BOUNDARY MAPPINGS OF GEOMETRIC ISOMORPHISMS OF FUCHSIAN GROUPS

BY

TAPANI KUUSALO

HELSINKI 1973
SUOMALAINEN TIEDEAKATEMIA

https://doi.org/10.5186/aasfm.1973.545
Communicated 9 April 1973 by Olli Lehto
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The object of the present paper is to apply certain ergodic theoretical results of E. Hopf ([2], [3]) to the study of boundary mappings of geometric isomorphisms of Fuchsian groups.

1. An isomorphism \( j : G_1 \rightarrow G_2 \) of two Fuchsian groups acting in the unit disc \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) is said to be geometric if there exists a homeomorphism \( \Phi : D \rightarrow D \) inducing the isomorphism \( j \), i.e. if we have

\[
\Phi \circ g = j(g) \circ \Phi
\]

for all \( g \in G_1 \). If both groups \( G_1, G_2 \) are the first kind, then \( \Phi \) has a unique homeomorphic extension \( \hat{\Phi} : D \rightarrow \bar{D} \), so that also the boundary mapping \( \varphi = \hat{\Phi} |_{\partial D} \) satisfies

\[
\varphi \circ g = j(g) \circ \varphi, \quad g \in G_1.
\]

Unlike \( \Phi \), the homeomorphism \( \varphi : T \rightarrow T \) of the unit circle \( T = BdD \) is uniquely determined by the isomorphism \( j \) ([5] §3, [6] 3.B). In the following, all Fuchsian groups are supposed to be of the first kind.

Occasionally we may study Fuchsian groups which act in the upper half plane \( H \) instead of \( D \). In that case we assume that the boundary mapping \( \psi \) fixes the point \( \infty \), so that \( \psi \) will be a strictly monotone mapping \( \psi : \mathbb{R} \rightarrow \mathbb{R} \).

2. We normalize the Lebesgue measure \( \tau_1 \) on \( T \) by \( \tau_1(T) = 1 \), and the torus \( T \times T \) has the product measure \( \tau_2 = \tau_1 \times \tau_1 \).

As a homeomorphism of the unit circle a boundary mapping \( \varphi : T \rightarrow T \) has a derivative \( \varphi' \in \mathbb{C} \) a.e. on \( T \). Similarly a real-valued boundary mapping \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) which corresponds to Fuchsian groups acting in \( H \) has a finite derivative \( \psi' \in \mathbb{R} \) a.e. on \( \mathbb{R} \). Because \( \psi \) is monotone, the derivative \( \psi' \) cannot change its sign.

Since the cross ratio \([z_1, z_2, z_3, z_4]\) is preserved under Moebius transformations it follows that also the differential

\[
dz_1 \dz_2 (z_1 - z_2)^{-2} = -[z_1, z_2, z_1 + dz_1, z_2 + dz_2].
\]
remains invariant. Let now \( \varphi : T \to T \) be the boundary mapping corresponding to a geometric isomorphism \( j : G_1 \to G_2 \). The invariance of (3) implies that also the expression

\[
\chi_\varphi(z_1, z_2) = \varphi'(z_1) \varphi'(z_2) \left( \frac{\varphi(z_1) - \varphi(z_2)}{z_1 - z_2} \right)^{-2}
\]

is invariant under Moebius transformations. Thus if \( h, k \) are two Moebius transformations, we have

\[
\chi_\xi(k(z_1), k(z_2)) = \chi_\varphi(z_1, z_2)
\]

for \( \xi = h \circ \varphi \circ k^{-1} : kT \to hT \). Since \( G_1 \) and \( G_2 \) have conjugate groups acting in \( H \), we see that \( \chi_\varphi : T \times T \to \mathbb{R} \) is a non-negative measurable function. Further it follows from (2) that \( \chi_\varphi \) is automorphic with respect to \( G_1 \); that is,

\[
\chi_\varphi(gz_1, gz_2) = \chi_\varphi(z_1, z_2)
\]

for all \( g \in G_1 \).

3. The class \( O_{HB} \). Suppose that the Riemann surface \( S = D/G \) corresponding to a Fuchsian group \( G \) is of class \( O_{HB} \), i.e. \( S \) does not have non-constant bounded harmonic functions, or equivalently that there is no non-constant \( G \)-automorphic bounded harmonic function in \( D \). Using the Poisson representation we see that all \( G \)-automorphic bounded harmonic functions are constant if and only if the action of \( G \) on \( T \) is metrically transitive, i.e. if and only if a measurable \( G \)-invariant subset \( E \subset T \) has either measure \( \tau_1(E) = 0 \) or \( \tau_1(E) = 1 \).

**Theorem 1.** Let \( \varphi \) be the boundary mapping of a geometric isomorphism \( j : G_1 \to G_2 \). If one of the Riemann surfaces \( S_i = D/G_i \), \( i = 1, 2 \), is of class \( O_{HB} \), then the mapping \( \varphi \) is either absolutely continuous or completely singular.

**Proof.** Suppose that \( S_2 \) is of class \( O_{HB} \). If \( \varphi \) is not absolutely continuous, there exists a Borel set \( E \subset T \) such that \( \tau_1(E) = 0 \), \( \tau_1(\varphi(E)) > 0 \). The set \( F_1 = G_1 E \) is invariant under \( G_1 \), and \( F_2 = \varphi(F_1) = G_2 \varphi(E) \) under \( G_2 \). Now \( \tau_1(F_1) = 0 \), and \( \tau_1(F_2) = 1 \) since \( G_2 \) is metrically transitive. Thus both \( \varphi \) and \( \varphi^{-1} \) are completely singular.

4. The Hopf classification. Let \( S \) be a hyperbolic Riemann surface, \( T(S) \) the tangent manifold of \( S \), and \( \sigma_d(v, w), x \in S, v, w \in T_d(S) \), the hyperbolic metric of \( S \). Since \( S \) is a complete Riemannian manifold with respect to the hyperbolic metric, the geodesic flow \( \beta \), determined
by the Lagrangian $L(x, \dot{x}) = \sigma(x, \dot{x})$ is globally defined on $T(S)$, i.e. $\beta_t : T(S) \to T(S)$, $t \in \mathbb{R}$, is a one-parameter transformation group. The surfaces $\mathcal{E}_c \subset T(S)$ of constant energy, $L(x, v) = c$, are invariant under the geodesic flow, and since the flow $\beta_t$ is essentially similar on every $\mathcal{E}_c$, $c > 0$, we can consider only $\mathcal{E} = \mathcal{E}_1$. The geodesic flow $\beta_t$ restricted to $\mathcal{E}$ is simply the flow of unit speed along geodesics.

E. Hopf has shown that the geodesic flow $\beta_t$ of a hyperbolic Riemann surface $\mathcal{S}$ always is either ergodic or dissipative on $\mathcal{E}$ ([2], [3]). The surface $\mathcal{S}$ is said to be of the first class in the ergodic case, and of the second class in the dissipative case. Suppose now that the surface $\mathcal{S}$ is represented by a Fuchsian group $G$ acting in $D$, $\mathcal{S} = D/G$. It follows then further that $\mathcal{S}$ is of the first class if and only if the action $G$ on the torus $\mathbf{T} \times \mathbf{T}$ is metrically transitive, i.e. if and only if each measurable $G$-invariant subset $E \subset \mathbf{T} \times \mathbf{T}$ has either measure $\tau(E) = 0$ or $\tau(E) = 1$ ([2], 8.1). It follows immediately that every surface of the first class is always of class $O_{HB}$.

Theorem 2. Suppose that one of the Riemann surfaces $\mathcal{S}_i = D/G_i$, $i = 1, 2$, is of the first class. Then for each geometric isomorphism $j : G_1 \to G_2$ either the boundary mapping $\varphi$ is completely singular or the isomorphism is induced by a Moebius transformation on $\mathbf{T}$.

Proof. Let $\mathcal{S}_1$ be of the first class, so that the boundary mapping is either absolutely continuous or completely singular by the preceding theorem. Since $\chi_{\varphi}$ is $G_1$-automorphic by (6), it is equal to a constant a.e. on $\mathbf{T} \times \mathbf{T}$. Obviously we must have $\chi_{\varphi} = 1$ a.e. in the case of absolute continuity, and $\chi_{\varphi} = 0$ a.e. in the singular case.

Suppose now that $\varphi$ is absolutely continuous. Using appropriate Moebius transformations $h, k$ we can find groups $G_1' = hG_1h^{-1}$, $G_2' = kG_2k^{-1}$ acting in $H$ with a real-valued boundary mapping

$$\psi = k \circ \varphi \circ h^{-1} : \mathbb{R} \to \mathbb{R}.$$ 

We may further suppose that $\psi(0) = 0, \psi'(0) = 1$, so that $\psi$ satisfies on $\mathbb{R}$ the differential equation

$$\psi'(x) = \psi(x)^2 / x^2$$

because $\chi_{\psi} = 1$ a.e. on $\mathbb{R} \times \mathbb{R}$. But given the initial value $\psi(0) = 0$, $\psi(x) = x$ is the only solution of (8) continuous on all of $\mathbb{R}$. Thus $\varphi = k^{-1} \circ h$, so that the isomorphism $j$ is induced on $\mathbf{T}$ by a Moebius transformation.
5. A Riemann surface $S = D/G$ can obviously be of the first class only if $G$ is a Fuchsian group of the first kind, but this condition is by far insufficient. If $S \subset \hat{C}$ is a hyperbolic planar surface, the covering group of $S$ is of the first kind if the complement $\hat{C} \setminus S$ is totally disconnected, but $S$ is of class $O_{\text{nn}}$ if and only if $\hat{C} \setminus S$ has vanishing logarithmic capacity.

If $A$ is the hyperbolic area of a hyperbolic Riemann surface $S$, the volume of $\mathcal{E}$ is $2\pi A$ (cf. n:o 4), so that all Riemann surfaces of finite hyperbolic area are of the first class by Poincaré's recurrence theorem ([2] 7.1, [3]). Now the hyperbolic area of a Riemann surface $S = D/G$ is finite if and only if $G$ is a finitely generated group of the first kind ([4], Theorem 5). Thus the Riemann surface $S = D/G$ is of the first class for all finitely generated Fuchsian groups $G$ of the first kind.

**Theorem 3.** Suppose that the geometric isomorphism $j : G_1 \rightarrow G_2$ of two finitely generated Fuchsian groups of the first kind acting in $H$ has an increasing boundary mapping $\psi : \mathbb{R} \rightarrow \mathbb{R}$. Then $\psi$ is either affine or a completely singular quasisymmetric function.

**Proof.** If $G$ is a finitely generated Fuchsian group of the first kind, the Riemann surface $S = (S_G, \ n_G) = H/G$ is a pointed surface of finite type, i.e. $S$ is a compact surface $S'$ with finitely many punctures; further, the support of $n_G$ is finite. Thus in the case of finitely generated groups of the first kind there always exists a quasiconformal mapping $\Phi : H \rightarrow H$ inducing the given isomorphism $j$ (cf. [5] Theorem 2.1, [6] 2.B), so that the boundary mapping $\psi : \mathbb{R} \rightarrow \mathbb{R}$ must be quasisymmetric, and the conclusion follows now from theorem 2.

Recently Sorvali has obtained results of a similar kind (cf. [5] Theorem 5.1). For quasisymmetric functions, cf. also Beurling — Ahlfors [1], for singular functions especially §7.

Current address:
Institut Mittag — Leffler
Djursholm, Sweden

University of Jyväskylä
Jyväskylä, Finland
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Printed August 1973