. définition 5, p. 117.) Thus $E \hat{\otimes}_\varepsilon F$, being a closed subspace of $L_e(E'_\tau, F)$, is semi-Montel.

Theorem 5.8. *If E and F are pre-Schwartz spaces of type (DF), then $E \otimes_\pi F$ is pre-Schwartz and its completion $E \hat{\otimes}_\pi F$ is a Montel space.*

Proof. It follows from theorem 4.9 and corollary 4.10 that $E \otimes_\pi F$ is pre-Schwartz and its completion is semi-Montel. As E and F are supposed to be of type (DF), the space $E \hat{\otimes}_\pi F$ is of type (DF) according to [4] part I, § 1, n° 3, proposition 5, p. 43. Thus $E \hat{\otimes}_\pi F$, being a semi-Montel space of type (DF), is Montel according to [13] teorema 2.

Theorem 5.9. *If the π-tensorproduct of two locally convex spaces E and F is pre-Schwartz, then E and F are pre-Schwartz spaces.*

Proof. Theorem 4.11. $\square$

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