ON THE DILATATION OF ISOMORPHISMS BETWEEN COVERING GROUPS

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Introduction

A group $G$ of Möbius transformations fixing a disk or half-plane $D$ is called a covering group if it is discontinuous in the following sense: For each point $z \in D$ there exists a neighborhood $U$ such that $g(z) \not\in U$ whenever $g \neq id$ lies in $G$. Hence a covering group may contain hyperbolic and parabolic transformations only.

In [3] we introduced the dilatation $\delta(j)$ of an isomorphism $j : G \to G'$ defined as follows: If $\kappa(g)$ denotes the multiplier of a Möbius transformation $g$, then $\delta(j)$ is the smallest number $1 \leq a \leq \infty$ for which $\kappa(g)^{1/a} \leq \kappa(j(g)) \leq \kappa(g)^a$ holds for all $g \in G$. As examples of the case where $\delta(j) < \infty$ we have the isomorphisms $j$ induced by quasiconformal mappings $f$, i.e. $j(g) = f \circ g \circ f^{-1}$ for all $g \in G$. On the other hand, if there is a parabolic $g \in G$ such that $j(g)$ is hyperbolic or vice versa, then $\delta(j) = \infty$.

In § 1 we consider isomorphisms $j$ between noncyclic covering groups with $\delta(j) = \infty$. We show that the dilatation of $j$ restricted to elements whose type is preserved is also infinite. In § 2 we consider parabolic elements under an isomorphism with a finite dilatation.

In § 3 we prove the following theorem: Let $\{g_1, g_2, \ldots\}$ be a set of generators of $G$. Suppose that an isomorphism $j : G \to G'$ preserves the multipliers of the elements of the type $(g_1^a \circ g_2^b)^{\alpha} \circ (g_3^\beta \circ g_4^\gamma)^{\varepsilon}$ where $\alpha, \beta, \gamma, \varepsilon$ are integers and $a = 1, 2$. Then $j$ is induced by a Möbius transformation.

Let $j : G \to G'$ be an isomorphism between covering groups acting on the upper half-plane $H$. A homeomorphism $\varphi : R \cup \{\infty\} \to R \cup \{\infty\}$, where $R$ is the set of the real numbers, is called a boundary mapping of $j$ if $\varphi \circ g = j(g) \circ \varphi$ holds for all $g \in G$. In § 4 we characterize $\delta(j)$ in terms of the local Hölder continuity of $\varphi$ and $\varphi^{-1}$. As a corollary we then obtain the following result: If $\varphi$ has a $K$-quasiconformal extension to the extended complex plane, then $\delta(j) \leq K$.

§ 1. Isomorphisms with an infinite dilatation

For a hyperbolic transformation $g$, let $\kappa(g)$ denote the multiplier and $P(g)$ and $N(g)$ the attracting and the repelling fixed point. The
parameters \( x(g) \), \( P(g) \), and \( N(g) \) determine \( g \) uniquely. We have \( x(g) = (z, g(z), P(g), N(g)) \) \( > 1 \), the cross ratio being defined as in \([3, \S 1]\). If \( g \) is parabolic, we define \( x(g) = 1 \) and \( P(g) = N(g) \) as the only fixed point of \( g \).

Let a parabolic or hyperbolic transformation \( g \) be given in the form \( z \mapsto g(z) = (az + b)/(cz + d) \) with \( ad - bc = 1 \). Then \( a + d \) is always real, and \( \chi(g) = |a + d| \) is the trace of \( g \). It follows that

\[
\chi(g) = x(g)^{1/2} + x(g)^{-1/2}.
\]

Hence \( \chi(g) \geq 2 \), where the equality holds if and only if \( g \) is parabolic.

Let \( \mathcal{M} \to \mathcal{M}' \) be a mapping between sets of hyperbolic and parabolic transformations. A calculation shows that the dilatation of \( j \) can also be defined in terms of \( \chi(g) \).

**Theorem 1.** Suppose that for any \( g \in \mathcal{M} \) the transformations \( g^n \), \( n = 2, 3, \ldots \), are in \( \mathcal{M} \), and suppose that \( j(g^n) = j(g)^n \). If \( 1 \leq a \leq \infty \) is the smallest number for which \( \chi(g)^{1/a} \leq \chi(j(g)) \leq \chi(g)^a \) holds for all \( g \in \mathcal{M} \), then \( a = \delta(j) \).

**Proof.** Let \( g \in \mathcal{M} \), \( k = x(g) \) and \( k' = x(j(g)) \). Suppose that we have \( \chi(j(g)^n) \leq \chi(g^n)^a \) for \( n = 1, 2, \ldots \). Then

\[
(k' - n^2) + (k')^{-n^2} \leq (k^n + k^{-n} + 2)^a,
\]

and hence

\[
(k')^n \leq (k')^n + (k')^{-n} + 2 \leq (k^n + k^{-n} + 2)^a \leq (2k^n)^a
\]

from some \( n = n_1 \) on. Therefore

\[
k' \leq (2k^n)^{a/n} = (2^{1/n} k)^a,
\]

and letting \( n \to \infty \) we obtain \( k' \leq k^a \). Similarly, if \( \chi(j(g)^n) \geq \chi(g^n)^{1/a} \) for \( n = 1, 2, \ldots \), then we get \( k \leq (k')^a \). Thus we have \( k^{1/a} \leq k' \leq k^a \).

Conversely, suppose that \( k^{1/a} \leq k' \leq k^a \). Then

\[
\chi(j(g)) = \sqrt{k'} + (1/\sqrt{k'}) \leq (\sqrt{k})^a + (1/\sqrt{k})^a \leq (\sqrt{k} + 1/\sqrt{k})^a = \chi(g)^a
\]

and similarly \( \chi(g) \leq \chi(j(g))^a \). \( \square \)

Let \( j : G \to G' \) be an isomorphism between covering groups \( G \) and \( G' \). If there is a parabolic \( g \in G \) such that \( j(g) \) is hyperbolic or vice versa, then \( \delta(j) = \infty \). By the following theorem, the dilatation of \( j \) restricted to elements whose type is preserved is also infinite.

**Theorem 2.** Let \( j : G \to G' \) be an isomorphism with \( \delta(j) = \infty \). Define \( G^* \) as the set of all hyperbolic elements \( g \in G \) for which \( j(g) \) is hyperbolic. If \( G \) is not cyclic, then \( \delta(j|G^*) = \infty \).
Proof. It follows from Lemma 3.1 in [3] that \( G^* \neq \emptyset \). If \( j \) preserves the type of all transformations of \( G \), then there is nothing to prove. In other cases choose a hyperbolic \( g_2 \in G \) such that \( j(g_1) \) is parabolic (if this is not possible, then we consider the isomorphism \( j^{-1}: G' \to G \)) and let \( g_2 \in G^* \). Then we have ([3, (4.11)]):

\[
\chi(g_1^n \circ g_2) = \left| \frac{k_1^n k_2 + 1 - x(k_1^n + k_2)}{(1-x)(k_1^n k_2)^{1/2}} \right|
\]

where \( k_1 = x(g_1) \) and \( x = 1 - (N(g_1), N(g_2), P(g_1), P(g_2)) \). Therefore

\[
\lim_{n \to \infty} \frac{\chi(g_1^n \circ g_2)}{k_1^n} = k_2^{-1/2} \frac{k_2 - x}{1 - x}.
\]

If \( k_2 - x = 0 \), we replace \( g_2 \) by \( g_2^2 \). Then there is a \( b \geq 1 \) such that we have for \( n = 1, 2, \ldots \)

\[
(1/b)k_1^n \leq \chi(g_1^n \circ g_2) \leq bk_1^n.
\]

We now consider the group \( G' \). Since \( g_1' = j(g_1) \) is parabolic and \( g_2' = j(g_2) \) hyperbolic, we may normalize such that

\[
g_1'(z) = z + \omega, \quad g_2'(z) = (kz)/((k-1)z + 1),
\]

where \( k = x(g_2') > 1 \). We may also assume that \( \omega > 0 \) since we can replace \( g_1' \) by \( (g_1')^{-1} \) if necessary. Then we have

\[
((g_1')^n \circ g_2') (z) = \frac{(k + n\omega(k-1))z + n\omega}{(k-1)z + 1},
\]

and hence

\[
\chi((g_1')^n \circ g_2') = \frac{1 + k + n\omega(k-1)}{k^{1/2}}.
\]

From (1.1) and (1.2) we conclude that \( g_1^n \circ g_2 \in G^* \) from some \( n = n_0 \) on.

By (1.2), \( \chi((g_1')^n \circ g_2') \leq 2n\omega k \) for sufficiently large \( n \). Then we have for any \( 1 \leq a < \infty \)

\[
\chi(g_1^n \circ g_2)^{1/a} \geq (k_1^n/b)^{1/a} > 2n\omega k \geq \chi((g_1')^n \circ g_2')
\]

from some \( n = n_a \) on. Therefore \( \delta(j | G^*) = \infty \) by Theorem 1. \( \square \)

§ 2. Distortion of parabolic transformations

Let \( j : G \to G' \) be an isomorphism between covering groups which act on the upper half-plane \( H \), and suppose that \( \delta(j) < \infty \). In this
section we consider the behavior of the parabolic elements of $G$ under $j$.

A parabolic transformation $g \in G$ fixing $\infty$ is of the type

$$g(z) = z + \omega. \tag{2.1}$$

If $g$ is parabolic with $P(g) \neq \infty$, then $g$ has a unique representation in the form

$$\frac{1}{g(z) - P(g)} = \frac{1}{z - P(g)} + \omega. \tag{2.2}$$

We call the number $\omega = o(g)$ defined by (2.1–2) the translation vector of $g$. From $g(H) = H$ it follows that $P(g)$ and $o(g)$ are real. If the transformation $g$ in (2.2) is given in the form $g(z) = (az + b)/(cz + d)$ with $a + d = 2$, then an elementary calculation shows that $o(g) = c$.

To interpret geometrically the translation vector $o(g)$, consider first the transformation (2.1) with $\omega > 0$. If we define the non-euclidean metric in $H$ by $(\text{Im } z)^{-1} |dz|$, then the non-euclidean length of the euclidean line segment $\{x + i | x_0 \leq x \leq x_0 + \omega\}$ is $\omega$. Since the non-euclidean distances are invariant under Möbius transformations, we then obtain from (2.2) the following interpretation for $o(g)$: Suppose that $P(g) \neq \infty$ and define $K(g)$ as the circle of diameter one through $P(g)$ and $P(g) + i$. If $z \in K(g)$, then $|o(g)|$ is the non-euclidean length of the part of $K(g)$ between $z$ and $g(z)$. From this it follows that we have $o(g) = o(h \circ g \circ h^{-1})$ for all translations $h: z \mapsto z + b$, $b$ real.

For a hyperbolic transformation $h$ fixing $H$, let $Ax(h)$ be the axis of $h$ (i.e. the circle through $P(h)$ and $N(h)$ orthogonal to $R$). If $z \in Ax(h)$, then $\log \kappa(h)$ is the non-euclidean length of the part of $Ax(h)$ between $z$ and $h(z)$. Thus $|o(g)|$ has some analogy with $\log \kappa(h)$. However, if we normalize such that $j$ fixes the translation $z \mapsto z + 1$, then $|o(g)|$ does not behave under $j$ as $\log \kappa(h)$ but like $\kappa(h)$.

**Theorem 3.** Suppose that the transformation $g_0: z \mapsto z + 1$ lies in $G \cap G'$. Let $j: G \to G'$ be an isomorphism such that $a = \delta(j) < \infty$. If $j(g_0) = g_0$, then $|o(g)|^{1/a} \leq |o(j(g))| \leq o(g)^{a}$ holds for all parabolic transformations $g$ of $G$.

**Proof.** We first note that for any parabolic element $h = g_0$ of $G$ we have $|o(h^{-1} \circ g_0 \circ h)| = o(h)^2$. To prove this, let $h$ be the transformation $z \mapsto ((1 + ox)z - ox^2)/(oz + 1 - ox)$, where $x = P(h)$ and $o = o(h)$. Then

$$(h^{-1} \circ g_0 \circ h)(z) = \frac{(1 + ox - o^2x)z + (1 - ox^2)}{-o^2z + 1 - ox + o^2x}.$$ 

Hence $o(h^{-1} \circ g_0 \circ h) = - o^2$. 


Let \( g \neq g_0 \) be a fixed parabolic transformation of \( G \). Define \( g_1 = g^{-1} \circ g_0 \circ g \) and inductively \( g_n = g_{n-1}^{-1} \circ g_0 \circ g_{n-1} \) for \( n = 2, 3, \ldots \). Then \( \{g_n\} \) is a sequence of parabolic elements of \( G \). By the above remark we have \( |\omega(g_n)| = \omega(g_{n-1})^2 \). Therefore
\[
(2.3) \quad |\omega(g_n)| = \omega(g)^{2^n}
\]
for \( n = 1, 2, \ldots \).

Since \( a = \delta(j) < \infty \), \( \{j(g_n)\} \) is a sequence of parabolic elements of \( G' \). Because \( j(g_0) = g_0 \), (2.3) holds if \( g_n \) and \( g \) are replaced by \( j(g_n) \) and \( j(g) \), respectively.

For any parabolic transformation \( h \neq g_0 \) of \( G \) we have
\[
(2.4) \quad \chi(g_0 \circ h) = |2 + \omega(h)|.
\]
Since \( \chi(g_0 \circ h) \geq 2 \), it follows that \( |\omega(h)| \geq 4 \). We apply (2.4) to the transformations \( g_n \) and \( j(g_n) \). Then by Theorem 1
\[
|2 + \omega(g_n)|^{1/a} \leq |2 + \omega(j(g_n))| \leq |2 + \omega(g_n)|^a.
\]
Formula (2.3) and the triangle inequality yield
\[
0 < (\omega(g)^{2^n} - 2)^{1/a} = (|\omega(g_n)| - 2)^{1/a} \leq |2 + \omega(g_n)|^{1/a} \leq |2 + \omega(j(g_n))| \leq 2 + \omega(j(g))^2^n.
\]
Hence
\[
[(\omega(g)^{2^n} - 2)^{1/a}]^{1/a} \leq [2 + \omega(j(g))^2^n]^{1/2^n},
\]
and letting \( n \to \infty \) we obtain \( |\omega(g)|^{1/a} \leq |\omega(j(g))| \). It follows similarly that \( |\omega(j(g))| \leq |\omega(j(g))|^{1/a} \).

Remark: Let \( G \) be a covering group containing the transformation \( g_0 : z \mapsto z + 1 \). As remarked above, it follows from (2.4) that \( |\omega(g)| \geq 4 \) for all parabolic elements \( g \neq g_0 \) of \( G \). This bound is sharp: Let \( g_1(z) = z/(4z + 1) \) and let \( G_1 \) be the group generated by \( g_0 \) and \( g_1 \). Then \( G_1 \) is a covering group and we have \( \omega(g_1) = 4 \).

§ 3. Isomorphisms with dilatation one

For a set \( M \) of Möbius transformations, let \( \text{Fix}(M) \) denote the set of fixed points of non-identity transformations of \( M \). If the set \( \text{Fix}(G) \) is dense in a circle or a straight line, then the covering group \( G \) is said to be of the first kind. If not, then \( G \) is of the second kind.

Let \( j : G \to G' \) be an isomorphism with \( \delta(j) = 1 \). If \( G \) and \( G' \) are of the first kind, then by Theorem 4.3 in [3] there is a Möbius transformation
Theorem 4. Let $E = \{g_1, g_2, \ldots \}$ be a set of generators of a covering group $G$. Let $F$ consist of the transformations of the form $(g_1^n \circ g_2^n \circ (g_1 \circ g_2)^a)$, where $\alpha, \beta, \gamma, \varepsilon$ are integers and $a = 1, 2$. If an isomorphism $j : G \to G'$ preserves the multipliers of the elements of $F$, then $j$ is induced by a Möbius transformation.

Proof. It suffices to show that there is a Möbius transformation $h$ such that $j(g_i) = h \circ g_i \circ h^{-1}$ for all $g_i \in E$.

(A) Suppose first that $E$ contains at least one hyperbolic element.

If $E = \{g_1\}$, $g_1$ hyperbolic, then $j$ is induced by any Möbius transformation sending $P(g_1)$ to $P(j(g_1))$ and $N(g_1)$ to $N(j(g_1))$. Let $E = \{g_1, g_2\}$ with $g_1$ parabolic and $g_2$ hyperbolic. We show that the Möbius transformation which sends $P(g_1)$ to $P(j(g_1))$, $i = 1, 2$, and $N(g_2)$ to $N(j(g_2))$ induces $j$. Since we can replace $G$ and $G'$ by conjugate groups $G_1 = hGh^{-1}$ and $G'_1 = h'G'h'^{-1}$, we may assume that $g_2$ and $j(g_2)$ both are the transformation $z \mapsto kz/((k - 1)z + 1)$ and that $P(g_1) = P(j(g_1)) = \infty$. Since we have $\chi(g_1^n \circ g_2^n) = \chi(j(g_1)^n \circ j(g_2))$, it follows from (1.2) that

$$|1 + k + n(k - 1)\omega(g_1)| = |1 + k + n(k - 1)\omega(j(g_1))|,$$

for $n = 1, 2, \ldots$. Therefore $\omega(g_1) = \omega(j(g_1))$, and the assertion follows.

Let $\text{Fix}(E)$ contain at least four distinct points. Choose $z_i \in \text{Fix}(E)$ such that $(z_1, z_2, z_3, z_4) > 1$. Suppose that $z_1 = N(h_1)$, $z_2 = N(h_2)$, $z_3 = P(h_3)$, $z_4 = P(h_4)$, where for each $i, i = 1, 2, 3, 4$, either $h_i \in E$ or $h_i^{-1} \in E$. If

$$w_1 = N(j(h_1)), \quad w_2 = N(j(h_2)), \quad w_3 = P(j(h_3)), \quad w_4 = P(j(h_4)),$$

then the points $w_i$ are well-defined and distinct. We show that

$$z_1, z_2, z_3, z_4 = (w_1, w_2, w_3, w_4). \quad (3.1)$$

To prove (3.1), set $g_1n = h_3^a \circ h_1^a$ and $g_2a = h_4^a \circ h_2^a$. Then by Lemma 3.1 in [3], $N(g_1n) \to z_i$, $P(g_1n) \to z_{i+2}$, and similarly $N(j(g_1)) \to w_i$, $P(j(g_1)) \to w_{i+2}$ as $n \to \infty$, $i = 1, 2$. Thus it suffices to show that

$$\begin{align*}
(N(g_1n, N(g_2a), P(g_1n), P(g_2a)) = &
(N(j(g_1)), N(j(g_2)), P(j(g_1)), P(j(g_2))))
\end{align*} \quad (3.2)$$

for sufficiently large values of $n$. Choose $n_0$ such that for $n \geq n_0$
(N(\(g_{1n}\)), N(\(g_{2n}\)), P(\(g_{1n}\)), P(\(g_{2n}\))) \geq 1.

Since \(j\) preserves the multipliers of \(g_{1n}, g_{2n}, g_{1n} \circ g_{2n}\) and \(g_{1n}^2 \circ g_{2n}^2\), we can apply the proof of Theorem 4.3 in [3] by replacing \(g_i\) by \(g_m\). Then it follows that (3.2) holds for \(n \geq n_0\), and (3.1) is proved. By (3.1) there is a Möbius transformation \(h\) such that \(h(P(g_{i}^{\pm1})) = P(j(g_i)^{\pm1})\) for all \(g_i \in E\). By the previous part of the proof we have \(j(g_i) = h \circ g_i \circ h^{-1}\) for \(g_i \in E\). Thus case (A) is proved.

(B) Suppose secondly that \(E\) contains only parabolic elements.

The case when \(E\) consists of one parabolic element is clear. Let \(E = \{g_1, g_2\}\) with \(g_1\) and \(g_2\) parabolic. We may suppose that \(g_1\) and \(j(g_1)\) both are the transformation \(z \mapsto z + 1\) and that \(P(g_2) = P(j(g_2))\). Since we have \(\chi(g_1 \circ g_2^*) = \chi(j(g_1) \circ j(g_2)^n)\), it follows from (2.4) that

\[
|2 + n \omega(g_2)| = |2 + n \omega(j(g_2))|
\]

for \(n = 1, 2, \ldots\). Therefore \(\omega(g_2) = \omega(j(g_2))\), and it follows that \(j = id\).

Let \(E = \{g_1, g_2, g_3\}\) with \(g_1, g_2, g_3\) parabolic. We show that the Möbius transformation sending \(P(g_i)\) to \(P(j(g_i))\) induces \(j\). We normalize such that

\[
P(g_1) = P(j(g_1)) = \infty, \quad P(g_2) = P(j(g_2)) = 0, \quad P(g_3) = P(j(g_3)) = -1.
\]

Then it suffices to show that \(j = id\).

Let \(\omega_i = \omega(g_i), \quad i = 1, 2, 3\). Then we have (cf. 2.4)

\[
\chi(g_i^* \circ g_i) = |2 + n \omega_i \omega_i|,
\]

for \(i = 2, 3\). A simple calculation yields

\[
(g_3 \circ g_2^*) (z) = \frac{(1 - n \omega_2 \omega_3 - \omega_3) z - \omega_3}{(n \omega_2 + n \omega_2 \omega_3 - \omega_3) z + \omega_3 + 1}.
\]

Hence

\[
\chi(g_3 \circ g_2^*) = |2 - n \omega_2 \omega_3|,
\]

and we also obtain similar expressions for

\[
\chi(j(g_1)^n \circ j(g_1)) \quad \text{and} \quad \chi(j(g_3) \circ j(g_2)^n).
\]

Let \(\omega_i' = \omega(j(g_i))\). Then we have the following equations

\[
|2 + n \omega_i \omega_i'| = |2 + n \omega_i' \omega_i'|, \quad i = 2, 3,
\]

\[
|2 - n \omega_2 \omega_3| = |2 - n \omega_2' \omega_3|,
\]

for \(n = 1, 2, \ldots\). Hence \(\omega_i \omega_k = \omega_i' \omega_k'\) holds for \(i \neq k\), and we have either \(\omega_i = \omega_i'\) or \(\omega_i = -\omega_i'\) for \(i = 1, 2, 3\). To verify that the latter
case is impossible, consider the transformation \( g_3 \circ g_2 \circ g_1^n \). It follows from (3.3) that
\[
\chi(g_3 \circ g_2 \circ g_1^n) = |2 - \omega_2 \omega_3 + n(\omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3)|.
\]
From \( \chi(g_3 \circ g_2 \circ g_1^n) = \chi(j(g_3) \circ j(g_2) \circ j(g_1)^n) \) we infer that \( \omega_i \) and \( \omega_i' \) have the same sign. Hence we have \( j(g_i) = g_i \) for \( i = 1, 2, 3 \) and it follows that \( j = \text{id} \) as asserted.

Suppose finally that \( \text{Fix}(E) \) contains at least four points. Similarly as in (A) we can show that there is a Möbius transformation \( h \) such that \( h(P(g_i^{1})) = P(j(g_i)^{1}) \) for all \( g_i \in E \). From the case of three generating transformations it then follows that \( h \) induces \( j \).

About results related to Theorem 4 we refer to \([2]\) pp. 150—151.

If we only know that \( \varphi(g_i) = \varphi(j(g_i)) \) for all \( g_i \in E \), then \( j \) need not be induced by any Möbius transformation. This is seen considering e.g. the case when the Riemann surfaces corresponding to \( G \) and \( G' \) are compact.

§ 4. The boundary mapping of an isomorphism with a finite dilatation

Let \( G \) and \( G' \) be covering groups acting on the upper half-plane \( H \). A homeomorphism \( \varphi: R \cup \{\infty\} \to R \cup \{\infty\} \) is called a boundary mapping of an isomorphism \( j: G \to G' \) if \( \varphi \circ g = j(g) \circ \varphi \) for all \( g \in G' \). Thus we have \( \varphi \circ (P(g)) = P(j(g)) \) for \( g \in G \). (Therefore, if \( G \) and \( G' \) are of the first kind, an isomorphism \( j: G \to G' \) has at most one boundary mapping.) In this section we consider the interrelation between \( \varphi \) and \( \delta(j) \).

Let \( K_1 \) and \( K_2 \) be circles or straight lines and \( \psi: K_1 \to K_2 \) a homeomorphism. Let \( z_0 \in K_1 \) be a finite point such that \( \psi(z_0) \neq \infty \). We say that \( \psi \) is Hölder continuous with the exponent \( \alpha \), \( 0 < \alpha \leq 1 \), at \( z_0 \) if there is a constant \( A \geq 1 \) and a neighborhood \( U \subset K_1 \) of \( z_0 \) such that
\[
(1/A) \, z^{-1/(1+\alpha)} \leq |\psi(z) - \psi(z_0)| \leq A \, z^{-1/(1+\alpha)}
\]
for all \( z \in U \). The mapping \( \psi \) is Hölder continuous with the exponent \( \alpha \) at the point \( \infty \) or at a point \( z_0 \) where \( \psi(z_0) = \infty \) if \( \psi(1/z) \) has this property at the origin or \( 1/\psi(z) \) at \( z_0 \), respectively. If \( \psi \) is Hölder continuous with the exponent \( \alpha = 1 \) at \( z_0 \), then we say that \( \psi \) is a Lipschitz mapping at \( z_0 \).

The Hölder continuity of \( \psi \) is invariant under Möbius transformations, i.e., if \( h_1 \) and \( h_2 \) are Möbius transformations and \( \psi \) is Hölder continuous with the exponent \( \alpha \) at \( z_0 \), then the same is true of \( h_2 \circ \psi \circ h_1^{-1} \) at the point \( h_1(z_0) \).
Theorem 5. Suppose that \( \varphi \) is a boundary mapping of an isomorphism \( j: G \to G' \). Let \( B(j) \) be the set of the real numbers \( \alpha \), \( 0 < \alpha \leq 1 \), such that \( \varphi \) is H"older continuous with the exponent \( \alpha \) at the fixed points of all hyperbolic elements of \( G \). Then \( B(j) \neq \emptyset \) if and only if \( \delta(j) < \infty \). If \( B(j) \neq \emptyset \), then we have \( 1/\delta(j) = \max \alpha \), \( \alpha \in B(j) \).

Proof. Let \( g \in G \) be hyperbolic. From the existence of \( \varphi \) we conclude that \( j(g) \) is also hyperbolic. Since the \( \alpha \)-H"older continuity of \( \varphi \) at a point is invariant under M"obius transformations, we may assume that

\[
N(g) = N(j(g)) = 0, \quad P(g) = P(j(g)) = \infty
\]

and \( \varphi(1) = 1 \).

Suppose that \( \alpha \in B(j) \). Then there is an \( A \geq 1 \) such that

\[
|\varphi(g^{-n}(1)) - \varphi(0)| = |\varphi(g^{-n}(1)) - j(g)^{-n}(1)| = \alpha(j(g))^{-n} \leq A |g^{-n}(1) - 0|^\alpha = A \alpha(g)^{-n\alpha}
\]

from some \( n = n_0 \) on. Thus \( \alpha(j(g)) \geq A^{-1} \alpha(g)^{\alpha} \), and letting \( n \to \infty \) we obtain \( \alpha(j(g)) \geq \alpha(g)^{\alpha} \). Similarly it follows that \( \alpha(g) \geq \alpha(j(g))^{\alpha} \). Hence \( \delta(j) \leq 1/\alpha \).

Conversely, suppose that \( \alpha = \delta(j) < \infty \). Choose \( t \) such that \( 0 < t < 1 \) and let \( n \) be the natural number for which \( 1/\alpha(g)^{n+1} \leq t < 1/\alpha(g)^n \). Since \( \varphi(1) = 1 \), we have \( 1/\alpha(j(g))^{n+1} \leq \varphi(t) < 1/\alpha(j(g))^n \). Hence

\[
\frac{\varphi(t)}{t^{1/\alpha}} \leq \frac{\alpha(g)^{(n+1)/\alpha}}{\alpha(j(g))^{n+1}} \leq \frac{\alpha(g)^{(n+1)/\alpha}}{\alpha(g)^{n/a}} = \alpha(g)^{1/\alpha},
\]

and similarly \( \varphi(t)/t^{1/\alpha} \geq 1/\alpha(g)^{\alpha} \). If \( -1 < t < 0 \), then we obtain \( |\varphi(t)|/|t|^{1/\alpha} \leq |\varphi(-1)|/\alpha(g)^{\alpha} \) and \( |\varphi(t)|/|t|^{1/\alpha} \geq |\varphi(-1)|/\alpha(g)^{\alpha} \). Hence

\[
1/\delta(j) \in B(j)
\]

and the first assertion is proved. Moreover, by the first part of the proof we have \( \delta(j) \leq 1/\alpha \) for all \( \alpha \in B(j) \). Thus \( 1/\delta(j) = \max \alpha \), \( \alpha \in B(j) \).

As in Theorem 4, let \( E = \{g_1, g_2, \ldots\} \) be a set of generators of \( G \) and let \( F \) be the set of the transformations \( (g_1^a \circ g_2^b) \circ (g_3^c \circ g_4^d)^e \). Then we have the following generalization for Theorem 5.1 in [3]:

Theorem 6. If an isomorphism \( j: G \to G' \) has a boundary mapping which is a Lipschitz mapping at the points of \( \text{Fix}(F) \), then \( j \) is induced by a M"obius transformation.

Theorem 6 follows from Theorem 4 and the proof of Theorem 5.

The following theorem shows that the H"older continuity of a boundary mapping \( \varphi \) of \( j: G \to G' \) at the fixed points of the parabolic elements of \( G \) does not depend on \( \delta(j) \).
Theorem 7. If \( g \in G \) is parabolic, then all boundary mappings of an isomorphism \( j : G \to G' \) are Lipschitz mappings at \( P(g) \).

Proof. We may assume that \( g \) and \( j(g) \) both are the transformation \( z \mapsto z/(z+1) \) and that \( q(\infty) = \infty \). Choose \( t \) such that \( 0 < t < 1 \) and let \( n \) be the natural number for which \( 1/(n+1) < t \leq 1/n \). Since \( g^n(\infty) = j(g)^n(\infty) = 1/n \), we have \( 1/(n+1) < q(t) \leq 1/n \). Therefore \( n/(n+1) \leq q(t)/t \leq (n+1)/n \), and it follows that \( t/2 \leq q(t) \leq 2t \).

Replacing \( g \) by \( g^{-1} \) we obtain \(|t|/2 \leq |q(t)| \leq 2|t| \) for \(-1 < t < 0 \).

By Theorem 4.1 in [3] we have \( \delta(j) \leq K \) if \( j \) is induced by a \( K \)-quasiconformal mapping \( f : H \to H \). This theorem is a special case of the following more general result:

Theorem 8. Let \( \psi : R U \{\infty\} \to R U \{\infty\} \) be a boundary mapping of \( j : G \to G' \). If there is a \( K \)-quasiconformal mapping \( f : H \to H \) such that \( f(R U \{\infty\}) = \psi \), then \( \delta(j) \leq K \).

Proof. Let \( h \) and \( h' \) be Möbius transformations mapping \( H \) onto the unit disk such that \( f_1 = h' \circ f \circ h^{-1} \) fixes the origin. By Theorem II.3.2 in [1], the restriction of \( f_1 \) to the unit circle is Hölder continuous with the exponent \( 1/K \). Then the same holds true of \( \psi \) at every point of \( R U \{\infty\} \) and we have \( \delta(j) \leq K \) by Theorem 5.

Let \( \varphi : R U \{\infty\} \to R U \{\infty\} \) be an increasing homeomorphism fixing \( \infty \). If for an interval \( I \subset \mathbb{R} \) there is a constant \( \lambda \), \( 1 \leq \lambda < \infty \), such that

\[
1/\lambda \leq \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leq \lambda
\]

holds whenever \( x + t \in I \), we say that \( \varphi \) is \( \lambda \)-quasisymmetric on \( I \). The mapping \( \varphi \) is called \( \lambda \)-quasisymmetric if (4.1) holds for all \( x \) and \( t \). Note that \( \varphi \) is \( 1 \)-quasisymmetric if and only if \( \varphi \) is the restriction of a Möbius transformation \( z \mapsto az + b \) with \( a > 0 \) and \( b \) real.

If an isomorphism \( j : G \to G' \) has a \( \lambda \)-quasisymmetric boundary mapping \( \varphi \), then

\[
\delta(j) \leq \log 2/\log (1 + 1/\lambda)
\]

by Theorem 4.2 in [3]. On the other hand, there is a \( K \)-quasiconformal extension \( f : H \to H \) of \( \varphi \) with \( K = \min (8\lambda, 1) \) (see [1, II.6.5]). Hence we have \( \delta(j) \leq \min (8\lambda, 1) \) by Theorem 8. However, one can verify by calculation that \( \log 2/\log (1 + 1/\lambda) \leq \min (8\lambda, 1) \) for all values \( \lambda \geq 1 \). Hence Theorem 8 implies (4.2) only if a \( \lambda \)-quasisymmetric \( \varphi \) always has a \( \log 2/\log (1 + 1/\lambda) \)-quasiconformal extension \( f : H \to H \).

By the following theorem, (4.2) can be deduced also from the local \( \lambda \)-quasisymmetry of \( \varphi \).
Theorem 9. Let \( \varphi : R \cup \{ \infty \} \to R \cup \{ \infty \} \) be a boundary mapping of an isomorphism \( j : G \to G' \). If for every hyperbolic \( g \in G \) satisfying \( P(g) \neq \infty \) there is an interval \( I \ni P(g) \) on which \( \varphi \) is \( \lambda \)-quasisymmetric, then \( \delta(j) \leq \log 2/\log (1 + 1/\lambda) \).

Proof. Let \( g \in G \) be hyperbolic, \( P(g) \neq \infty \) and \( h, h' \) Möbius transformations fixing \( H \) such that \( h(P(g)) = h'(P(j(g))) = 0 \), \( h(N(g)) = h'(N(j(g))) = \infty \). For every \( \varepsilon > 0 \) there is an interval \( I \) containing the origin such that the mapping \( q_1 = h' \circ \varphi \circ h^{-1} \) is \((\lambda + \varepsilon)\)-quasisymmetric on \( I \). Then there are \( 1 \)-quasisymmetric mappings \( h_1 \) and \( h_1' \) fixing the origin such that \( q_1' = h_1' \circ q_1 \circ h_1^{-1} \) is \((\lambda + \varepsilon)\)-quasisymmetric on the closed unit interval. Replacing \( \varphi \) by \( q_1' \) and \( \lambda \) by \( \lambda + \varepsilon \) in the proof of Theorem 4.2 in [3] we can show that \( \kappa(g)^{1/\lambda} \leq \kappa(j(g)) \leq \kappa(g)^{\lambda} \) holds for

\[
a = \log 2/\log (1 + 1/(\lambda + \varepsilon)).
\]

Suppose that all boundary mappings of an isomorphism \( j : G \to G' \) are increasing and fix the point \( \infty \). To our knowledge, it is an open question whether \( \delta(j) < \infty \) then implies that \( j \) has a boundary mapping which is \( \lambda \)-quasisymmetric for some fixed \( \lambda \geq 1 \) in a neighborhood of the attracting fixed point of every hyperbolic element of \( G \). However, the following theorem tells that all boundary mappings of \( j \) have a quasisymmetry property at the fixed points of the parabolic elements of \( G \).

Theorem 10. Suppose that the transformation \( g_0 : z \mapsto z + 1 \) lies in \( G \cap G' \). Let \( \varphi : R \cup \{ \infty \} \to R \cup \{ \infty \} \) be a boundary mapping of an isomorphism \( j : G \to G' \) for which \( j(g_0) = g_0 \). If \( g \neq g_0 \) is a parabolic element of \( G \), \( x_0 = P(g) \) and \( a = \delta(j) < \infty \), then we have for all \( t > 0 \)

\[
\omega(g)^{-a} \leq \frac{\varphi(x_0 + t) - \varphi(x_0)}{\varphi(x_0) - \varphi(x_0 - t)} \leq \omega(g)^{a}.
\]

Proof: It means no restriction to consider only the case when \( \varphi \) is increasing. Using \( 1 \)-quasisymmetric mappings of the type \( z \mapsto z + b \) we normalize such that \( P(g) = P(j(g)) = 0 \). Then \( \omega(g) \), \( \omega(j(g)) \) and \( \omega(g_0) \) are not changed. We may assume that \( \omega(g) \) and \( \omega(j(g)) \) are positive. Then by Theorem 3, \( \omega(g)^{1/\alpha} \leq \omega(j(g)) \leq \omega(g)^{\alpha} \).

Let \( t > 1 \) and \( n \) be the natural number for which \( n \leq t < n + 1 \). From \( n = n^{\pm n}(0) = j(g_0)^{\pm n}(0) \) we infer that \( n \leq \pm \varphi(\pm t) < n + 1 \). It follows that \( n/(n + 1) \leq \varphi(t)/(\varphi(t) - t) \leq (n + 1)/n \), and we have

\[
1/2 \leq \varphi(t)/(\varphi(-t)) \leq 2.
\]

Let \( 1/\omega(g) < t \leq 1 \). Since \( g(\infty) = 1/\omega(g) \), we obtain

\[
1/\omega(j(g)) < \varphi(t) \leq 1,
\]

and similarly \(-1/\omega(j(g)) > \varphi(-t) \geq -1 \). Hence
\[
\omega(g)^{-a} \leq \varphi(t)/( - \varphi(-t)) \leq \omega(g)^a.
\]

Finally, let \(0 < t \leq 1/\omega(g)\) and \(n\) be the natural number for which \(1/((n + 1)\omega(g)) < t \leq 1/(n\omega(g))\). From \(g^{\pm n}(x) = 1/(\pm n\omega(g))\) it follows that \(1/((n + 1)\omega(j(g))) \leq \pm \varphi(\pm t) \leq 1/(n\omega(j(g)))\). Hence
\[
\frac{n\omega(j(g))}{(n + 1)\omega(j(g))} \leq \frac{\varphi(t)}{- \varphi(-t)} \leq \frac{(n + 1)\omega(j(g))}{n\omega(j(g))},
\]
and we conclude that \(1/2 \leq \varphi(t)/( - \varphi(-t)) \leq 2\).

Since \(\omega(g) \geq 4\) (cf. Remark in § 2), it follows that
\[
\omega(g)^{-a} \leq \varphi(t)/( - \varphi(-t)) \leq \omega(g)^a
\]
for all \(t > 0\). \(\square\)

Observe that Theorem 10 does not follow from Theorem 7.
References


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