ON VARIATIONAL INTEGRALS IN "THE BORDERLINE CASE"

BY

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1. Introduction

In this paper we study variation in \( I(v) = \int F(x, v(x), \nabla v(x)) \, dm_n(x) \)

in the borderline case \( F(x, y, z) \approx |z|^n \). The behavior of (1.1) is well-known for \( n = 2 \), see e.g. [3], [5]. Our main contribution is to show how conformal technics in \( \mathbb{R}^2 \) can be replaced by the use of quasiconformal mappings in \( \mathbb{R}^n \), \( n \geq 3 \). The main result, Theorem 4.1, is an extension of [5, Theorem 4.3.5, p. 111] which states that if \( G \) is a Lipschitz domain in \( \mathbb{R}^n \) and \( u_0 \in C(G) \cap W^1_n(G) \), then there exists \( u \in C(G) \cap W^1_n(G) \), \( u \mid \partial G = u_0 \mid \partial G \), and \( u \) minimizes (1.1) among all similar \( v \). We replace the condition \( \text{G Lipschitz} \) by \( \text{G quasiconformally collared} \). Our method is based on the fact that a quasiconformal mapping between two domains \( G \) and \( G' \) preserves the classes \( C(G) \cap W^1_n(G) \) and \( C(G') \cap W^1_n(G') \).

Section 2 contains notation and assumptions on \( F \). It also includes a lower-semicontinuity theorem relevant to our case and some basic properties of quasiconformally collared domains. Equicontinuity properties of monotone functions are studied in Section 3 and Section 4 contains the main theorem.

2. Notation and preliminaries

2.1. Notation. The real number system is denoted by \( \mathbb{R} \), \( \mathbb{R}^+ = \{ x \in \mathbb{R} \mid x \geq 0 \} \), and \( \bar{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{ \infty \} \). We let \( \mathbb{R}^n \), \( n \geq 1 \), denote the euclidean \( n \)-space with a fixed orthonormal bases \( e_1, \ldots, e_n \). For \( x \in \mathbb{R}^n \) and \( r > 0 \), \( B^n(x, r) \) denotes the open ball centered at \( x \) with radius \( r \), and \( S^{n-1}(x, r) = \partial B^n(x, r) \). We shall use the abbreviations \( B^n(r) = B^n(0, r) \), \( B^n = B^n(1) \), \( S^{n-1}(r) = S^{n-1}(0, r) \). \( H^+_n \) denotes the upper half-space \( \{ x \in \mathbb{R}^n \mid x_n > 0 \} \).

If \( A \subset \mathbb{R}^n \) we let \( C(A) \) denote the class of real valued continuous functions on \( A \). If \( A \subset \mathbb{R}^n \) is a Lebesgue measurable set and \( u : A \to \bar{\mathbb{R}}^+ \) a measurable function we let
\[ \int_A u \, dm \quad \text{or} \quad \int_A u(x) \, dm_n(x) \]
denote the integral of \( u \) over \( A \). The integral has values in \( \mathbb{R}^+ \cdot L^p(A) \), \( 1 \leq p < \infty \), denotes the Banach space of (equivalence classes of) measurable functions \( u : A \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\} \) with the norm
\[
\|u\|_{p,A} = \left( \int_A u^p \, dm_n \right)^{1/p} < \infty.
\]

Suppose that \( G \subseteq \mathbb{R}^n \) is a domain and \( 1 \leq p < \infty \). \( W^1_p(G) \) means the Sobolev space of all functions \( u \) in \( L^p(G) \) with the first generalized partial derivatives \( \partial_i u, 1 \leq j \leq n \), in \( L^p(G) \). We let \( \nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u) \). The norm in \( W^1_p(G) \) is denoted by
\[
\|u\|_{1,p,G} = \|u\|_{p,G} + \|\nabla u\|_{p,G} = \|u\|_{p,G} + \left( \sum_{i=1}^n |\partial_i u|^2 \right)^{1/2}_{p,G}.
\]

We shall make use of the following reflection principle in the Sobolev space. It can be easily proved by using the ACL-properties of functions in \( W^1_p(G) \), see e.g. [5, pp. 66–67].

**2.2. Lemma.** Suppose \( u \in W^1_p(B^+ \cap H^+_n) \cap C(B^+ \cap H^+_n) \). Let
\[
u^*(x) = u(x), \quad x \in B^+ \cap H^+_n = u(x - 2x_e) \in B^+ \cap B^+ \cap H^+_n.
\]
Then \( u^* \in C(B^+) \cup W^1_p(B^+) \) and
\[
2 \int_{B^+ \cap H^+_n} |\nabla u|^p \, dm_n = \int_{B^+} |\nabla u^*|^p \, dm_n.
\]

**2.3. Variational integrals.** Let \( G \subseteq \mathbb{R}^n \) be a bounded domain and \( F : G \times R \times R^+ \to \mathbb{R}^+ \). We shall make use of three sets of assumptions on \( F \):

\[
\begin{align*}
\text{Either} \\
&\begin{cases}
(i) \ F \text{ is continuous and into } \mathbb{R}^+.
&z \mapsto F(x,y,z) \text{ is convex for all } (x,y) \in G \times R.
&
(ii) \ F \text{ is Borel-measurable.}
&\text{for every } \varepsilon > 0 \text{ there exists a compact set } S \text{ in } G \text{ such that}
&\quad m_n(G \setminus S) < \varepsilon \text{ and } F \cdot S \times R^{n+1} \text{ is continuous.}
&\quad F(x,\cdot) : R^{n+1} \to \mathbb{R}^+ \text{ is convex for a.e. } x \in G.
&\end{cases}
\end{align*}
\]

or
\[
(2.4) \quad \begin{cases}
\text{For a.e. } x \in G \ F(x,y,z) \geq F(x,y,0) \text{ for all } y \in R.
\end{cases}
\]

For a.e. \( x \in G \ F(x,y,0) \geq F(x,y,0) \text{ for all } y \in R.\)
There exists $C > 0$ and $w \in L^1(G)$ such that for a.e. $x \in G$
\begin{equation}
F(x, y, z) \geq C|z|^a - w(x)
\end{equation}
for all $(y, z) \in R \times R^n$ (the borderline case).

2.7. Example. If $F(x, y, z) = |z|^a$ then $F$ satisfies (2.4)–(2.6).

We shall use the abbreviation
\begin{equation}
I(u) = \int_G F(x, u(x), \nabla u(x)) \, dm_n(x)
\end{equation}
if $u \in W^1_n(G)$. The following lower-semicontinuity theorem will be used several times.

2.8. Lemma. Suppose that $F$ satisfies (2.4). If $(u_i)$ is a bounded sequence in $W^1_p(G)$ and $u_i \rightarrow u \in W^1_p(G)$ in $L^p(G)$, then
\begin{equation}
I(u) \leq \liminf_{i \rightarrow \infty} I(u_i).
\end{equation}

Proof. If $F$ satisfies (2.4) (i), then (2.9) is a special case of [9, Theorem 13] (see also [5, Theorem 4.1.2] and [8, Theorem 1.3]) and if $F$ satisfies (2.4) (ii), then (2.9) follows from [8, Theorem 1.2].

2.10. Remark. Both sides of (2.9) may be infinite.

2.11. Quasiconformally collared domains. Let $G \subset R^n$ be a bounded domain. $G$ is called quasiconformally collared, if every $x \in \partial G$ has arbitrary small neighborhoods $U$ such that $U \cap G$ can be mapped quasiconformally onto $B^n \cap H^+_n$. This means that there is a homeomorphism $f: U \cap G \rightarrow B^n \cap H^+_n$ such that the coordinate functions of $f$ belong to $W^1_n(U \cap G)$ and
\begin{equation}
|f'(x)|^a \leq KJ(x, f) \text{ a.e. in } G
\end{equation}
for some $K \geq 1$, for more details see [1] or [10].

2.12. Remarks. (a) A Lipschitz-domain is quasiconformally collared since every bi-Lipschitz homeomorphism is quasiconformal but not vice versa, see Remark 4.2 (b).

(b) For $n = 2$, $G$ is quasiconformally collared if and only if $\partial G$ consists of a finite number of disjoint Jordan curves.

2.13. Remark. Instead of the above definition for a quasiconformally collared domain we may also use the equivalent definition: Every $x \in \partial G$
has arbitrary small neighborhoods $U$ such that there exists a homeomorphism $f: \overline{U \cap G} \to B^n \cap \partial H^n_+\uparrow$ with the properties (1) $f|U \cap G$ is quasiconformal, (2) $f(x) = 0$, and (3) $f(\partial G \cap U) = B^n \cap \partial H^n_+$. This can be seen as follows: Let $U$ be as in the first definition. Then $f$ can be extended to a homeomorphism of $U \cap G$ [7, Lemma 2.3]. Denote by $l'$ the $x$-component of $\partial G \cap U$. Then $fV$ is open on $\partial (B^n \cap H^n_+)$ and clearly we may assume $fV \subset \partial H^n_+$ and $f(x) = 0$. Thus $U' = f^{-1}(B^n(r) \cap H^n_+)$ for sufficiently small $r$ can be used in the second definition with an obvious modification of $f$.

3. Monotone functions

Let $A \subset \mathbb{R}^n$, $u \in C(A)$, and $S \subset A$. The oscillation of $u$ on $S$ is denoted by

$$\omega(f, S) = \sup_{x \in S} f(x) - \inf_{x \in S} f(x).$$

Suppose that $G \subset \mathbb{R}^n$ is a domain. A function $u \in C(G)$ is called monotone (in the sense of Lebesgue) if

$$\sup_{x \in D} f(x) = \sup_{x \in D} f(x) \quad \text{and} \quad \inf_{x \in D} f(x) = \inf_{x \in D} f(x)$$

for every domain $D \subset G$.

3.1. Lemma. ([6, Lemma 4.1], [1, Lemma 1]) Let $u \in C^1(G)$ and let $B^n(x, r) \subset G$. Then

\begin{equation}
(3.2) \quad \omega(u, S^{n-1}(x, r))^n \leq Ar \int_{S^{n-1}(x, r)} |\nabla u|^n dS
\end{equation}

where $A = A(n)$ and $dS$ denotes the $(n-1)$-measure on $S^{n-1}(x, r)$.

By approximation Fubini's theorem implies

3.3. Corollary. Let $u \in W^1_n(G) \cap C(G)$. If $B^n(x_0, r_0) \subset G$, then (3.2) holds for $u$ for a.e. $r \in (0, r_0)$.

3.4. Theorem. Suppose that $G$ is a bounded quasiconformally collared domain and that $\mathcal{U}$ is a family of functions $u$ on $G$ such that

(1) $u \in C(G) \cap W^1_n(G)$.
(2) $u$ is monotone.
(3) $\|\nabla u\|_{n, G} \leq M$ for all $u \in \mathcal{U}$.

Then $\mathcal{U}$ is equicontinuous if and only if $\mathcal{U} | \partial G$ is equicontinuous.
Proof. The only if part is trivial. For the other direction we split the proof into two parts.

A. Equicontinuity in \( G \). Let \( x \in G \) and pick \( \delta > 0 \) such that \( B^n(x, \delta) \subset G \). Choose \( \alpha > \delta \) so that \( B^n(x, \alpha) \subset G \). Since \( u \in \mathcal{M} \) is monotone, Corollary 3.3 implies for a.e. \( r \in (\delta, \alpha) \)

\[
\omega(u, B^n(x, \delta))^n \leq \omega(u, B^n(x, r))^n = \omega(u, S^{n-1}(x, r))^n \leq A r \int_{S^{n-1}(x, r)} |\nabla u|^n \, dS.
\]

Multiplying by \( r^{-1} \) and integrating from \( \delta \) to \( \alpha \) yields

\[
\omega(u, B^n(x, \delta))^n \log \frac{\alpha}{\delta} \leq A \int_{\delta}^{\alpha} |\nabla u|^n \, dS \leq A M^n.
\]

This shows that \( \mathcal{M} \) is equicontinuous at \( x \).

B. Equicontinuity at points on \( \partial G \). Let \( x \in \partial G \). By Remark 2.12 there exists a neighborhood \( U \) of \( x \) and a homeomorphism \( f : U \cap G \to B^n \cap H^+_n \) with properties (1)—(3) in 2.11. For \( r \in (0, 1) \) let \( D(r) \) denote the disk \( B^n(r) \cap \partial H^+_n \).

It is enough to show that \( \mathcal{M} f^{-1} = \{ v \mid v = u \circ f^{-1}, u \in \mathcal{M} \} \) is equicontinuous at \( 0 \). Suppose that \( \mathcal{M} f^{-1} \) is not equicontinuous at \( 0 \). Then for all \( \delta \in (0, 1) \) there exists \( v \in \mathcal{M} f^{-1} \) such that

\[
(3.5) \quad \omega(v, B^n(\delta) \cap H^+_n) \geq \varepsilon > 0.
\]

Pick \( r_0 > 0 \) so small that

\[
(3.6) \quad \omega(v, D(r_0)) < \varepsilon/2
\]

for all \( v \in \mathcal{M} f^{-1} \). This is possible since \( \mathcal{M} \big| \partial G \) is equicontinuous and hence \( \mathcal{M} f^{-1} \big| B^n \cap \partial H^+_n \) is equicontinuous. Since every \( u \in \mathcal{M} \) is monotone and \( f \) is a homeomorphism, \( v = u \circ f^{-1} \) is also monotone, and (3.5) and (3.6) imply

\[
\omega(v, S^{n-1}(r) \cap H^+_n) \geq \varepsilon/2
\]

for all \( r \in [\delta, 1] \). On the other hand \( v \in C(B^n \cap H^+_n) \cap W^1(B^n \cap H^+_n) \) [11]. Hence \( v \) has an extension \( v^* \) to \( \bar{B}^n \) described in Lemma 2.2. Now Corollary 3.3 yields for a.e. \( r \in [\delta, 1] \)

\[
(\varepsilon/2)^n \leq \omega(v, S^{n-1}(r) \cap H^+_n)^n = \omega(v^*, S^{n-1}(r))^n \leq A r \int_{S^{n-1}(r)} |\nabla v^*|^n \, dS.
\]

Multiplying by \( r^{-1} \) and integrating from \( \delta \) to \( 1 \) gives
(3.7) \((e/2)^n \log 1/\delta \leq A \int_{B^n} |\nabla v|^n \, dm_n \leq 2A \int_{B^n \cap H_n^+} |\nabla v|^n \, dm_n\).

Since \(f \mid U \cap G\) is quasiconformal, \(f^{-1} \mid B^n \cap H_n^+\) is also quasiconformal for some \(K \geq 1\) [1, Theorem 4]. This gives

\[
\int_{B^n \cap H_n^+} |\nabla v|^n \, dm_n \leq \int_{B^n \cap H_n^+} |\nabla u(f^{-1}(y))|^n \|(f^{-1})'(y)\|^n \, dm_n(y)
\leq K \int_{B^n \cap H_n^+} |\nabla u(f^{-1}(y))|^n J(y, f^{-1}) \, dm_n(y)
= K \int_{U \cap G} |\nabla u|^n \, dm_n \leq K \int_{G} |\nabla u|^n \, dm_n \leq K M^n.
\]

This estimate together with (3.7) implies

\((e/2)^n \leq 2AKM^n (\log 1/\delta)^{-1}\)

which is a contradiction for \(\delta\) small enough.

3.8. Remark. Theorem 3.4 is an extension of [5, Theorem 4.3.4].

The idea in the next theorem is due to Lebesgue [3].

3.9. Theorem. Suppose that \(F\) satisfies (2.4) and (2.5). Let \(u_0 \in C(\bar{G}) \cap W^1_0(G)\). Then there exists \(u \in C(\bar{G}) \cap W^1_0(G)\) such that \(u\) is monotone, \(u \mid \partial G = u_0 \mid \partial G\), and \(I(u) \leq I(u_0)\).

Proof. Using Lebesgue's method [3] (see also [1], [5], and [6]) it is possible to construct a sequence of functions \(u_i, i = 0, 1, \ldots,\) such that (i) \(u_i \in C(\bar{G})\), (2) \(u_i \mid \partial G = u_0 \mid \partial G\), (3) \(u_{i-1} = u_i\) except on an open set \(V_i \subset G\) and \(u_i\) is a constant on the components of \(V_i\), and (4) \(u_i\) converges uniformly on \(\bar{G}\) to a monotone function \(u\). From (3) it follows that \(o(u_i, \triangle) \leq o(u_0, \triangle)\) on each line segment \(\triangle \subset G\). This implies that each \(u_i\) is ACL since \(u_0\) is. Moreover, if \(U\) is a component of \(V_i\), then \(u_i\) is a constant on \(\bar{U}\), and hence by [4, p. 254] \(\nabla u_i = 0\) a.e. in \(\bar{U}\). This implies

(3.10) \(\int_G |\nabla u_i|^n \, dm_n \leq \int_G |\nabla u_0|^n \, dm_n < \infty\),
and since $F$ satisfies (2.5), it also follows that

\[(3.11) \quad I(u_i) \leq I(u_0).\]

Since $u_i$ is ACL and (3.10) holds, $u_i \in W^1_n(G)$. Furthermore, by (4) and (3.10) $(u_i)$ is a bounded sequence in $W^1_n(G)$ converging in $L^p(G)$ to $u$. Consequently $u \in W^1_n(G)$. Finally Lemma 2.8 and (3.11) yield

\[I(u) = \lim_{i \to \infty} I(u_i) \leq I(u_0).\]

4. Main theorem

4.1. Theorem. Suppose that $G$ is a bounded quasiconformally collared domain and $F$ satisfies (2.4)–(2.6). Let $u_0 \in C(\tilde{G}) \cap W^1_n(G)$. Then there exists $u \in C(\tilde{G}) \cap W^1_n(G)$ such that $u \mid \partial G = u_0 \mid \partial G$ and $u$ minimizes the integral $I(\nu)$ among all similar $\nu$.

Proof. Let

\[\mathcal{J} = \{v \in C(\tilde{G}) \cap W^1_n(G) \mid v \mid \partial G = u_0 \mid \partial G\}\]

and denote

\[I_0 = \inf_{v \in \mathcal{J}} I(v).\]

If $I_0 = +\infty$, we may take $u = u_0$. Suppose $I_0 < \infty$. Then there exists a sequence $(u_i)$, $u_i \in \mathcal{J}$, such that

\[(4.2) \quad I(u_i) \to I_0.\]

We may assume

\[(4.3) \quad I_0 \leq I(u_i) < I_0 + 1.\]

By Theorem 3.9 we can replace $(u_i)$ by $(u_i^*)$ such that $u_i^* \in \mathcal{J}$, $u_i^*$ is monotone, and $u_i^*$ satisfies both (4.2) and (4.3). Then (2.6) and (4.3) imply

\[(4.4) \quad \|\nabla u_i^*\|_{n, G}^n \leq (I_0 + 1 + \|w\|_{1, \infty})/C, \quad i = 1, 2, \ldots.\]

Hence by Theorem 3.4 $(u_i^*)$ is equicontinuous and since the functions $u_i^*$ are monotone, $(u_i^*)$ is also bounded. By Ascoli’s theorem there exists a subsequence $(u_j^*)$ converging uniformly on $\tilde{G}$ to $u \in C(\tilde{G})$. Since, by (4.4), the sequence $(u_j^*)$ is bounded in $W^1_n(G)$ and $u_j^* \to u$ in $L^p(G)$, $u \in W^1_n(G)$. Thus $u \in \mathcal{J}$. Finally Lemma 2.8 implies
\[ I_0 = \lim_{j \to \infty} I(u_j^*) \geq I(u) \geq I_0. \]

This completes the proof.

4.2. Remarks. (a) Theorem 4.1 (and Theorems 3.4 and 3.9) can be extended to vector valued functions \( u : G \to \mathbb{R}^m \).

(b) For instance a smooth domain in \( \mathbb{R}^n, n \geq 3 \), with an outward directed spire is quasiconformally collared (for details see [2]). However, such a domain is not a Lipschitz-domain.

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References


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