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EXTENSION OF BOUNDARY HOMEOMORPHISMS OF DISCRETE GROUPS OF THE UNIT DISK

BY

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Extension of boundary homeomorphisms of discrete groups of the unit disk

In this paper we consider discrete groups of the unit disk, i.e. groups whose elements are directly or indirectly conformal self-equivalences of the closed unit disk $E^1 = \{z \in \mathbf{C} : |z| \leq 1\}$, acting discontinuously in the open unit disk $E = \{z \in \mathbf{C} : |z| < 1\}$. If F and F' are two such groups and $\varphi : F \to F'$ an isomorphism, it is said to be geometric if there is a homeomorphism f of the open unit disk E such that (1) below is true.

(1) $$f(T(x)) = \varphi(T)(f(x)) \quad \text{for } x \in E \text{ and } T \in F .$$

In this case we say that f induces φ.

If T is a hyperbolic transformation of the unit disk we denote by $\mathrm{Ax}(T)$ the hyperbolic line joining the attractive and the repelling fixed point of T including the endpoints. If T is parabolic, $\mathrm{Ax}(T)$ is also defined; it consists of one point only, of the fixed point of T. Then (2) below is the axis condition for φ. It is always fulfilled if φ is geometric.

(2) $\mathrm{Ax}(T)$ *and* $\mathrm{Ax}(S)$ *intersect if and only if* $\mathrm{Ax}(\varphi(T))$ *and* $\mathrm{Ax}(\varphi(S))$ *intersect,* $T, S \in F$ *hyperbolic or parabolic.*

We shall show that condition (2) implies the existence of a homeomorphism of E inducing φ if F is a group whose limit set contains more than two points, i.e. if F contains a free subgroup with two generators (cf. p. 15). This is previously known for a large class of groups (cf. Tukia [4] pp. 32—33) including those not containing reflections.

It was also shown in [4] (pp. 31—32) that if F and F' are groups of the first kind and f is a homeomorphism of E inducing φ, then f admits a continuous extension to a homeomorphism g of the closed unit disk E^1. The restriction $h = g \mid S^1$ of g to the unit circle $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ is a homeomorphism of S^1 such that the following is true.

(3) $$h(T(x)) = \varphi(T)(h(x)) \quad \text{for } x \in S^1 \text{ and } T \in F .$$

In general if $\varphi : F \to F'$ is an isomorphism of discrete groups of the unit disk, not necessarily of the first kind, a homeomorphism $h : S^1 \to S^1$ of the unit circle is called a *boundary homeomorphism* of F and F' inducing φ if it fulfils (3). We remark that h is uniquely determined by φ only if F and F' are of the first kind.

The existence of a boundary homeomorphism inducing φ and the axis condition (2) for φ are equivalent if the groups are of the first kind but this is not true in general. The axis condition is weaker and is implied by the existence of a boundary homeomorphism. Again, as we have shown above, with Fuchsian groups of the first kind, a boundary homeomorphism can always be extended to a homeomorphism of the closed unit disk E^1 that induces φ. Our second result will be (cf. theorem 3 p. 12 later) that this is true also for groups of the second kind.

We shall use notation of [4] (cf. especially section 1.A). All the references, too, are to [4] except when otherwise stated. We remark that it is not possible to read this paper without the knowledge of [4].

The types of boundary of a pointed surface. Let F be a discrete group of the unit disk. We denote by

$$\lim(F)$$

the set of limit points of F and by

$$L(F)$$

the union of $\lim(F)$ and the endpoints of the axes of reflections of F.

In [4] we assumed that a discrete group F of the unit disk acted in the open unit disk E and the quotient space S was E/F. As a consequence S had a boundary if and only if F had reflections. In the present case we allow for the possibility of F acting in a slightly larger space E' where $E \subset E' \subset E^1 \setminus L(F)$ and $E' \cap S^1$ is open in S^1. When we say that F acts in E' we always mean that F acts discontinuously in E'. Thus the quotient space E'/F has two kinds of boundary points: those that belong to $(E' \setminus E)/F$ and the quotient sets of points x in E that are on the axes of reflections of F. If we have no knowledge about the group F, we cannot, as a rule (cf. theorem 2 later), know whether a boundary point is in the set $(E' \setminus E)/F$ or not. It is perfectly possible to have homeomorphic pointed surfaces $S = E/F$ and $S' = E'/F$ where F has reflections and F' not. That is why we divide the boundary of S into two distinct sets B_1 and B_2 where B_1 originates from the reflections of F and B_2 is the quotient set $(E' \setminus E)/F$. We call B_1 the reflective boundary of S and B_2 the ordinary boundary of S. Thus the group F determines a pointed surface denoted $(S, B_2, n) = (S_F, B_F, n_F)$ (cf. the definition of (S_F, n_F) on pp. 10—11 in [4] when F acts in E).

Thus we are led to consider triples of the form (S, B, n) where S is a countable bordered surface, B a union of the components of the boundary ∂S of S and $n : S \to \mathbf{N}$ a mapping with discrete support $|n| = \{x : n(x) > 1\}$ disjoint from B. The surface S may or may not have a conformal structure. Given such a triple (S, B, n) we ask if there exists a discrete group F of the unit disk acting in E', $E \subset E' \subset E \setminus L(F)$ such that $(S, B, n) = (E'/F, (E' \setminus E)/F, n_F)$ where $n_F(\mathrm{cl}\, x)$ depends on the stabilizer of x (cf. [4] pp. 10—11). This can be answered easily by considerations of [4]. We remark that the analogies of proposition 2.4 and corollaries 2.4.1, 2.4.2 and 2.4.3 in [4] hold also true for pointed surfaces of the form (S, B, n). These together with theorem 2.8 show that if $B \neq \emptyset$, (S, B, n) has a representation of the desired form. This can be seen as follows. We first consider the triple $(S \setminus B, \emptyset, n)$, S endowed with some conformal structure of a bordered Riemann surface. The results mentioned above show that this triple can be represented in the form $(S_F, \emptyset, n_F)$ for some discrete group of the unit disk. After that we simply add suitably intervals of S^1 to E to form the space E' such that E'/F is conformally equivalent to S.

So far we have not considered the set $E^1 \setminus E'$. It is clear that it may differ from the set $L(F)$. When starting with a given discrete group F of the unit disk it is most natural to consider the triple of the form (S_F, B_F, n_F) where F acts in $E^1 \setminus L(F)$. We ask now, given a pointed surface (S, B, n), if it can be represented in this form if we do not demand conformal equivalence of the pointed surfaces (S, B, n) and (S_F, B_F, n_F). Simple examples of the form $(E', E' \setminus E, 1)$ where $E \subset E' \subset E^1$ and E' is open in E^1 show that this is not always possible. But later we shall show that we can demand that F acts in E' where $\mathrm{cl}(E' \cap S^1) = S^1$, supposing $B \neq \emptyset$.

The technique for proving the following theorems is basically the same as that used in [4]. Thus we first consider representations of pointed surfaces.

Representations for pointed surfaces. Let (S, B, n) be a pointed surface such that all boundary components of S are included in B, i.e. $B = \partial S$. A representation of (S, B, n) is defined as follows. Let A be a plane domain bounded by a Jordan curve J. Let E' be a subset of $A \cup J$ with $A \subset E'$. Further, we suppose that two sets of intervals of J are given; let these be $\{I_i\}$, $i \in K$, and $\{I_j\}$, $j \in L$, $(K \cap L = \emptyset)$. The intervals may be open, half-open or closed (but not points) and their union will be $E' \cap J$. We require that:

(*i*) $I_i \cap I_j$ *is either empty or is a common endpoint,* $i \neq j$, i, $j \in K \cup L$.

(ii) I_k, $k \in K$, *is a component of* $(E' \cap J) \setminus \bigcup_{i \in L} I_i$ *when its (possible) endpoints are removed.*

(iii) If x *is an endpoint of* I_i, $i \in L$, $x \in I_i$, *then it is an endpoint for another* I_j, $j \in K \cup L$, $x \in I_j$.

(iv) I_i, $i \in L$, *intersects with at most one* I_j, $j \neq i$, $j \in L$.

The set of vertices is

$$V = \cup (I_i \cap I_j)\,, \; i \neq j\,, \; i\,, j \in L\,.$$

Let

$$i : \bigcup_{i \in L} I_i \to \bigcup_{i \in L} I_i$$

be a homeomorphism such that

$$i^2 = id\,,$$

$$i(I_i) = I_j \text{ for another } j \in L \; (j \neq i) \text{ when } i \in L\,.$$

$$i \mid V = id\,.$$

Let S' be the surface obtained from E' by identifying x and $i(x)$ for $x \in \bigcup_{i \in L} I_i$. If there is a homeomorphism $f : S' \to S$ such that $|n| = f(k(V))$ when $k : E' \to S'$ is the canonical projection, we say that (E', i, f) is a representation for (S, B, n). We remark that the intervals I_i, $i \in K \cup L$, as well as the sets $\{I_i\}$, $i \in L$ and $\{I_j\}$, $j \in K$, are uniquely determined by (E', i, f).

Theorem 1. *Let* (S, B, n) *be a pointed surface such that* B *is the boundary of* S *and* S *non-compact or* $B \neq \emptyset$. *Then there is a representation* (E', i, f) *of* (S, B, n) *where* $E \subset E' \subset E^1$, E' *is open in* E^1, *and such that any two intervals identified by* i *have equal length. If the pointed surface* (S, B, n) *is not equivalent to* $(E, \emptyset, 1)$ *we may besides assume that* $E' \cap S^1$ *is dense in* S^1.

We shall suppose that $B \neq \emptyset$ for otherwise theorem 1 is a consequence of proposition 2.7 in [4]. We can also assume that S is not compact for in this case theorem 1 is easily seen to be true by using the classification theorems for compact bordered surfaces. We shall consider sequences $K_1 \subset K_2 \subset \ldots$ satisfying the following conditions.

(i) $S = \bigcup_{i=1}^{\infty} K_i$.

(ii) Each K_i, $i \geq 1$, *is a compact bordered subsurface of* S *such that* $K_i \cap \partial S$ *is a finite union of Jordan arcs or Jordan curves of* ∂S.

(iii) $K_i \subset \operatorname{int} K_{i+1}$, $i \geq 1$.

(iv) The support of n *does not intersect with the boundary of* K_i , $i \geq 1$.

(v) Each component of $\operatorname{cl}(S \setminus K_i)$ *is non-compact,* $i \geq 1$.

(vi) Each component of $\operatorname{cl}(S \setminus K_i)$ *that does not intersect with* B *has in common with* K_i *exactly one Jordan curve.*

(vii) If C *is a component of* $\operatorname{cl}(S \setminus K_i)$ *intersecting with* B *then each component of* $C \cap K_i$ *is a Jordan arc whose endpoints lie on* B .

An exhaustion fulfilling conditions (i)—(vii) is called a *canonical exhaustion* of (S , B , n) . We should verify that such exhaustions exist. The construction method is that of Ahlfors-Sario [1] ch. 1 section 29 for non-compact surfaces without boundary. It is easy to see that there exist exhaustions satisfying (i)—(v). The condition (vi) is then added like in Ahlfors-Sario ch. 1 29 C. (If a component C of $\operatorname{cl}(S \setminus K_i)$ has two or more Jordan curves in common with K_i their number can be reduced as follows. Let J_1 and J_2 be two Jordan curves in $C \cap K_i$. Connect them with a Jordan arc J with one endpoint in J_1 and the other in J_2 , all inner points of J lying in $S \setminus K_i$. Now J is slightly enlargened to a strip A whose boundary in $\operatorname{cl}(S \setminus K_i)$ together with subarcs of J_1 and J_2 form a new Jordan curve J' . Now $K_i \cup A$ has fewer Jordan curves in common with $\operatorname{cl}(S \setminus (K_i \cup A))$ than K_i) . Similarly, if an exhaustion $K_1 , K_2 \ldots$ does not satisfy (vii) we can proceed as follows. Let $C , C \cap B \neq \emptyset$, be a component of $\operatorname{cl}(S \setminus K_i)$ which contains a Jordan curve J not intersecting with B . We connect J by a Jordan arc to B . Then one endpoint of J' lies on B , the other lies on J and the inner points belong to $S \setminus K_i$. This arc J' is slightly enlargened to a strip A whose boundary in $S \setminus K_i$ together with a subarc of J is a Jordan arc whose endpoints lie on B . The number of Jordan curves of $\operatorname{cl}(S \setminus (K_i \cup A))$ in common with $K_i \cup A$ is smaller than the number of Jordan curves of $\operatorname{cl}(S \setminus K_i)$ that are in common with K_i .

It is a consequence of (vii) that each component of $\operatorname{bd}(K_i)$, $i \geq 1$, intersects with B unless it is contained in a component of $\operatorname{cl}(S \setminus K_i)$ whose intersection with B is empty. In particular there are arcs of $\operatorname{bd}(K_1)$ that intersect with B , since B is assumed to be non-empty.

We have in [4] treated the case of (S , B , n) with $B \neq \emptyset$. (Cf. proposition 2.7 p. 21). To avoid repeating the argumentation used there we define a new exhaustion $K_1' \subset K_2' \subset \ldots$ of (S , B , n) as well as formulate a lemma that is the form in which we use proposition 2.7.

We set

$$K_1' = K_1$$
$$K_i' = K_i \cup \Big(\bigcup_{j \in I_j} S_j \Big) , \; i > 1 .$$

where $S_j , j \in I_i$, are components of $\mathrm{cl}(S \setminus K_i)$ that do not intersect with B .

Lemma 1. *Let C be a circle arc orthogonal to S^1 in E^1 and let E' be the closure of a component of $E^1 \setminus C$. Let (S , B , n) be a pointed surface with B homeomorphic to S^1 . Let $g : C/\{x , y\} \to B$ be a homeomorphism where x and y are the endpoints of C . Then there is a representation (E'' , i , f) of (S , B , n) with $g = f \circ k \mid C$ when k is the canonical projection $k : E'' \to E''/i$, $\mathrm{cl}\, E'' = E'$ and $\mathrm{cl}(E'' \cap S^1) = E' \cap S^1$.*

The validity of the above lemma follows from proposition 2.7. For, according to this proposition, (S , B , n) has a representation (E_1 , i , f) where $\mathrm{cl}\, E_1$ is homeomorphic to a closed disk from which an open disk has been removed and the boundary of this removed disk corresponds to B .

After that we begin the construction of the representation for (S , B , n) . This is done step by step constructing first a representation $(E_j , i_j , f_j , S_j , k_j)$ for $(K_j' , \partial K_j' , n \mid K_j') , j \geq 1$, where we have for clarity written the triple (E_j , i_j , f_j) as a quintuple and where $S_j = E_j/i_j$ and k_j the canonical projection $E_j \to S_j$. As we have already observed $K_1' \cap B \neq \emptyset$. Using this result and the classification theorems for compact bordered surfaces we see that there is a representation $(E_1 , i_1 , f_1 , S_1 , k_1)$ for $(K_1 , \partial K_1 , n \mid K_1)$ satisfying the following conditions.

(i) E_1 is a closed Jordan domain bounded by S^1 and a set of disjoint closed circle arcs orthogonal to S^1 .

(ii) $k_1^{-1} f_1^{-1}(K_1 \cap B)$ is a set of disjoint closed intervals of S^1 .

(iii) $k_1^{-1} f_1^{-1}(\partial K_1 \setminus B)$ is a disjoint set of open or closed circle arcs orthogonal to S^1 whose endpoints belong to $k_1^{-1} f_1^{-1}(B)$ unless its endpoints are identified by i_1 .

(iv) The intervals identified by i_1 are either closed circle arcs orthogonal to S^1 whose endpoints belong to $k_1^{-1} f_1^{-1}(B)$ or closed intervals of S^1 at least one endpoint of which belongs to $k_1^{-1} f_1^{-1}(B)$. The other endpoint if it does not belong to $k_1^{-1} f_1^{-1}(B)$ belongs either to the set of vertices of the representation (to the set V p. 6) or is an endpoint of the circle arc orthogonal to S^1 whose endpoints are identified by i_1 .

After that we extend $(E_1 , i_1 , f_1 , S_1 , k_1)$ to a representation of $(K' , \partial K' , n \mid K')$ where K' is K_1 added by a component K'' of $\mathrm{cl}(K_2' \setminus K_1)$. We first consider the case in which this component is compact. We then construct a representation $(E'' , i'' , f'' , S'' , k'')$ of $(K'' , \partial K'' , n \mid K'')$ satisfying conditions similar to (i)—(iv). Besides we may assume that $E_1 \cap E''$ is a circle arc orthogonal to S^1 and that $f_1 k_1 \mid E_1 \cap E'' = f''k'' \mid E_1 \cap E''$. Now we define the representation (E', i', f', S', k') with $E' = E_1 \cup E''$ where we set

$$i' \mid \text{domain of } i_1 = i_1 ,$$
$$i' \mid \text{domain of } i'' = i'' .$$

Besides there may be sets of circle arcs $I_k \subset E_1$ and $J_k \subset E''$, $k \in N$, orthogonal to S^1 such that $f_1 k_1(I_k) = f''k''(J_k)$. Then we define i' on $\bigcup_{k \in N}(I_k \cup J_k)$ in such a way that the resulting quotient space is homeomorphic to K' by a homeomorphism f'.

If the component K'' added to K_1 is not compact we obtain by the above lemma 1 a representation for K''. Other details are similar and are omitted. There are only a finite number of components to be added to K_1 to obtain K_2, so after a finite number of steps we have a representation $(E_2, i_2, f_2, S_2, k_2)$ for $(K_2', \partial K_2', n \mid K_2')$. It should be remarked that this representation fulfils conditions similar to (i)—(iv) except that we must remember that non-compact components may be added to K_1. Therefore we add to conditions (i)—(iv) the condition (v). Let I and J be two intervals of S^1 identified by i_2. Then I may also be open or half-open.

(*v*) *If* I *is half-open, the endpoint belonging to it is either a vertice of the representation or belongs to the set* $k_2^{-1} f_2^{-1}(B)$.

The above discussion shows that we can form a series of representations $(E_n, i_n, f_n, S_n, k_n)$ for $(K_n', \partial K_n', n \mid K_n')$, $n \geq 1$, so that conditions similar to (i)—(v) hold true and that for $n \geq 2$

$$E_{n-1} \subset E_n ,$$
$$i_n \mid \textit{domain of } i_{n-1} = i_{n-1} ,$$
$$f_n \mid S_{n-1} = f_{n-1} .$$

Besides we may demand:

(*vi*) *The diameter of components of* $E_n \setminus E_{n-1} < 1/n$, $n \geq 2$.

Then the representation (E', i, f) is defined as follows.

$$E' = \bigcup_{j=1}^{\infty} E_j ,$$

$$\textit{domain of } i = \bigcup_{j=1}^{\infty} \textit{domain of } i_j ,$$

$$i \mid \textit{domain of } i_j = i_j , j \geq 1 ,$$

$$f \mid S_j = f_j , j \geq 1 .$$

The triple is not yet the representation we have sought. First of all it may be that $E \not\subset E'$. This is remedied by substituting for every circle arc orthogonal to S^1 belonging to the boundary of E' the corresponding arc of S^1 and modifying i suitably. We suppose this is done. After that $\mathrm{cl}(E' \cap S^1) = S^1$ but two intervals identified by i may not have equal length. However, by the following lemma 2, the representation (E, i, f) can be replaced by another that has this property.

Lemma 2. *Let (E', i, F), $E \subset E' \subset E^1$, be a representation for some pointed surface (S, B, n). Then there is a homeomorphism f of S^1 such that if I and J are two intervals identified by i then $f(I)$ and $f(J)$ have equal length.*

Proof: Let V be the set of vertices of the representation and let I_k, J_k, $k \in k$, $i(I_k) = J_k$, be the intervals of $E' \cap S^1 \setminus V$ identified by i. We choose a $k \in K$ and define a homeomorphism $f_k : S^1 \to S^1$ as follows. Fo simplicity we assume that the endpoints of I_k are e° and e^{xi} and those of J_k $e^{\pi i}$ and $e^{(\pi-y)i}$, $o < x \leq (\pi - y) < \pi$. Let $f_k = id$ restricted to the lower half of $y \leq o$ of the circle. Let $f_k \mid (\exp o, \exp xi)$ be the linear stretch to the interval $(\exp o, \exp (x+y)i/2)$, $f_k \mid (\exp (\pi-y)i, \exp \pi i)$ be the linear stretch to the interval $(\exp (\pi-(x+y)/2)i, \exp \pi i)$ and let f_k restricted to $(\exp xi, \exp (\pi - y)i)$ be the rotation to the interval $(\exp (x+y)i/2, \exp (\pi-(x+y)/2)i)$. We remark that f_k does not change the lengths of the other intervals $I_l, J_l, k \neq l, l \in K$, and that the difference between arc x and arc $f_k(x)$ is not more than $1/2$ |length of I_k — length of J_k|.

We shall denote this mapping by f_1. After that we choose another index $l \in K$ and define the mapping f_2 in the same manner with respect to the intervals $f_1(I_l)$ and $f_1(J_l)$. We continue in this way. If K is finite, $K = \{k_1, \ldots k_n\}$,

$$g_n = f_n o \ldots o f_1 .$$

will be the required homeomorphism. Otherwise there will be a limit $\lim_{n\to\infty} g_n$ that is the required homeomorphism.

Theorem 2. *Let (S, B, n) be a pointed surface such that $S \setminus B$ is non-compact. Then it can be represented, up to a homeomorphism, in the form (S_F, B_F, n_F) by means of a discrete group F of the unit disk acting in a set E', $E \subset E' \subset E^1 \setminus L(F)$, E' open in E^1. Moreover, we can assume that either $E' \cap S^1 = \emptyset$ or $\mathrm{cl}(E' \cap S^1) = S^1$. If $\lim F$ contains more than two points, $\lim F = S^1$ if $B = \emptyset$. In addition, if $\partial S = B$, i.e. F does not contain reflections we may assume that F is formed as a free combination of cyclic groups.*

In case $B = \partial S$, the proof of theorem 2 is quite straightforward, once we have theorem 1. It is a repetition of the arguments used to prove theorem 2.8 p. 25 in [4]. We shall not do it here again. Only the case with $\lim F$ containing at most two points requires a comment. But before we do this we treat the case with non-empty reflective boundary.

Let (S, B, n) be a pointed surface with $\partial S \neq B$. We represent the triple $(S, \partial S, n \mid \operatorname{int} S)$ by means of a discrete group F'' of the unit disk acting in E', $E \subset E' \subset E^1$, E' open in E^1, where $\operatorname{cl}(E' \cap S^1) = S^1$. The conformal structure of S is defined to be equal with that of $S_{F''} = E'/F''$. This can be done since we know that ∂S is non-empty. Let $B_1 = p^{-1}(B)$ when $p : E' \to S$ is the canonical projection. Define

$$S^* = (E' \coprod E')/(E' \cap S^1 \setminus B_1)$$

i.e. we take two copies of E' and identify the boundary points x for which $p(x) \in \partial S \setminus B$. This is a bordered surface whose boundary consists of two copies of B_1 denoted $B^* = B_1 \coprod B_1$. A mapping $n^* : S^* \to \mathbf{N}$ is defined by setting $n^*(x) = n(p(p^*(x)))$ if $x \in E' \cap S^1$ and $= 1$ otherwise when $p^* : S^* \to E'$ is the canonical projection. Since either $B^* \neq \emptyset$ or S^* is non-compact, there is by the first part of the proof of theorem 2 a discrete group F' acting in E'', $E \subset E'' \subset E^1 \setminus L(F')$ such that $(S_{F'}, B_{F'}, n_{F'}) = (S^*, B^*, n^*)$. We denote the projection $E'' \to S^*$ by p'. It may be that the combined projection $p_1 = p \circ p^* \circ p'$ is not a local conformal equivalence at points of $p_1^{-1}(|n| \cup (\partial S \setminus B))$ but the conformal structure of E'' can be redefined so that it is. Then the homeomorphisms f of E'' for which the triangle

$$\begin{tikzcd}
E'' \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E'' \arrow[dl, "p_1"] \\
& S &
\end{tikzcd}$$

commutes, form by analogy of corollary 2.4.1 in [4] for the bordered case a group F of conformal self-equivalences of E'' so that (S_F, B_F, n_F) is equivalent, up to a homeomorphism, to (S, B, n).

If F contains a free subgroup with two generators then E'' minus its boundary is conformally equivalent to E. So we may assume that it is E added with a union of intervals of S^1. Moreover it is seen that $E'' \cap S^1$ is empty if and only if $\lim F$ is dense in S^1, for the emptiness of $E'' \cap S^1$ is equivalent to the emptiness of B_1 and thus with the emptiness of B^* and B. If $\lim F$ is not dense in S^1 we see that $E'' \cap S^1 = p^{*-1}(B^*) = p_1^{-1}(B)$ is dense in S^1.

After that we treat the case in which F does not contain a free subgroup with two generators. If $B = \partial S$, then the discussion in theorem 2.8 p. 25 in [4] shows that this can happen if and only if $|n| = \emptyset$ and

$S \setminus B$ is a sphere punctured in two points or a projective plane punctured in one point or of the type $(E, \emptyset, n)$ where $|n| = x$ or $n = \{x, y\}$ with $n(x) = n(y) = 2$. Suppose then that $\partial S \setminus B \neq \emptyset$. If $|n| = \emptyset$ we see that $(S \setminus B, \emptyset, n)$ is homeomorphic either to the complex plane with the open unit disk removed or with the strip $\{z \in \mathbf{C} : o \leq y \leq 1\}$. If $|n| \neq \emptyset$, $(S \setminus B, \emptyset, n)$ is homeomorphic to the half plane $\{z \in \mathbf{C} : y \geq o\}$ with $|n| = \{o\}$ or $|n| = \{i\}$ and $n(i) = 2$ or $|n| = \{-1, 1\}$ with $n(-1) = n(1) = 2$. In any case we see that the terms of the theorem can be met.

The boundary homeomorphism. In this section we discuss the case of a boundary homeomorphism inducing an isomorphism $\varphi : F \to F'$ between two discrete groups of the unit disk. First we formulate a lemma for use in theorem 3.

Lemma 3. *Let* φ, F *and* F' *be as above and suppose that* F *acts in* E', $E \subset E' \subset E^1$, E' *open in* E^1 *and that* F' *acts in* E'', *with similar properties. If* $\mathrm{cl}(E' \cap S^1) = S^1 = \mathrm{cl}(E'' \cap S^1)$ *or both groups are of the first kind and if* $f : E' \to E''$ *is a homeomorphism such that*

$$(*) \qquad f(T(x)) = \varphi(T)(f(x)) \quad \textit{for} \quad x \in E' \quad \textit{and} \quad T \in F$$

then f *can be extended to a homeomorphism* $g : E^1 \to E^1$ *such that* $(*)$ *is valid for all* $x \in E^1$, $T \in F$ *if* f *is replaced* g.

Proof: If F and F' are of the second kind, proof is obvious. If they are of the first kind see [4] pp. 31—32.

Theorem 3. *Let* F, F' *and* φ *be as above and suppose that there exists a boundary homeomorphism* $h : S^1 \to S^1$ *satisfying*

$$(1) \qquad h(T(x)) = \varphi(T)(h(x)) \quad \textit{for} \quad x \in S^1 \quad \textit{and} \quad T \in F.$$

Then there exists a homeomorphism $f : E^1 \to E^1$ *extending* h *such that* (1) *is valid for all* $x \in E^1$ *and* $T \in F$, h *replaced by* f.

Proof: We assume that E/F is non-compact. For E/F compact see references [2]—[5]. In this case any isomorphism φ is induced by a homeomorphism $f : E \to E$ that can be extended to E^1 ([4] p. 32).

We first assume that F and consequently F' does not contain reflections. Let (S_F, B_F, n_F) be the pointed surface defined by F, acting in $E' \setminus L(F)$. By theorem 2 (S_F, B_F, n_F) is up to a homeomorphism equivalent to a pointed surface (S_G, B_G, n_G) where G is either of the first kind or it acts in E', $E \subset E' \subset E^1$, E' open in E^1, and that $\mathrm{cl}(E' \cap S^1) = S^1$ and that it is formed as a free combination of cyclic groups. In both cases by corollary 2.4.3 in [4] for bordered surfaces there exists a homeomorphism $g : E' \to E^1 \setminus L(F)$ such that

$$g(T(x)) = \varphi'(T)(g(x) \quad \text{for} \quad x \in E' \quad \text{and} \quad T \in G \tag{2}$$

(if F is of the first kind $E' = E$) where $\varphi' : G \to F$ is an isomorphism. By lemma 3 g can be extended to a homeomorphism $E^1 \to E^1$ so that (2) is satisfied by all $x \in E^1$ and $T \in G$. Thus if we replace F by G, φ by $\varphi \circ \varphi'$ and the boundary homeomorphism h by $h \circ g \mid S^1$ we may assume that the group in question is formed as a free combination of cyclic groups. We suppose this is done.

Let F_i, D_i, T_i, $i \in I$, form F as a free combination of cyclic groups where these notations are used as in p. 16 of [4]. Then we define F_i', D_i', T_i', $i \in I$, by means of the boundary homeomorphism h as was done on pp. 34—35 of [4]. The proof is complete if we can show that F_i', D_i', T_i', $i \in I$, form indeed F' as a free combination of cyclic groups. (The details of defining the homeomorphism inducing φ can then be found on p. 35 of [4] and can be readily modified to the present case.) According to the proof of theorem 2.6 in [4] this is equivalent with the fact that D' below is the fundamental domain for F'. Let

$$\begin{aligned} D' &= \bigcap_{i \in I} D_i' \\ D'' &= \bigcup_{T \in F'} T(D') . \end{aligned} \tag{3}$$

By theorem 2.6, D' is the fundamental domain for F' if and only if D'' is the whole open unit disk. Assume the contrary and let x be an element of $E \setminus D''$. Then define a chain $\{D_n'\}$, $n \geq o$, by (i) and (ii) p. 18 in [4]. Let $\{D_n\}$, $n \geq o$, be the chain defined by $D_n = T_n(D)$ if $D_n' = \varphi(T_n)(D')$ and $D = \bigcap_{i \in I} D_i$, $n \geq o$. Since D is a fundamental domain for F this chain converges towards a point y of S^1. But then $\{D_n'\}$ converges towards the point $h(y)$ in contradiction with the definition of $\{D_n'\}$.

After that we assume that there are reflections in the groups F and F'. Let E_0 be the closure of a component of $p^{-1}(S \setminus \partial S)$ where $p : E^1 \setminus L(F) \to S$ is the canonical projection. Clearly E_0 is a hyperbolically convex, closed Jordan domain. We may assume that there is a point $y \in E_0 \cap S^1$ since otherwise S would be compact and this case was treated in [4].

Next we define E_0' which is a closure of a component of $p'^{-1}(S' \setminus \partial S')$ when $p' : E^1 \setminus L(F') \to S'$ is the canonical projection. Let L be a (closed) hyperbolic line with the endpoint y such that a neighbourhood of y in L is contained in $\operatorname{int} E_0 \cup \{y\}$. Let L' be L transformed by h. Then E_0' shall be the closure of a component of $p'^{-1}(S' \setminus \partial S')$ for which a neighbourhood of $h(y)$ in L' is contained in $\operatorname{int} E_0' \cup \{h(y)\}$. It is clear that E_0' is uniquely determined regardless of the choice of y or L.

Suppose then that $T \in F$ is a reflection for which $\operatorname{Ax}(T) \setminus E_0$ is a line segment. We claim that $\operatorname{Ax}(\varphi(T)) \cap E_0'$ is also a line segment. This is obvious if there is a point $y \in \operatorname{Ax}(T) \cap E_0 \cap S^1$. If $\operatorname{Ax}(T) \cap E_0 \subset E$ we choose a point $y \in E_0 \cap S^1$ and denote by L the hyperbolic line (closed) that joins the points y and $T(y)$. Then if L is not the axis of a reflection of F, a neighbourhood of y in L is contained in $\operatorname{int} E_0 \cup \{y\}$ and L does not intersect in E with other axes of reflections of F than T. This proves our claim in this case. And it is not difficult to modify the argument if L is the axis of a reflection of F.

We need the following fact. If T and S are reflections of F such that the intersection of $\operatorname{Ax}(T) \cap E_0 \cap E$ with $\operatorname{Ax}(S) \cap E_0 \cap E$ is a point then the same is true of $\operatorname{Ax}(\varphi(T)) \cap E_0' \cap E$ and $\operatorname{Ax}(\varphi(S)) \cap E_0' \cap E$ and conversely. For if this is not the case we could find three reflections of F such that their axes bound a hyperbolic triangle in E and this is impossible since S was assumed to be non-compact.

Thus we have the Jordan domains E_0 and E_0'. Let F_0 be the subgroup of F leaving E_0 fixed. This is equivalent with the fact that F_0 leaves $\operatorname{bd} E_0$ invariant. So $F_0' = \varphi(F_0)$ is the subgroup leaving E_0' invariant. Next, we define a homeomorphism $h' : \operatorname{bd} E_0 \to \operatorname{bd} E_0'$ such that

$$h'(T(x)) = \varphi(T)(h'(x)) \quad \text{for} \quad T \in F_0 \quad \text{and} \quad x \in \operatorname{bd} E_0 . \tag{4}$$

We set $h' \mid E_0 \cap S^1 = h \mid E_0 \cap S^1$. Let T be a reflection of F such that $E_0 \cap \operatorname{Ax}(T)$ consists of more than one point. Then $E_0 \cap \operatorname{Ax}(T)$ is a (closed) hyperbolic segment if there are two reflections T_1 and T_2 such that $\operatorname{Ax}(T_i) \cap E_0$ is homeomorphic to a line segment and that $\operatorname{Ax}(T) \cap \operatorname{Ax}(T_i)$ is an endpoint of $\operatorname{Ax}(T) \cap E_0$, $i = 1, 2$, that lies in E. If there is only one such reflection $\operatorname{Ax}(T) \cap E_0$ is a hyperbolic ray (closed) and if there are not such reflections of F at all $E_0 \cap \operatorname{Ax}(T)$ is a hyperbolic line (closed). Thus we can always define a homeomorphism $f_T : E_0 \cap \operatorname{Ax}(T) \to E_0' \cap \operatorname{Ax}(\varphi(T))$ in such a way that any two mappings coincide on common points and with $h' \mid E_0 \cap S^1$. Besides it is clear that we may suppose

$$f_T(S(x)) = \varphi(S)(f_{S^{-1}TS}(x)) \quad \text{for} \quad x \in \operatorname{Ax}(S^{-1}TS) \cap E_0 , S \in F_0 .$$

Thus together these mappings f_T and $h \mid E_0 \cap S^1$ define a homeomorphism satisfying (4). Then we can by the first part of this proof extend it to a homeomorphism $f' : E_0 \to E_0'$ such that (4) holds for all $T \in F_0$ and $x \in E_0$.

By proposition 2.4 (or by its corollary 2.4.1 which, however, does not take account of the boundary $p^{-1}(\partial S \setminus B))$ (S_F, B_F, n_F) is equivalent with

$((E_0 \setminus L(F_0))/F_0, ((E_0 \cap S^1) \setminus L(F_0))/F_0, n)$ where $n(x)$ is determined by the stabilizer of y, $p(y) = x$, and a similar result is valid for $(S_{F'}, B_{F'}, n_{F'})$. Thus we can by corollary 2.4.3 find a lifting f of the homeomorphism $f'' : (S_F, B_F, n_F) \to (S_{F'}, B_{F'}, n_{F'})$ defined by f' in the quotient set $(E_0 \setminus L(F_0))/F_0$. Then f is a homeomorphism $E^1 \setminus L(F) \to E^1 \setminus L(F')$ and it can be chosen so that

$$f(T(x)) = \varphi'(T)(f(x)) \quad \text{for all } x \in E^1 \setminus L(F) \text{ and } T \in F, \tag{5}$$

$$f \mid E_0 \setminus L(F) = f' \mid E_0 \setminus L(F) \tag{6}$$

where φ' is an isomorphism $F \to F'$. By lemma 3 it can be extended to the whole closed unit disk so that (5) holds for all $x \in E^1$ and $T \in F$. It is clear that $\varphi \mid F_0 = \varphi' \mid F_0$ and that $\varphi(T) = \varphi'(T)$ for such reflections $T \in F$ for which $\mathrm{Ax}(T) \cap E_0$ is a line segment. Since these generate F, we have $\varphi = \varphi'$. This proves the theorem. In proving it we have also proved the fact that a group whose fundamental domain is non-compact and does not contain reflections can be formed as a free combination of cyclic groups. This is perhaps worth noting as a proposition.

Proposition. *Let F be a discrete group of the unit disk such that E/F is non-compact and that F does not contain reflections. Then F has a fundamental domain D bounded by hyperbolic lines equivalent in pairs and by hyperbolic rays equivalent in pairs so that if P is such a ray, $P = T(Q)$ where Q is another ray bounding D and T is an elliptic transformation of F with a fixed point that is the common endpoint of P and Q. Let P_i and Q_i, $i \in I$, be the hyperbolic lines or rays bounding D so that $P_i = T_i(Q_i)$ where $T_i \in F$. Let F_i be the cyclic group generated by T_i. Then F is the free product of the groups F_i, $i \in I$.*

A corollary of theorem 3 is the following.

Corollary. *Let F and F' be discrete groups of the unit disk such that F contains a free subgroup with two generators, i.e. the limit set of F contains more than two points. Let $\varphi : F \to F'$ be an isomorphism. Then it is geometric if and only if it satisfies the axis condition* (2) p. 3.

Proof: By lemma 3.4 in [4] (2) is necessary. It is also sufficient. For by lemma 3.4 and theorem 2 above we can assume that F and F' are groups of the first kind. The conclusion follows now by theorem 3, since the boundary homeomorphism exists by proposition 3.5.

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