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INTEGRATION IN A SPACE OF MEASURES

BY

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Preface

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Introduction

When studying analytic functions, Ahlfors and Beurling [1] introduced the concept of extremal length $\lambda(\Gamma)$ for path families Γ of the complex plane. This was defined by

$$\lambda(\Gamma) = \sup_{\varrho} \frac{\left(\inf\limits_{\gamma \in \Gamma} \int\limits_{\gamma} \varrho ds\right)^2}{\int\limits_{R^2} \varrho^2 dm},$$

where supremum is taken over all non-negative Borel functions ϱ such that the numerator and denominator are not simultaneously 0 or ∞. The significance of the extremal length is largely due to its invariance under conformal mappings.

The number $M(\Gamma) = \lambda(\Gamma)^{-1}$ is called the modulus of the path family Γ. Fuglede [2] defined for measure families E a p-modulus $M_p(E)$, which is a generalization of the modulus $M(\Gamma)$. The number p, $1 \leq p < \infty$, corresponds to the exponent 2 in the definition of $M(\Gamma)$. The modulus $M_n(\Gamma)$ plays an important role in the theory of n-dimensional quasiconformal mappings, and it has turned out to be more convenient than the corresponding extremal length. The basic properties of M_p are given in section 1.

In section 2 we define an integral with respect to the modulus M_p. This is motivated by the fact that M_p is an outer measure (Theorem 1.2). However, it is useless to define this modulus integral in the usual way by means of measurable subfamilies, because in many cases there are too few of these (cf. [3] and [5]). The definition of the integral is based on the countable additivity of M_p in the case of separate subfamilies (Theorem 1.4).

In section 3 we consider the basic properties of the modulus integral. In section 4 we define measurable families and thus obtain an integral by the usual Lebesgue definition. This, when it exists, turns out to be equal to the modulus integral. The last section deals with the modulus integral in path families.

Notation

We use the following notation: The set of positive integers is denoted by N. The real axis is R^1, $\bar{R} = R^1 \cup \{\infty\} \cup \{-\infty\}$ is the extended real number system and R^n, $n \geq 2$, is the n-dimensional euclidean space. If $x = (x_1, \ldots, x_n) \in R^n$, its norm is $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. If A and B are subsets of R^n, the distance between them is $d(A, B) = \inf\{|x-y| \mid x \in A, y \in B\}$. By a domain in R^n we mean a non-empty, open and connected subset of R^n.

If the space under consideration is R^n, m is the n-dimensional Lebesgue measure and m_k, $1 \leq k \leq n-1$, the k-dimensional Lebesgue measure. The corresponding outer measures are m^* and m_k^*.

If m is a measure in an arbitrary space X and $p \geq 1$, we denote by L^p, $L^p(X)$ or $L^p(m)$ the set of all m-measurable functions $f: X \to \bar{R}$ for which $|f|^p$ is integrable over X.

The characteristic function of a set A is denoted by χ_A.

1. The modulus of a measure family

Let X be an arbitrary set and m a measure in X. We denote by M the family of all measures in X whose σ-algebra of definition contains that of m. These concepts will be kept fixed throughout the whole paper. In the following, we shall consider measure families which all are subfamilies of M.

If $E \subset M$, we denote by $F(E)$ the set of all non-negative, m-measurable functions $f: X \to \bar{R}$ such that

$$\int_X f d\mu \geq 1$$

for all $\mu \in E$. For each real number $p \geq 1$, the *p-modulus* $M_p(E)$ of the measure family E is defined by

$$M_p(E) = \inf_{f \in F(E)} \int_X f^p dm . \tag{1.1}$$

If $F(E) = \emptyset$, then $M_p(E) = \infty$. This case occurs only when the measure $\mu \equiv 0$ belongs to E. Otherwise $F(E)$ contains at least the function $f \equiv \infty$.

The above definition is due to Fuglede [2]. He considers only σ-finite measures. This restriction is, however, unnecessary, since the results of the theory of measure and integral which are involved hold for arbitrary measures (see [4]).

In the following theorems we give the basic properties of the modulus. Fuglede [2] has proved the results in Theorems 1.2—1.6 and Ziemer [7] and [8] Theorem 1.7. Unless otherwise stated, we shall always assume that $p \geq 1$.

1.2. Theorem. *The modulus* M_p *is an outer measure in* M. *That is,*

(1) $M_p(\emptyset) = 0$,

(2) *If* $E \subset E'$, *then* $M_p(E) \leq M_p(E')$,

(3) $M_p(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} M_p(E_i)$.

A measure family E' is said to *minorize* a family E if for each measure $\mu \in E$ there is a measure $\mu' \in E'$ such that $\mu' \leq \mu$ (that is, $\mu'(A) \leq \mu(A)$ for all m-measurable sets A). We shall then write $E' < E$.

1.3. Theorem. *If* $E' < E$, *then* $M_p(E) \leq M_p(E')$.

Families $E_1, E_2, \ldots$ are said to be *separate* if there are disjoint m-measurable sets $S_1, S_2, \ldots$ such that $\mu(X \setminus S_i) = 0$ for all $\mu \in E_i$.

1.4. Theorem. *If the families* $E_1, E_2, \ldots$ *are separate, then*

$$M_p(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} M_p(E_i).$$

A family $E \subset M$ is called p*-exceptional* if $M_p(E) = 0$. A property concerning measures of a family $E \subset M$ is said to hold for *almost every* $\mu \in E$ *of order* p (abbreviated: p—*a.e.* $\mu \in E$) if the family of all measures $\mu \in E$ for which the property fails is p-exceptional.

1.5. Theorem. *A family* E *is* p*-exceptional if and only if there exists a function* $f \in L^p(m)$ *such that* $\int_X f d\mu = \infty$ *for all* $\mu \in E$.

1.6. Theorem. *If a sequence* (f_j), $f_j \in L^p(m)$, *converges to a function* f *in* L^p*-metric, then there is a subsequence* (f_{j_i}) *such that*

$$\lim_{i \to \infty} \int_X |f_{j_i} - f| \, d\mu = 0 \quad \textit{for } p\text{—}a.e. \ \mu \in M.$$

1.7. Theorem. *If* $p > 1$ *and* $E_1 \subset E_2 \subset \ldots$, *then*

$$M_p(\bigcup_{i=1}^{\infty} E_i) = \lim_{i\to\infty} M_p(E_i) .$$

In Example 1.10 we shall show that the preceding theorem is invalid if $p = 1$.

1.8. *The modulus of a path family.* The most important special case of the modulus of a measure family is the modulus of a path family. By a *path* we mean a continuous mapping $\gamma : I \to R^n$, where I is a closed interval of the real axis. (The restriction to closed intervals is only for convenience. Allowing I to be open or half-open would not cause any essential changes.)

The *locus* $|\gamma|$ *of a path* $\gamma : I \to R^n$ is the image set $\gamma(I) \subset R^n$, and the *locus* $|\Gamma|$ *of a path family* Γ is the set $\bigcup_{\gamma \in \Gamma} |\gamma|$. A path $\gamma : I \to R^n$ is a *segment* if it is of the form $\gamma(t) = ta + b$, where a , $b \in R^n$. It is parallel to x_k -axis, $1 \leq k \leq n$, if only the k : th coordinate of a is different from zero. A path γ is *rectifiable* if it is of bounded variation. The total variation of γ is called its *length* and is denoted by $l(\gamma)$. The *normal representation* of a rectifiable path γ (or the parametrization of γ by means of its arc length) is denoted by γ° ; see [6]. Let A , B and C be subsets of R^n . A path $\gamma : [a , b] \to R^n$ is said to *join* A and B in C if one of the end points $\gamma(a)$, $\gamma(b)$ belongs to A and the other to B and $\gamma(t) \in C$ for $a < t < b$.

To each rectifiable path γ , we assign a measure μ_γ , defined by

$$\mu_\gamma(A) = m_1(\gamma^{\circ-1}(A))$$

for all Borel sets $A \subset R^n$. If $f : R^n \to \bar{R}$ is a non-negative Borel function, then

$$\int_{R^n} f d\mu_\gamma = \int_\gamma f ds .$$

Let Γ be a family of rectifiable paths in R^n . In the definition (1.1), we choose $X = R^n$ and as the measure m the Borel measure, that is, the restriction of the Lebesgue measure to the σ -algebra of Borel sets and define the p -modulus $M_p(\Gamma)$ of Γ by

$$M_p(\Gamma) = M_p(\{\mu_\gamma \mid \gamma \in \Gamma\}) .$$

This is equivalent to

$$M_p(\Gamma) = \inf_{f \in F(\Gamma)} \int_{R^n} f^p dm ,$$

where $F(\Gamma)$ is the set of all non-negative Borel functions $f : R^n \to \bar{R}$ such that $\int_\gamma f ds \geq 1$ for $\gamma \in \Gamma$ and m is the Lebesgue measure. We may also replace R^n by any Borel set X such that $|\Gamma| \subset X$.

Let $B \subset R^{n-1}$ and $h > 0$. The set

$$H = \{(x, y) \in R^n \mid x \in B, 0 < y < h\}$$

is called a *cylinder*, the sets $\{(x, 0) \mid x \in B\}$ and $\{(x, h) \mid x \in B\}$ its *bases* and the number h its *height*. If B is a Borel set, we call H a *Borel cylinder*. We denote by $\Delta(H)$ the family of all segments parallel to x_n-axis which join the bases of H and by $\Delta_a(H)$ the family of all rectifiable paths which join the bases of H in H. They have the same modulus (cf. [6] 7.2 and [5] Satz 2):

$$M_p(\Delta(H)) = M_p(\Delta_a(H)) = m_{n-1}^*(B)h^{1-p}. \tag{1.9}$$

If R is a rectangle in the plane R^2, we denote by $\Delta_1(R)$ the family of the segments parallel to x-axis which join the vertical sides of R and by $\Delta_2(R)$ the family of the corresponding segments parallel to y-axis.

We next show by an example that Theorem 1.7 does not hold if $p = 1$.

1.10. Example. Let $R_k = \{(x, y) \mid 0 < x < 1/k, 0 < y < 1\}$, $\Gamma_k' = \Delta_1(R_k)$ and $\Gamma_k = \bigcup_{i=1}^{k} \Gamma_i'$ for all $k \in N$. Then $\Gamma_1 \subset \Gamma_2 \subset \ldots$, $\Gamma_k' \subset \Gamma_k$ and $\Gamma_k' < \Gamma_k$. Therefore $M_1(\Gamma_k) = M_1(\Gamma_k') = 1$ and also $\lim_{k\to\infty} M_1(\Gamma_k) = 1$. Let $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$. We show that $M_1(\Gamma) = \infty$.

Let f be a non-negative Borel function such that $\int_{R_1} f dm < \infty$. Then by the absolute continuity of integral, $\lim_{k\to\infty} \int_{R_k} f dm = 0$. If we choose k so that $\int_{R_k} f dm < 1$, it follows from Fubini's theorem that $\int_0^{1/k} f(x, y) dx < 1$ for some $y \in [0, 1]$, which means that $f \notin F(\Gamma)$. Hence $\int_{R_1} g dm = \infty$ for all $g \in F(\Gamma)$ and $M_1(\Gamma) = \infty$.

2. The modulus integral

Let $E \subset M$ be a measure family. By a *separate partition* of E we mean any finite collection $D = \{E_1, \ldots, E_k\}$ of separate subfamilies E_i of E. If u is a non-negative function $E \to \bar{R}$, we define the p-*modulus integral* $\int_E u dM_p$ of u over E by

$$\int_E u dM_p = \sup_D \sum_{i=1}^{k} \inf u(E_i) M_p(E_i) , \tag{2.1}$$

where the supremum is taken over all separate partitions of E . Here we use the agreement $0 \cdot \infty = \infty \cdot 0 = 0$. If u may also have negative values, we write $u = u^+ - u^-$, where $u^+ = \sup(u , 0)$ and $u^- = \sup(-u , 0)$. The integral $\int_E u dM_p$ is defined by

$$\int_E u dM_p = \int_E u^+ dM_p - \int_E u^- dM_p , \tag{2.2}$$

if at least one of the integrals on the right is finite. Otherwise $\int_E u dM_p$ is not defined.

We denote by $D(E , p)$ the set of all functions u for which $\int_E u dM_p$ is defined. Then $D(E , p)$ contains all non-negative functions defined on E , and $E' \subset E$ implies $D(E , p) \subset D(E' , p)$.

The modulus integral is not an integral in the Lebesgue sense. The definition (2.1) resembles the definition of the lower Lebesgue integral, but there is a noticeable difference. In the definition of the lower integral, the whole set of integration is partitioned into measurable subsets, while the definition (2.1) may be interpreted in such a way that an arbitrary subfamily of E is partitioned into measurable parts, in a sense (cf. section 4). Thus one often obtains much more admissible partitions and a greater value to the integral.

2.3. Example. A Dirac measure δ_x associated with $x \in X$ is defined by $\delta_x(B) = \chi_B(x)$ for all $B \subset X$. If $A \subset X$, we denote $E(A) = \{\delta_x \mid x \in A\}$. If A is m -measurable, then $M_p(E(A)) = m(A)$ ([2] p. 176). We show that this connection holds even for integrals.

Let f be a non-negative, m -measurable function defined on A and put $u(\delta_x) = f(x)$ for $x \in A$. If $\{A_1 , \ldots , A_k\}$ is an m -measurable partition of A , that is, $A = \bigcup_{i=1}^{k} A_i$ and the sets A_i are m -measurable and disjoint, then $\{E(A_1) , \ldots , E(A_k)\}$ is a separate partition of $E(A)$. Hence

$$\sum_{i=1}^{k} \inf f(A_i) m(A_i) = \sum_{i=1}^{k} \inf u(E(A_i)) M_p(E(A_i)) \leq \int_{E(A)} u dM_p$$

and

$$\int_A fdm \leq \int_{E(A)} udM_p \,.$$

On the other hand, let $\{E(A_1), \ldots, E(A_k)\}$ be any separate partition of $E(A)$. Then there exist disjoint, m-measurable sets $S_1, \ldots, S_k$ such that $A_i \subset S_i \subset A$. Since f is m-measurable, the sets $B_i = \{x \in A \mid f(x) \geq \inf f(A_i)\} \cap S_i$ are m-measurable, and $A_i \subset B_i \subset S_i$. Moreover, $\inf f(A_i) = \inf f(B_i)$. Hence $\inf u(E(A_i)) = \inf u(E(B_i))$ and thus

$$\sum_{i=1}^{k} \inf u(E(A_i))M_p(E(A_i)) \leq \sum_{i=1}^{k} \inf u(E(B_i))M_p(E(B_i)) =$$

$$\sum_{i=1}^{k} \inf f(B_i)m(B_i) \leq \int_A fdm \,.$$

This gives the opposite inequality and therefore

$$\int_{E(A)} udM_p = \int_A fdm \,.$$

2.4. Example. Let $H = \{(x, y) \in R^n \mid x \in B, 0 < y < h\}$ be a Borel cylinder, $\Gamma = \Delta(H)$, f a non-negative Borel function defined on H and $u(\gamma) = \int_\gamma fds$ for $\gamma \in \Gamma$. We compute the integral $\int_\Gamma udM_p$.

Denote by P the projection $R^n \to R^{n-1}$ for which $P(x, y) = x$. Let $\{B_1, \ldots, B_k\}$ be a partition of B into disjoint Borel sets and Γ_i the unique subfamily of Γ for which $P(|\Gamma_i|) = B_i$, $i = 1, \ldots, k$. Then $\{\Gamma_1, \ldots, \Gamma_k\}$ is a separate partition of Γ and $M_p(\Gamma_i) = m_{n-1}(B_i)h^{1-p}$, whence

$$\sum_{i=1}^{k} (\inf_{x \in B_i} \int_0^h f(x, y)dy)m_{n-1}(B_i) = \sum_{i=1}^{k} \inf u(\Gamma_i)h^{p-1}M_p(\Gamma_i) \leq h^{p-1} \int_\Gamma udM_p$$

Therefore $\int_H fdm = \int_B (\int_0^h f(x, y)dy)dm_{n-1}(x) \leq h^{p-1} \int_\Gamma udM_p$.

In order to prove the opposite inequality, we choose a separate partition $\{\Gamma_1, \ldots, \Gamma_k\}$ of Γ and, for each $i = 1, \ldots, k$, a measurable set A_i such that $P(|\Gamma_i|) \subset A_i$ and $m^*_{n-1}(P(|\Gamma_i|)) = m_{n-1}(A_i)$. As in Example 2.3, we see that the set A_i may be chosen so that $\inf_{x \in A_i} \int_0^h f(x, y)dy = \inf_{x \in P(|\Gamma_i|)} \int_0^h f(x, y)dy$. Since the families Γ_i are separate, we obtain

$$\sum_{i=1}^{k} m_{n-1}(A_i) = h^{p-1} \sum_{i=1}^{k} M_p(\Gamma_i) = h^{p-1} M_p(\bigcup_{i=1}^{k} \Gamma_i) \leq m_{n-1}(\bigcup_{i=1}^{k} A_i) .$$

Thus $\sum_{i=1}^{k} m_{n-1}(A_i) = m_{n-1}(\bigcup_{i=1}^{k} A_i)$. Set $C_i = A_i \setminus \bigcup_{j \neq i} A_j$. Then $m_{n-1}(C_i) = m_{n-1}(A_i) = h^{p-1} M_p(\Gamma_i)$ and $\inf u(\Gamma_i) = \inf_{x \in A_i} \int_0^h f(x\,,\, y) dy \leq \inf_{x \in C_i} \int_0^h f(x\,, y) dy$. Since the sets C_i are disjoint, we get

$$\sum_{i=1}^{k} \inf u(\Gamma_i) M_p(\Gamma_i) \leq h^{1-p} \sum_{i=1}^{k} \left(\inf_{x \in C_i} \int_0^h f(x\,,\, y) dy \right) m_{n-1}(C_i) \leq h^{1-p} \int_H f dm .$$

This yields the desired inequality and proves

$$\int_\Gamma u dM_p = h^{1-p} \int_H f dm .$$

3. General properties of the modulus integral

Here we shall show that many of the basic properties of the Lebesgue integral are also true for the modulus integral. However, one of the most important properties, linearity, is not valid. In this section u , u_j and v are always functions from some subfamily of M into $\bar{R}$ and E , E_j and E' subfamilies of M .

The following theorems are easy consequences of the definition.

3.1. Theorem. (a) *If* $c \in R^1$ *and* $u \in D(E\,,p)$, *then* $\int_E cu dM_p = c \int_E u dM_p$.

(b) $\int_E \chi_{E'} dM_p = M_p(E \cap E')$ *for all* $E\,,\, E' \subset M$, *and, in particular,* $\int_E dM_p = M_p(E)$.

3.2. Theorem. (a) *If* $u\,,\, v \in D(E\,,\, p)$ *and* $u \leq v$, *then* $\int_E u dM_p \leq \int_E v dM_p$.

(b) *If* $E \subset E'$ *and* $u \geq 0$, *then* $\int_E u dM_p \leq \int_{E'} u dM_p$.

The following examples show that the modulus integral is not linear. In fact, neither of the inequalities

$$\int_E (u + v) dM_p \leq \int_E u dM_p + \int_E v dM_p\,, \int_E u dM_p + \int_E v dM_p \leq \int_E (u + v) dM_p$$

holds generally. In these examples, 3.3 and 3.8, it is easy to define non-negative Borel functions f and g such that $u(\gamma) = \int_\gamma f ds$ and $v(\gamma) = \int_\gamma g ds$. Thus the above inequalities do not hold even for these kinds of functions.

3.3. Example. Let $Q = \{(x, y) \mid 0 < x < 1, \ 0 < y < 1\}$, $R = \{(x, y) \mid 0 < x < 1/2, \ 0 < y < 2\}$, $\Gamma_1 = \Delta_1(Q)$, $\Gamma_2 = \Delta_2(R)$ and $\Gamma = \Gamma_1 \cup \Gamma_2$. Then $M_p(\Gamma_1) = 1$, $M_p(\Gamma_2) = 2^{-p}$ and $M_p(\Gamma) = 1$, since $\Gamma_1 \subset \Gamma$ and $\chi_Q \in F(\Gamma)$. Put $u = \chi_{\Gamma_1}$ and $v = \chi_{\Gamma_2}$. Then $u + v = \chi_\Gamma$. By 3.1(b), we have

$$\int_\Gamma u dM_p + \int_\Gamma v dM_p = 1 + 2^{-p} > 1 = \int_\Gamma (u + v) dM_p .$$

In the following example we need an inequality, which we state as a lemma.

3.4. Lemma. *Let* $p > 1$ *and* $h(p) = 2(1 + 2^{1/(p-1)})^{1-p}$. *Then*

(a) $t^p + 2(1 - t)^p \geq h(p)$ *for all* $t \in [0, 1]$ *and*

(b) $2^{1-p} < h(p) < 1$.

There is equality in (a) *for* $t = t_p = (1 + 2^{1/(1-p)})^{-1}$.

Proof. We write $\varphi(t) = t^p + 2(1 - t)^p$ for $0 \leq t \leq 1$. Then $\varphi'(t) = pt^{p-1} - 2p(1 - t)^{p-1} = 0$ if and only if $t = t_p$. Since $\varphi'(0) = -2p < 0$ and $\varphi'(1) = p > 0$, φ attains its minimum in the interval $[0, 1]$ at t_p. Hence $\varphi(t) \geq \varphi(t_p) = h(p)$ for all $t \in [0, 1]$ and $h(p) < \varphi(1) = 1$. This proves (a) and the right side of (b). The left side of (b) follows from

$$h(p) = 2(1 + 2^{1/(p-1)})^{1-p} > 2(2^{1/(p-1)} + 2^{1/(p-1)})^{1-p} = 2^{1-p} .$$

3.5. Example. Let B be any set of real numbers, $H_1 = \{(x, y) \mid x \in B, \ 0 < y < 2\}$, $H_2 = \{(x, y) \mid x \in B, \ 1 < y < 3\}$, $\Gamma_1 = \Delta(H_1)$, $\Gamma_2 = \Delta(H_2)$ and $\Gamma = \Gamma_1 \cup \Gamma_2$. We compute the modulus $M_p(\Gamma)$, when $p > 1$.

Let $f \in F(\Gamma)$. The functions $x \mapsto \int_0^2 f(x, y) dy$ and $x \mapsto \int_1^3 f(x, y) dy$ are measurable in R^1, and therefore the set

$$A = \left\{ x \in R^1 \mid \inf \left\{ \int_0^2 f(x, y) dy , \int_1^3 f(x, y) dy \right\} \geq 1 \right\}$$

is measurable. Moreover, $B \subset A$. Fubini's theorem and Hölder's inequality yield

$$(3.6) \quad \int_{R^2} f^p dm \geq \int_A \left(\int_0^1 f(x\,,\,y)^p dy + \int_1^2 f(x\,,\,y)^p dy + \int_2^3 f(x\,,\,y)^p dy \right) dx \geq$$

$$\int_A \left(\left(\int_0^1 f(x\,,\,y) dy \right)^p + \left(\int_1^2 f(x\,,\,y) dy \right)^p + \left(\int_2^3 f(x\,,\,y) dy \right)^p \right) dx\,.$$

For each $x \in A$, we have

$$(3.7) \quad \left(\int_0^1 f(x\,,\,y) dy \right)^p + \left(\int_1^2 f(x\,,\,y) dy \right)^p + \left(\int_2^3 f(x\,,\,y) dy \right)^p \geq h(p)\,.$$

If $\int_1^2 f(x\,,\,y) dy > 1$, this follows from 3.4(b). If $\int_1^2 f(x\,,\,y) dy \leq 1$, we choose $t = \int_1^2 f(x\,,\,y) dy$ in 3.4(a) and obtain (3.7), since $\int_0^1 f(x\,,\,y) dy \geq 1 - t$ and $\int_2^3 f(x\,,\,y) dy \geq 1 - t$ by the definition of A. By combining (3.6) and (3.7), we get

$$\int_{R^2} f^p dm \geq h(p) m_1(A) \geq h(p) m_1^*(B)\,.$$

Thus
$$M_p(\Gamma) \geq h(p) m_1^*(B)\,.$$

Let C be a Borel set such that $B \subset C$ and $m_1^*(B) = m_1(C)$. Define a Borel function f by $f(x\,,\,y) = t_p$ if $x \in C$ and $1 \leq y \leq 2$, $f(x\,,\,y) = 1 - t_p$ if $x \in C$ and $0 < y < 1$ or $2 < y < 3$, and $f(x\,,\,y) = 0$ otherwise. Then $f \in F(\Gamma)$ and

$$M_p(\Gamma) \leq \int_{R^2} f^p dm = h(p) m_1(C) = h(p) m_1^*(B)\,.$$

This proves that

$$M_p(\Gamma) = h(p) m_1^*(B)\,.$$

3.8. Example. Let $R_1 = \{(x\,,\,y) \mid 0 < x < 1\,,\; 0 < y < 2\}$, $R_2 = \{(x\,,\,y) \mid 0 < x < 1\,,\; 1 < y < 3\}$, $\Gamma_1 = \Delta_2(R_1)$, $\Gamma_2 = \Delta_2(R_2)$ and $\Gamma = \Gamma_1 \cup \Gamma_2$. Then $M_p(\Gamma_1) = M_p(\Gamma_2) = 2^{1-p}$ and, by the preceding example, $M_p(\Gamma) = h(p)$ when $p > 1$. Define u and v by

$$u(\gamma) = \begin{cases} 2^{p-1}h(p) & \text{for } \gamma \in \Gamma_1 , \\ 1 & \text{for } \gamma \in \Gamma_2 \end{cases}$$

and

$$v(\gamma) = \begin{cases} 1 & \text{for } \gamma \in \Gamma_1 , \\ 2^{p-1}h(p) & \text{for } \gamma \in \Gamma_2 . \end{cases}$$

Let $\{\Gamma_1' , \ldots , \Gamma_k'\}$ be a separate partition of Γ. We add to each Γ_i' all the paths $\gamma'' \in \Gamma$ such that $|\gamma''| \cap |\gamma'| \neq \emptyset$ for some $\gamma' \in \Gamma_i'$ and denote the family we thus obtain by Γ_i''. The families Γ_i'', $i = 1 , \ldots , k$, are also separate. By 3.5, $M_p(\Gamma_i'') = m_1^*(B_i)h(p)$, where B_i is the common projection of $|\Gamma_i'|$ and $|\Gamma_i''|$ at x-axis. If $\Gamma_2 \cap \Gamma_i' = \emptyset$, then $M_p(\Gamma_i') = m_1^*(B_i)2^{1-p}$ and

$$\inf u(\Gamma_i')M_p(\Gamma_i') = 2^{p-1}h(p)m_1^*(B_i)2^{1-p} = M_p(\Gamma_i'') .$$

If $\Gamma_2 \cap \Gamma_i' \neq \emptyset$, then

$$\inf u(\Gamma_i')M_p(\Gamma_i') = M_p(\Gamma_i') \leq M_p(\Gamma_i'') .$$

Hence

$$\sum_{i=1}^{k} \inf u(\Gamma_i')M_p(\Gamma_i') \leq \sum_{i=1}^{k} M_p(\Gamma_i'') \leq M_p(\Gamma) = h(p)$$

and

$$\int_\Gamma u dM_p \leq h(p) .$$

Actually, this holds as equality, since $u \geq 1$ by 3.4(b). Similarly, one can see that $\int_\Gamma v dM_p = h(p)$. Since $u + v \equiv 1 + 2^{p-1}h(p) > 2$, we get

$$\int_\Gamma (u + v)dM_p = (1 + 2^{p-1}h(p))M_p(\Gamma) > 2h(p) = \int_\Gamma u dM_p + \int_\Gamma v dM_p .$$

3.9. Theorem. *If* $E = \bigcup_{i=1}^{\infty} E_i$ *and* $u \geq 0$, *then*

$$\int_E u dM_p \leq \sum_{i=1}^{\infty} \int_{E_i} u dM_p .$$

Proof. Let $\{F_1 , \ldots , F_k\}$ be a separate partition of E. Denote $F_{ij} = E_i \cap F_j$ for all $i \in N$ and $j = 1 , \ldots , k$. Then $\{F_{i1} , \ldots , F_{ik}\}$ is a separate partition of E_i for each $i \in N$ and $F_j = \bigcup_{i=1}^{\infty} F_{ij}$ for each $j = 1 , \ldots , k$. This implies

$$\sum_{j=1}^{k} \inf u(F_j)M_p(F_j) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{k} \inf u(F_{ij})M_p(F_{ij}) \leq \sum_{i=1}^{\infty} \int_{E_i} u dM_p .$$

The assertion follows now if we take supremum over all separate partitions of E.

3.10. Theorem. *If* $E = \bigcup_{i=1}^{\infty} E_i$, *the families* E_i *are separate and* $u \in D(E, p)$, *then*

$$\int_E u dM_p = \sum_{i=1}^{\infty} \int_{E_i} u dM_p .$$

Proof. Suppose first that $u \geq 0$. By 3.9. it suffices to show that

$$\int_E u dM_p \geq \sum_{i=1}^{\infty} \int_{E_i} u dM_p .$$

We may assume that $\int_{E_i} u dM_p < \infty$ for all $i \in N$, since otherwise the assertion follows immediately from 3.2(b). Let $\varepsilon > 0$. For every positive integer i, we can find a separate partition $\{E_{i1}, \ldots, E_{ik_i}\}$ of E_i such that

$$\int_{E_i} u dM_p \leq \sum_{j=1}^{k_i} \inf u(E_{ij})M_p(E_{ij}) + \varepsilon 2^{-i} .$$

For every $n \in N$, the set $\{E_{ij} \mid j = 1, \ldots, k_i, i = 1, \ldots, n\}$ is a separate partition of E. Hence

$$\sum_{i=1}^{n} \int_{E_i} u dM_p \leq \sum_{i=1}^{n} \sum_{j=1}^{k_i} \inf u(E_{ij})M_p(E_{ij}) + \varepsilon \leq \int_E u dM_p + \varepsilon .$$

Letting $n \to \infty$ and $\varepsilon \to 0$ yields the desired inequality.

If $u \in D(E, p)$ is arbitrary, the theorem is valid for u^+ and u^-. Considering the cases $\left|\int_E u dM_p\right| < \infty$ and $\left|\int_E u dM_p\right| = \infty$ separately, one easily verifies the asserted equality.

3.11. Remark. Not even absolute convergence of the series in the above theorem implies that $u \in D(E, p)$. This can be seen, for example, if we choose $E_i = \Delta_1(R_i) \cup \Delta_2(R_i)$, where $R_i = \{(x, y) \mid 0 < x < 1, i < y < i + 1\}$, and $u(\gamma) = 1$ for $\gamma \in \bigcup_{i=1}^{\infty} \Delta_1(R_i)$ and $u(\gamma) = -1$ for $\gamma \in \bigcup_{i=1}^{\infty} \Delta_2(R_i)$.

3.12. Theorem. *Suppose that* $E' \subset E$ *and* $u(\mu) = 0$ *for* p—*a.e.* $\mu \in E \setminus E'$. *Then* $u \in D(E, p)$ *if and only if* $u \in D(E', p)$, *and then* $\int_E u dM_p = \int_{E'} u dM_p$.

Proof. It is obvious by definition that $u \in D(E \setminus E', p)$ and

$$\int_{E \setminus E'} u dM_p = 0 .$$

Therefore, if $u \geq 0$, Theorems 3.9 and 3.2(b) imply that

$$\int_E u dM_p \leq \int_{E'} u dM_p + \int_{E \setminus E'} u dM_p = \int_{E'} u dM_p \leq \int_E u dM_p ,$$

or

$$\int_E u dM_p = \int_{E'} u dM_p$$

The theorem follows immediately from this for arbitrary functions.

3.13. Theorem. *Suppose that* $u(\mu) = v(\mu)$ *for* p—*a.e.* $\mu \in E$. *Then* $u \in D(E, p)$ *if and only if* $v \in D(E, p)$, *and then* $\int_E u dM_p = \int_E v dM_p$.

Proof. Denote $E_0 = \{\mu \in E \mid u(\mu) \neq v(\mu)\}$. Then $M_p(E_0) = 0$. From 3.12 we conclude that $u \in D(E, p)$ if and only if $v \in D(E, p)$ and that in this case

$$\int_E u dM_p = \int_{E \setminus E_0} u dM_p = \int_{E \setminus E_0} v dM_p = \int_E v dM_p .$$

3.14. Theorem. *Suppose that* $u \geq 0$. *Then* $\int_E u dM_p = 0$ *if and only if* $u(\mu) = 0$ *for* p-*a.e.* $\mu \in E$.

Proof. If $u(\mu) = 0$ for p-*a.e.* $\mu \in E$, 3.13 implies that $\int_E u dM_p = \int_E 0 dM_p = 0$.

Conversely, suppose that $\int_E u dM_p = 0$. If u does not vanish p-*a.e.*, it follows from the subadditivity of the modulus that for some $a > 0$ the family $E_a = \{\mu \in E \mid u(\mu) \geq a\}$ has a positive p-modulus. Then by 3.2(a) and (b) $\int_E u dM_p \geq a M_p(E_a) > 0$, which is a contradiction.

3.15. Fatou's lemma. *If* $p > 1$ *and* (u_j) *is a sequence of non-negative functions defined on a measure family* E , *then*

$$\int_E \liminf_{j\to\infty} \; u_j dM_p \leq \liminf_{j\to\infty} \int_E u_j dM_p \, .$$

Proof. Set $u = \liminf\limits_{j\to\infty} u_j$. Let $\{E_1 , \ldots , E_k\}$ be a separate partition of E and $c_i < \inf u(E_i)$ for $i = 1 , \ldots , k$. Denote $E_{in} = \{\mu \in E_i \mid u_j(\mu) \geq c_i \text{ for all } j \geq n\}$ for $n \in N$ and $i = 1 , \ldots , k$. Then $E_i = \bigcup\limits_{n=1}^{\infty} E_{in}$. By 1.7, we have $M_p(E_i) = \lim\limits_{n\to\infty} M_p(E_{in})$. The families E_{in} , $i = 1 , \ldots , k$, are separate for all $n \in N$ and $\inf u_j(E_{in}) \geq c_i$ for all $j \geq n$. Hence

$$\sum_{i=1}^{k} c_i M_p(E_{in}) \leq \int_E u_j dM_p$$

for $j \geq n$, and therefore

$$\sum_{i=1}^{k} c_i M_p(E_{in}) \leq \inf_{j\geq n} \int_E u_j dM_p \, .$$

Since this is true for every $n \in N$, we obtain

$$\sum_{i=1}^{k} c_i M_p(E_i) = \lim_{n\to\infty} \sum_{i=1}^{k} c_i M_p(E_{in}) \leq \liminf_{j\to\infty} \int_E u_j \, dM_p \, .$$

We conclude the proof by letting $c_i \to \inf u(E_i)$ for each $i = 1 , \ldots , k$ and taking supremum over all separate partitions of E .

3.16. Monotone convergence theorem. *If* $p > 1$ *and* (u_j) *is an increasing sequence of non-negative functions defined on a measure family* E , *then*

$$\int_E \lim_{j\to\infty} u_j dM_p = \lim_{j\to\infty} \int_E u_j dM_p \, .$$

Proof. This follows directly from 3.2(a) and Fatou's lemma.

3.17. Theorem. *Let* $p > 1$. *If* $E_1 \subset E_2 \subset \ldots$, $E = \bigcup\limits_{i=1}^{\infty} E_i$ *and* $u \in D(E , p)$, *then*

$$\int_E u dM_p = \lim_{i\to\infty} \int_{E_i} u dM_p \, .$$

Proof. We may assume that $u \geq 0$. Application of the monotone convergence theorem to the functions $u_i = u \cdot \chi_{E_i}$ gives by 3.12,

$$\int_E u dM_p = \lim_{i \to \infty} \int_E u_i dM_p = \lim_{i \to \infty} \int_{E_i} u_i dM_p = \lim_{i \to \infty} \int_{E_i} u dM_p .$$

3.18. Remark. None of the three last theorems is valid for $p = 1$. This can be seen if we use Example 1.10 and Theorem 3.1(b).

3.19. Example. Let Q be a unit square in the plane R^2, $\Gamma = \{\gamma \mid |\gamma| \subset Q, \quad 1/2 < l(\gamma) < 1\}$ and $\Gamma_j = \{\gamma \in \Gamma \mid 1 - 1/j < l(\gamma) < 1\}$ for $j \in N$. Since $l(\gamma) > 1/2$ for all $\gamma \in \Gamma$, $M_p(\Gamma) \leq 2^p$, and since $\Gamma_j < \Delta_1(Q)$, $M_p(\Gamma_j) \geq M_p(\Delta_1(Q)) = 1$ for all $j \in N$. If $u_j = \chi_{\Gamma_j}$, then $u_1 \geq u_2 \geq \ldots \geq 0$, $\lim u_j = 0$ and $M_p(\Gamma) = \int_\Gamma u_1 dM_p \geq \int_\Gamma u_j dM_p = M_p(\Gamma_j) \geq 1$ for all $j \in N$. So we see that $\lim_{j \to \infty} \int_\Gamma u_j dM_p \neq 0$, although $M_p(\Gamma) < \infty$, $\int_\Gamma u_1 dM_p < \infty$, the sequence (u_j) is uniformly bounded and converges monotonely to zero.

The preceding example shows that, for instance, Lebesgue's bounded convergence theorem does not hold. The following theorem is, however, easily established.

3.20. Theorem. *If* $M_p(E) < \infty$ *and the sequence* (u_j), $u_j \in D(E, p)$, *converges uniformly to a finite function* $u \in D(E, p)$, *then*

$$\lim_{j \to \infty} \int_E u_j dM_p = \int_E u dM_p .$$

Proof. Since the sequences (u_j^+) and (u_j^-) converge uniformly to u^+ and u^-, we may assume that the functions u_j are non-negative. Furthermore, we may assume that $M_p(E) > 0$. Let $\varepsilon > 0$. By the uniform convergence, there is a positive integer j_0 such that for all $\mu \in E$ and $j \geq j_0$

$$u(\mu) - \varepsilon/M_p(E) \leq u_j(\mu) \leq u(\mu) + \varepsilon/M_p(E) .$$

For any separate partition $\{E_1, \ldots, E_k\}$ of E, we get

$$\sum_{i=1}^k \inf u(E_i) M_p(E_i) - \varepsilon \leq \sum_{i=1}^k \inf u_j(E_i) M_p(E_i) \leq \sum_{i=1}^k \inf u(E_i) M_p(E_i) + \varepsilon$$

for $j \geq j_0$, and therefore

$$\int_E u dM_p - \varepsilon \leq \int_E u_j dM_p \leq \int_E u dM_p + \varepsilon ,$$

or

$$\lim_{j\to\infty} \int_E u_j dM_p = \int_E u dM_p .$$

4. Measurability

In the following we shall consider subfamilies of a fixed measure family $K \subset M$. We shall define a σ-algebra $\mathcal{K}$ such that the restriction of the modulus M_p to $\mathcal{K}$ is a measure. Since M_p is an outer measure, one σ-algebra of this kind consists of those subfamilies E of K for which the equation

$$M_p(F) = M_p(F \cap E) + M_p(F \setminus E) \tag{4.1}$$

holds for all families $F \subset K$. We call these families *measurable of order p*. However, we choose as $\mathcal{K}$ a smaller σ-algebra, which has a closer relation to the separate partitions in the definition of the modulus integral.

4.2. Definition. *Let $\mathcal{K}$ be the set of all measure families $E \subset K$ that are separate with their complements $K \setminus E$. The families of $\mathcal{K}$ are called measurable families.*

The following theorem justifies this definition.

4.3. Theorem. *The restriction of M_p to $\mathcal{K}$ is a measure.*

Proof. We first show that $\mathcal{K}$ is a σ-algebra. Obviously, $O \in \mathcal{K}$ and $\mathcal{K}$ contains the complement of each of its families. Let $E_i \in \mathcal{K}$, $i = 1, 2, \ldots$. We have to show that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{K}$ i.e. $\bigcup_{i=1}^{\infty} E_i$ and $K \setminus \bigcup_{i=1}^{\infty} E_i$ are separate. Since for each i E_i and $K \setminus E_i$ are separate, there are disjoint m-measurable sets S_i and T_i such that $\mu(X \setminus S_i) = 0$ for $\mu \in E_i$ and $\mu(X \setminus T_i) = 0$ for $\mu \in K \setminus E_i$. Denote $S = \bigcup_{i=1}^{\infty} S_i$ and $T = \bigcap_{i=1}^{\infty} T_i$. Then $S \cap T = \emptyset$ and it is easy to see that $\mu(X \setminus S) = 0$ for $\mu \in \bigcup_{i=1}^{\infty} E_i$ and $\mu(X \setminus T) = 0$ for $\mu \in K \setminus \bigcup_{i=1}^{\infty} E_i$. Hence $\bigcup_{i=1}^{\infty} E_i$ and $K \setminus \bigcup_{i=1}^{\infty} E_i$ are separate.

By the general theory of measure, $M_p \mid \mathcal{K}$ is a measure if the families of $\mathcal{K}$ satisfy the condition (4.1). To see this choose $E \in \mathcal{K}$ and $F \subset K$. Since $F \cap E \subset E$ and $F \setminus E \subset K \setminus E$ and since E and $K \setminus E$ are separate, the families $F \cap E$ and $F \setminus E$ are separate, too, and (4.1) follows.

4.4. Example. Let H be a Borel cylinder in R^n. If $K = \Delta(H)$, then all subfamilies of K whose loci are Borel sets are measurable. But if $K = \Delta_a(H)$ and H is a domain, then the only measurable subfamilies of K are $\emptyset$ and K. To prove the second assertion, suppose that E is a measurable, non-empty subfamily of K. If $K \setminus E$ is not empty, choose $\gamma \in E$ and $\gamma' \in K \setminus E$. Then there is a path $\gamma'' \in K$ such that γ and γ'' have a common subpath γ_1 with $l(\gamma_1) > 0$ and γ' and γ'' have a common subpath γ_2 with $l(\gamma_2) > 0$. But then E and $K \setminus E$ cannot be separate. Hence $K \setminus E$ must be empty.

4.5. Remarks. The measurable subfamilies of K are always measurable of order p for all $p \geq 1$, but not conversely, since, for example, the p-exceptional families and their complements are measurable of order p for all $p \geq 1$, but not always in the sense of Definition 4.2.

If K is the family of all rectifiable paths in R^n and $p > 1$, then no family $\Gamma \subset K$ such that $0 < M_p(\Gamma) < \infty$ is measurable of order p. Renggli [5] has proved this in the special case $p = n = 2$ and Hesse [3] in the general case.

4.6. By 4.3, all results of the general theory of measure are applicable to $M_p \mid \mathcal{K}$. We recall the basic definitions: A function $u : K \to \bar{R}$ is measurable if the inverse image $u^{-1}(G)$ of every open set $G \subset R^1$ and the sets $u^{-1}(\infty)$ and $u^{-1}(-\infty)$ are measurable, or equivalently, if the sets $E_\alpha = \{\mu \in K \mid u(\mu) > \alpha\}$ are measurable for all real numbers α. If u is non-negative and measurable, the integral of u over K, $I_p(K, u)$, according to the general theory of measure, is,

$$I_p(K, u) = \sup_D \sum_{i=1}^{k} \inf u(E_i) M_p(E_i) ,$$

where $D = \{E_1, \ldots, E_k\}$ ranges over all measurable partitions of K, i.e. the families E_i are disjoint and measurable and $K = \bigcup_{i=1}^{k} E_i$. If u may also have negative values and either $I_p(K, u^+)$ or $I_p(K, u^-)$ is finite, then

$$I_p(K, u) = I_p(K, u^+) - I_p(K, u^-) .$$

4.7. Remark. From Definition 4.2 it follows immediately that if $E \subset K' \subset K$ and E is measurable in K, then E is measurable in K'. Observe that K' is not necessarily measurable in K. Similarly, if u is a measurable function in K, then it is also measurable in K'.

4.8. Theorem. *If* $I_p(K, u)$ *is defined, then* $I_p(K, u) = \int_K u dM_p$.

Proof. We may assume that $u \geq 0$. By the definition of $\mathcal{K}$, every measurable partition of K is separate. Hence

$$I_p(K, u) \leq \int_K u dM_p .$$

To prove the opposite inequality, let $\{E_1, \ldots, E_k\}$ be a separate partition of K. Put $c_i = \inf u(E_i)$ for $i = 1, \ldots, k$. Rearranging the families E_i, we may assume that $c_1 \leq c_2 \leq \ldots \leq c_k$. The families

$$F_j = \{\mu \in K \mid c_j \leq u(\mu) < c_{j+1}\}, j = 1, \ldots, k-1,$$
$$F_k = \{\mu \in K \mid c_k \leq u(\mu)\}$$

are measurable and disjoint. Hence, for each fixed $i = 1, \ldots, k$, the families $E_{ij} = E_i \cap F_j$, $j = 1, \ldots, k$, are separate as subfamilies of separate families F_j. Similarly, the families E_{ij}, $i = 1, \ldots, k$, are separate for each $j = 1, \ldots, k$. Since $E_i = \bigcup_{j=1}^{k} E_{ij}$ and $\bigcup_{i=1}^{k} E_{ij} \subset F_j$, we have

$$M_p(E_i) = \sum_{j=1}^{k} M_p(E_{ij}) \quad \text{and} \quad \sum_{i=1}^{k} M_p(E_{ij}) \leq M_p(F_j)$$

for all $i, j = 1, \ldots, k$. If $j < i$, then $E_{ij} = \emptyset$, and if $i \leq j$, then $c_i \leq c_j$, whence

$$\sum_{j=1}^{k} c_i M_p(E_{ij}) \leq \sum_{j=1}^{k} c_j M_p(E_{ij})$$

for $i = 1, \ldots, k$. Combining these inequalities, we see that

$$\sum_{i=1}^{k} c_i M_p(E_i) \leq \sum_{j=1}^{k} c_j M_p(F_j) \leq I_p(K, u),$$

which gives the desired inequality.

4.9. Remark. By the preceding theorem, we may apply the results of the general theory of integral to the modulus integral. Thus we have, for example,

$$\int_K (u+v)dM_p = \int_K u dM_p + \int_K v dM_p$$

if $I_p(K, u)$ and $I_p(K, v)$ are defined.

4.10. Definition. *If $E \subset M$, $u : E \to \bar{R}$ and if there exists an m-measurable function f such that $u(\mu) = \int_X f d\mu$ for all $\mu \in E$, we say that u is an integral. Especially, we denote $l(\mu) = \int_X d\mu = \mu(X)$.*

4.11. Theorem. *If l is measurable and $0 < l(\mu) < \infty$ in K, then every non-negative, measurable function $K \to \bar{R}$ is an integral.*

Proof. Let $u : K \to \bar{R}$ be non-negative and measurable. Since u and l are measurable, so is u/l, and we can find an increasing sequence (u_j) of simple measurable functions such that $u/l = \lim_{j\to\infty} u_j$. The functions u_j may be represented in the form:

$$u_j = \sum_{k=1}^{p_j} a_{jk} \chi_{E_{jk}},$$

where the families E_{jk} are measurable, non-empty and disjoint, $K = \bigcup_{k=1}^{p_j} E_{jk}$ and each family E_{jk} is a subfamily of some family $E_{(j-1)h}$ for $j \geq 2$. The families E_{jk}, $k = 1, \ldots, p_j$, are separate for all $j \in N$. The disjoint »separating» sets S_{jk} may be chosen so that $S_{jk} \subset S_{(j-1)h}$ if $E_{jk} \subset E_{(j-1)h}$. We define m-measurable functions f_j by

$$f_j(x) = \begin{cases} a_{jk} & \text{for } x \in S_{jk} \\ \infty & \text{for } x \in X \setminus \bigcup_{k=1}^{p_j} S_{jk} \end{cases}$$

Then the sequence (f_j) is increasing and converges to some m-measurable function f. If $\mu \in K$, then, for every $j \in N$, μ belongs to some E_{jk} and hence

$$\int_X f_j d\mu = \int_{S_{jk}} f_j d\mu = u_j(\mu)\mu(S_{jk}) = u_j(\mu) l(\mu).$$

Thus

$$\int_X f d\mu = \lim_{j\to\infty} \int_X f_j d\mu = \lim_{j\to\infty} u_j(\mu) l(\mu) = u(\mu),$$

which proves the theorem.

For a rectifiable path γ , $l(\mu_\gamma) = l(\gamma)$. So the measurability of l in a path family means, roughly speaking, that the paths of different lengths do not intersect too much. Nevertheless, it is possible that l is not measurable, even if the loci of all paths were disjoint (cf. 4.19).

4.12. Example. The constant function $u \equiv 1$ is always measurable. It is not an integral if $K = \{\mu , 2\mu\}$ for any measure μ (finite, infinite or zero).

4.13. Definition. *A family Γ of rectifiable paths in R^n is said to be continuous if for every $\gamma \in \Gamma$ and $\varepsilon > 0$ there is $\delta > 0$ such that $|\gamma^\circ(t) - \gamma'^\circ(t)| < \varepsilon$ for all $t \in [0 , l(\gamma)] \cap [0 , l(\gamma')]$ and $|l(\gamma) - l(\gamma')| < \varepsilon$ whenever $\gamma' \in \Gamma$ and $d(|\gamma| , |\gamma'|) < \delta$.* (γ° and γ'° are the normal representations of γ and γ').

For path families, we obtain a partial converse of Theorem 4.11:

4.14. Theorem. *Suppose that Γ is a continuous family of rectifiable paths in R^n and that $|\Gamma|$ is a Borel set. If u is an integral in Γ and $p \geq 1$, then Γ has a subfamily Γ' such that $M_p(\Gamma \setminus \Gamma') = 0$ and u is measurable in Γ' .*

Proof. If $\gamma , \gamma' \in \Gamma$ and $|\gamma| \cap |\gamma'| \neq \emptyset$, then by the continuity of Γ , $\gamma^\circ = \gamma'^\circ$ and so $\mu_\gamma = \mu_{\gamma'}$. Therefore, since $|\Gamma|$ is a Borel set, the subfamilies of Γ whose loci are Borel sets are measurable.

Let f be a Borel function on R^n such that $u(\gamma) = \int_\gamma f ds$ for $\gamma \in \Gamma$. We first assume that f is non-negative, continuous and vanishes outside some compact set. Then f is bounded and uniformly continuous. Choose $B > 0$ such that $f \leq B$. Let α be a real number and $\Gamma_\alpha = \{\gamma \in \Gamma \mid u(\gamma) > \alpha\}$. We show that $|\Gamma_\alpha|$ is open relative to $|\Gamma|$.

Let $x \in |\Gamma_\alpha|$. Then there exists $\gamma \in \Gamma_\alpha$ so that $x \in |\gamma|$. If $l(\gamma) = 0$, then $0 = u(\gamma) > \alpha$, whence $\Gamma_\alpha = \Gamma$ and $|\Gamma_\alpha| = |\Gamma|$. Suppose that $l(\gamma) > 0$ and denote $\varepsilon = (u(\gamma) - \alpha)/l(\gamma)$. Since f is uniformly continuous, we can find $\delta > 0$ such that

(4.15) $$|f(y) - f(z)| < \varepsilon/2 \quad \text{for} \quad |y - z| < \delta .$$

The continuity of Γ implies that there is $\varrho > 0$ such that

(4.16) $$|\gamma^\circ(t) - \gamma'^\circ(t)| < \delta \quad \text{for} \quad t \in [0 , l(\gamma)] \cap [0 , l(\gamma')]$$

and

(4.17) $$|l(\gamma) - l(\gamma')| < \varepsilon l(\gamma)/2B$$

whenever $\gamma' \in \Gamma$ and $d(|\gamma| , |\gamma'|) < \varrho$.

In order to prove that $|\Gamma_\alpha|$ is open in $|\Gamma|$, we choose $y \in |\Gamma|$ such that $|x - y| < \varrho$ and show that $y \in |\Gamma_\alpha|$. Since $y \in |\Gamma|$, there is $\gamma' \in \Gamma$ such that $y \in |\gamma'|$. The inequalities (4.16) and (4.17) hold for γ and γ', because $d(|\gamma|, |\gamma'|) \leq |x - y| < \varrho$. By (4.17), we obtain

$$\int_{l(\gamma')}^{l(\gamma)} f \circ \gamma^\circ(t)dt < B\varepsilon l(\gamma)/2B = \varepsilon l(\gamma)/2 \, , \text{ if } \; l(\gamma') \leq l(\gamma) \, ,$$

and, by (4.16) and (4.15),

$$f \circ \gamma'^\circ(t) > f \circ \gamma^\circ(t) - \varepsilon/2 \quad \text{for} \quad t \in [0 \, , l(\gamma)] \cap [0 \, , l(\gamma')] \, .$$

Considering separately the cases $l(\gamma') \leq l(\gamma)$ and $l(\gamma) < l(\gamma')$, we see now by direct calculation that $u(\gamma') = \int_0^{l(\gamma')} f \circ \gamma'^\circ(t)dt > \alpha$. Hence $\gamma' \in \Gamma_\alpha$ and therefore $y \in |\Gamma_\alpha|$. As an open set of the subspace $|\Gamma|$, the locus $|\Gamma_\alpha|$ is a Borel set of $|\Gamma|$ and, since $|\Gamma|$ is a Borel set of R^n, so is $|\Gamma_\alpha|$. This shows that Γ_α is a measurable family and, accordingly, u is a measurable function in Γ.

We next assume that $f \in L^p(R^n)$. Since the continuous functions with compact support are dense in $L^p(R^n)$, there exists a sequence (f_j) of such functions converging to f in L^p-metric. By 1.6, the sequence (f_j) has a subsequence (f_{j_i}) and the family Γ a subfamily Γ' such that $M_p(\Gamma \setminus \Gamma') = 0$ and $\lim_{i \to \infty} \int_\gamma f_{j_i} ds = \int_\gamma f ds$ for $\gamma \in \Gamma'$. By the first part of this proof, the functions $u_{j_i} = \int_\gamma f_{j_i} ds$ are measurable in Γ and hence in Γ' (cf. 4.7). Since $u(\gamma) = \lim_{i \to \infty} u_{j_i}(\gamma)$ for $\gamma \in \Gamma'$, u is likewise measurable in Γ'.

If f is an arbitrary, non-negative Borel function, there is an increasing sequence (f_j) of L^p-functions converging pointwise to f. Then

$$u(\gamma) = \lim_{j \to \infty} \int_\gamma f_j ds \, .$$

Finally, in the general case we have

$$u(\gamma) = \int_\gamma f^+ ds - \int_\gamma f^- ds$$

This proves the theorem.

4.18. Example. If H is a Borel cylinder in R^n, the conditions of Theorem 4.14 hold for $\Delta(H)$. They hold also for the following families Γ_1 and Γ_2.

Let $0 < a < b$ and $Y \subset \{x \in R^n \mid |x| = 1\}$ be a Borel set. For all $y \in Y$ we define a radial segment $\gamma_y : [a, b] \to R^n$, which joins the spheres of radii a and b, by $\gamma_y(t) = ty$ and choose

$$\Gamma_1 = \{\gamma_y \mid y \in Y\} .$$

Let $A \subset (0, \infty)$ be a Borel set. If $r \in A$, we denote by γ_r the circle in R^2 defined by $\gamma_r(t) = r(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ and take

$$\Gamma_2 = \{\gamma_r \mid r \in A\} .$$

4.19. Example. Let A be a non-measurable subset of the interval $[0, 1]$ and

$$H_1 = \{(x, y) \mid 0 < x < 1 , y \in [0, 1] \setminus A\} ,$$
$$H_2 = \{(x, y) \mid 0 < x < 1/2 , y \in A\} ,$$
$$H_3 = \{(x, y) \mid 1/2 < x < 1 , y \in A\} .$$

If $\Gamma_i = \Delta(H_i)$, $i = 1, 2, 3$, and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, then the families Γ_1 and $\Gamma \setminus \Gamma_1$ are not separate, and thus Γ_1 is not measurable in Γ. Since $\Gamma_1 = \{\gamma \in \Gamma \mid l(\gamma) > 1/2\}$, l is not measurable in Γ, though it is an integral and $|\Gamma|$ is a Borel set. Moreover, the loci of the paths of Γ are disjoint with the exception of the end points.

To conclude this section, we prove an inequality for integral functions. If f is a non-negative, m-measurable function $X \to \bar{R}$ and $E \subset M$ is a measure family, we denote

$$L_f(E) = \inf_{\mu \in E} \int_X f d\mu .$$

4.20. Lemma. *If $L_f(E) = \infty$ and if $p \geq 1$, then either $M_p(E) = 0$ or $\int_X f^p dm = \infty$.*

Proof. If $\int_X f^p dm < \infty$ and $L_f(E) = \infty$, then, by 1.5, $M_p(E) = 0$. This is equivalent to the assertion.

4.21. Theorem. *Let $E \subset M$ be a measure family, f a non-negative m-measurable function $X \to \bar{R}$ and $u(\mu) = \int_X f d\mu$ for $\mu \in E$. Then*

$$\int_E u^p dM_p \leq \int_X f^p dm .$$

Proof. Let $\{E_1, \ldots, E_k\}$ be a separate partition of E. Then there are disjoint m-measurable sets $S_1, \ldots, S_k$ such that $\mu(X \setminus S_i) = 0$ for $\mu \in E_i$. Suppose that $0 < L_f(E_i) < \infty$ for each $i = 1, \ldots, k$. The function g_i, defined by $g_i(x) = f(x)/L_f(E_i)$ for $x \in S_i$ and $g_i(x) = 0$ otherwise, belongs to $F(E_i)$. Hence

$$M_p(E_i) \leq \int_X g_i^p dm = L_f(E_i)^{-p} \int_{S_i} f^p dm .$$

This gives

$$\sum_{i=1}^{k} L_f(E_i)^p M_p(E_i) \leq \sum_{i=1}^{k} \int_{S_i} f^p dm \leq \int_X f^p dm .$$

It is clear that this inequality remains valid if $L_f(E_i) = 0$ for some i and, by 4.20, also if $L_f(E_i) = \infty$. We conclude the proof by taking supremum over all separate partitions of E.

4.22. Remarks. There is equality in the above theorem if f is an extremal function of E i.e. $f \in F(E)$ and $\int_X f^p dm = M_p(E)$. Then $\int_E u^p dM_p = M_p(E)$. Another case of equality is given in 5.1. In the situations of 5.4(a) and (b) the inequality is strict.

5. Integration in path families

In this section we examine the modulus integral over families of rectifiable paths. We use the following notation: If $A \subset R^n$ is a Borel set, we denote by $\Gamma(A)$ the family of all rectifiable paths whose loci are subsets of A and, if $\varepsilon > 0$, by $\Gamma_\varepsilon(A)$ the family of all segments which belong to $\Gamma(A)$, are parallel to x_n-axis and whose lengths are less than ε. Of course, we might as well consider the segments parallel to any fixed line.

The main result of this section is the following theorem, which, in a way, resembles Fubini's theorem.

5.1. Theorem. *Suppose that $G \subset R^n$ is open and that Γ is a path family such that $\Gamma_\varepsilon(G) \subset \Gamma \subset \Gamma(G)$ for some $\varepsilon > 0$. If $f : G \to \bar{R}$ is a nonnegative Borel function and $u(\gamma) = \int_\gamma f ds$ for $\gamma \in \Gamma$, then*

$$\int_\Gamma u^p dM_p = \int_G f^p dm .$$

Proof. By 4.21, we have

$$\int_{\Gamma} u^p dM_p \leq \int_{G} f^p dm .$$

To prove the opposite inequality, we shall approximate f with simple functions. Suppose first that f is the characteristic function of a Borel set $A \subset G$. Let $\delta > 0$ and denote by $\mathcal{V}$ the family of all closed cubes $Q \subset G$ parallel to the coordinate axis such that the side length of Q is less than ε and

$$\left(\frac{m(A \cap Q)}{m(Q)}\right)^{p-1} > 1 - \delta . \tag{5.2}$$

Then $\mathcal{V}$ is a Vitali covering of the set A' consisting of the points of density of A. By Vitali's covering theorem, there exist disjoint cubes $Q_k \in \mathcal{V}$, $k = 1, 2, \ldots$, such that

$$m(A' \setminus \bigcup_{k=1}^{\infty} Q_k) = 0 .$$

Since A is measurable, almost all of its points are points of density, whence

$$m(A \setminus \bigcup_{k=1}^{\infty} Q_k) = 0 .$$

For each $k \in N$, we write $Q_k = I_k \times J_k$, $I_k \subset R^{n-1}$ and $J_k \subset R^1$, and $d_k = m_1(J_k)$. If $x \in R^{n-1}$, we denote

$$A_{xk} = \{y \in R^1 \mid (x, y) \in Q_k \cap A\} .$$

Then, by Fubini's theorem,

$$m(A \cap Q_k) = \int_{I_k} m_1(A_{xk}) dm_{n-1}(x) . \tag{5.3}$$

We partition I_k into disjoint Borel sets E_{ki}, $i = 1, \ldots, n_k$, and set

$$H_{ki} = \{(x, y) \in Q_k \mid x \in E_{ki}\} ,$$
$$\Gamma_{ki} = \Delta(H_{ki}) .$$

The families Γ_{ki}, $i = 1, \ldots, n_k$, are separate and $\Gamma_{ki} \subset \Gamma_{\varepsilon}(Q_k)$, whence $\{\Gamma_{k1}, \ldots, \Gamma_{kn_k}\}$ is a separate partition of $\Gamma_{\varepsilon}(Q_k)$. By (1.9),

$$M_p(\Gamma_{ki}) = m_{n-1}(E_{ki}) d_k^{1-p} .$$

Since $u(\gamma) = \int_{\gamma} \chi_A ds = m_1(A_{xk})$, if $x \in R^{n-1}$ is the first coordinate of the points of $|\gamma|$, we obtain

$$\int\limits_{\Gamma_\varepsilon(Q_k)} u^p dM_p \geq \sum_{i=1}^{n_k} (\inf u(\Gamma_{ki}))^p M_p(\Gamma_{ki}) = d_k^{1-p} \sum_{i=1}^{n_k} \inf_{x \in E_{ki}} m_1(A_{xk})^p m_{n-1}(E_{ki}) .$$

By taking supremum over all partitions of I_k into Borel sets, we get

$$\int\limits_{\Gamma_\varepsilon(Q_k)} u^p dM_p \geq d_k^{1-p} \int\limits_{I_k} m_1(A_{xk})^p dm_{n-1}(x)$$

and from here, by Hölder's inequality, (5.3) and (5.2),

$$\int\limits_{\Gamma_\varepsilon(Q_k)} u^p dM_p \geq (d_k m_{n-1}(I_k))^{1-p} \Bigl(\int\limits_{I_k} m_1(A_{xk}) dm_{n-1}(x) \Bigr)^p =$$

$$\left(\frac{m(A \cap Q_k)}{m(Q_k)}\right)^{p-1} m(A \cap Q_k) \geq (1 - \delta) m(A \cap Q_k) .$$

Since the families $\Gamma_\varepsilon(Q_k)$, $k = 1, 2, \ldots$, are separate and $\bigcup_{k=1}^{\infty} \Gamma_\varepsilon(Q_k) \subset \Gamma$, Theorems 3.2(b) and 3.10 yield

$$\int\limits_{\Gamma} u^p dM_p \geq \sum_{k=1}^{\infty} \int\limits_{\Gamma_\varepsilon(Q_k)} u^p dM_p \geq (1 - \delta) \sum_{k=1}^{\infty} m(A \cap Q_k)$$

$$= (1 - \delta) m(A \cap (\bigcup_{k=1}^{\infty} Q_k)) = (1 - \delta) m(A) .$$

By letting $\delta \to 0$, we obtain the desired inequality:

$$\int\limits_{\Gamma} u^p dM_p \geq m(A) = \int\limits_{G} f^p dm .$$

If $f \equiv 0$, the assertion is trivial. Suppose next that f is a simple Borel function, not identically zero:

$$f = \sum_{i=1}^{k} a_i \chi_{A_i} , \ \sum_{i=1}^{k} a_i^p > 0 .$$

Let $\delta > 0$. Then there exist closed sets $F_i \subset A_i$ such that

$$m(A_i \setminus F_i) \leq \delta \, (\sum_{i=1}^{k} a_i^p)^{-1} , \ i = 1, \ldots, k .$$

Since the sets F_i are closed and disjoint, we can find disjoint open sets G_i such that $F_i \subset G_i \subset G$. The families $\Gamma_\varepsilon(G_i)$ are separate and $\bigcup_{i=1}^{k} \Gamma_\varepsilon(G_i) \subset \Gamma$. We define functions g and v by

$$g(x) = \begin{cases} f(x) & \text{for } x \in \bigcup_{i=1}^{k} (A_i \cap G_i) \\ 0 & \text{for } x \in G \setminus \bigcup_{i=1}^{k} (A_i \cap G_i) \end{cases}$$

and $v(\gamma) = \int_\gamma g ds$ for $\gamma \in \Gamma$. Then $g \leq f$ and, accordingly, $v \leq u$. Hence

$$\int_\Gamma u^p dM_p \geq \sum_{i=1}^{k} \int_{\Gamma_\varepsilon(G_i)} u^p dM_p \geq \sum_{i=1}^{k} \int_{\Gamma_\varepsilon(G_i)} v^p dM_p .$$

Since $g | G_i = a_i \chi_{A_i} | G_i$, we deduce from the first part of this proof that

$$\int_{\Gamma_\varepsilon(G_i)} v^p dM_p \geq a_i^p m(A_i \cap G_i) .$$

Combining the above inequalities and recalling that $F_i \subset A_i \cap G_i$ and $m(F_i) \geq m(A_i) - \delta(\sum_{i=1}^{k} a_i^p)^{-1}$, we obtain

$$\int_\Gamma u^p dM_p \geq \int_G f^p dm - \delta$$

and, letting $\delta \to 0$,

$$\int_\Gamma u^p dM_p \geq \int_G f^p dm .$$

If f is an arbitrary non-negative Borel function, approximation with simple functions concludes the proof.

5.4. Remarks. (a) In Theorem 5.1 the open set G cannot be replaced by an arbitrary Borel set. For example, if B consists of those points in a unit square $Q \subset R^2$ whose coordinates are irrational, then every component of B is a one-point-set and, consequently, all paths in $\Gamma(B)$ are constant functions. Then $l \equiv 0$ and $\int_{\Gamma(B)} l^p dM_p = 0$, but $\int_B 1^p dm = 1$.

(b) The condition $\Gamma_\varepsilon(G) \subset \Gamma$ for some $\varepsilon > 0$ in 5.1 cannot be replaced by the condition $G \subset |\Gamma|$. This can be seen if we let Γ be the family of all non-constant paths $\gamma \in \Gamma(R^n)$ such that $|\gamma|$ contains the origin and use the fact that $M_n(\Gamma) = 0$ ([6] 7.9).

(c) Consider the integral $\int_{\Gamma(G)} u^q dM_p$ for $q < p$ in the situation of 5.1. Suppose that $f(x) > 0$ in a subset of G with positive measure. Set $f_1 =$

inf $(f, 1)$ and $u_1(\gamma) = \int_\gamma f_1 ds$ for $\gamma \in \Gamma(G)$. Then $u_1 \leq l$ and $u_1 \leq u$. Using 5.1, we obtain for every $\varepsilon > 0$

$$0 < \int_G f_1^p dm = \int_{\Gamma_\varepsilon(G)} u_1^p dM_p \leq \int_{\Gamma_\varepsilon(G)} l^{p-q} u_1^q dM_p \leq \varepsilon^{p-q} \int_{\Gamma(G)} u^q dM_p .$$

This shows that

$$\int_{\Gamma(G)} u^q dM_p = \infty .$$

5.5. Theorem. *Suppose that $G \subset R^n$ is open and that Γ is a path family such that $\Gamma_\varepsilon(G) \subset \Gamma \subset \Gamma(G)$ for some $\varepsilon > 0$. If f and g are non-negative Borel functions $G \to \bar{R}$, and $u(\gamma) = \int_\gamma f ds$ and $v(\gamma) = \int_\gamma g ds$ for $\gamma \in \Gamma$, then*

$$\int_\Gamma (u^p + v^p) dM_p = \int_\Gamma u^p dM_p + \int_\Gamma v^p dM_p .$$

Proof. Suppose first that f and g are simple Borel functions. Then they can be presented in the form

$$f = \sum_{i=1}^k a_i \chi_{C_i} , \quad g = \sum_{i=1}^k b_i \chi_{C_i} ,$$

where the sets C_i are disjoint Borel sets.

Let $\delta > 0$. As in the proof of 5.1, we can find disjoint open sets G_i , $i = 1, \ldots, k$, such that

$$\tag{5.6} \begin{aligned} \int_G f^p dm - \delta &\leq \sum_{i=1}^k a_i^p m(C_i \cap G_i) \\ \int_G g^p dm - \delta &\leq \sum_{i=1}^k b_i^p m(C_i \cap G_i) . \end{aligned}$$

Define

$$f_0(x) = \begin{cases} f(x) & \text{for } x \in \bigcup_{i=1}^k (C_i \cap G_i) \\ 0 & \text{for } x \in G \setminus \bigcup_{i=1}^k (C_i \cap G_i) \end{cases}$$

and

$$g_0(x) = \begin{cases} g(x) & \text{for } x \in \bigcup_{i=1}^k (C_i \cap G_i) \\ 0 & \text{for } x \in G \setminus \bigcup_{i=1}^k (C_i \cap G_i) , \end{cases}$$

$u_0(\gamma) = \int_\gamma f_0 ds$ and $v_0(\gamma) = \int_\gamma g_0 ds$ for $\gamma \in \Gamma$. By 5.1, we obtain from the inequalities (5.6)

$$\int_\Gamma u^p dM_p - \delta \leq \sum_{i=1}^k \int_{\Gamma(G_i)} u_0^p dM_p \tag{5.7}$$
$$\int_\Gamma v^p dM_p - \delta \leq \sum_{i=1}^k \int_{\Gamma(G_i)} v_0^p dM_p .$$

Since $u_0(\gamma) = a_i \int_\gamma \chi_{C_i} ds$ and $v_0(\gamma) = b_i \int_\gamma \chi_{C_i} ds$ for $\gamma \in \Gamma(G_i)$, we have

$$(u_0(\gamma)^p + v_0(\gamma)^p)^{1/p} = (a_i^p + b_i^p)^{1/p} \int_\gamma \chi_{C_i} ds = \int_\gamma (f_0^p + g_0^p)^{1/p} ds$$

and, by 5.1,

$$\int_{\Gamma(G_i)} (u_0^p + v_0^p) dM_p = \int_{G_i} (f_0^p + g_0^p) dm = \int_{G_i} f_0^p dm + \int_{G_i} g_0^p dm =$$
$$\int_{\Gamma(G_i)} u_0^p dM_p + \int_{\Gamma(G_i)} v_0^p dM_p .$$

This gives, by (5.7) and the inequalities $u_0 \leq u$ and $v_0 \leq v$,

$$\int_\Gamma u^p dM_p + \int_\Gamma v^p dM_p - 2\delta \leq \int_\Gamma (u^p + v^p) dM_p .$$

Hence

$$\int_\Gamma u^p dM_p + \int_\Gamma v^p dM_p \leq \int_\Gamma (u^p + v^p) dM_p .$$

In the general case this follows by 5.1 and the usual method of approximation.

In order to prove the opposite inequality, we apply the reversed Minkowski's inequality (see [4] p. 192) to the functions f^p and g^p and the exponent $1/p$, obtaining

$$\left(\int_\gamma f ds\right)^p + \left(\int_\gamma g ds\right)^p \leq \left(\int_\gamma (f^p + g^p)^{1/p} ds\right)^p \text{ for } \gamma \in \Gamma . \tag{5.8}$$

Let $\{\Gamma_1, \ldots, \Gamma_k\}$ be a separate partition of Γ. Then there exist disjoint Borel sets $S_i \subset G$, $i = 1, \ldots, k$, such that $\mu_\gamma(G \setminus S_i) = 0$

for $\gamma \in \Gamma_i$. Set $a_i = \inf\limits_{\gamma \in \Gamma_i} \left((\int\limits_{\gamma} f ds)^p + (\int\limits_{\gamma} g ds)^p \right)$. Assume that $0 < a_i < \infty$ for all $i = 1, \ldots, k$. If $\gamma \in \Gamma_i$, (5.8) gives $a_i^{1/p} \leq \int\limits_{\gamma} (f^p + g^p)^{1/p} ds$. This implies that $a_i^{-1/p}(f^p + g^p)^{1/p} \in F(\Gamma_i)$. Hence

$$a_i M_p(\Gamma_i) \leq \int\limits_{S_i} (f^p + g^p) dm \,. \tag{5.9}$$

Obviously, this holds also if $a_i = 0$. If $a_i = \infty$, then, by (5.8), $\int\limits_{\gamma} (f^p + g^p)^{1/p} ds = \infty$ for all $\gamma \in \Gamma_i$ and, by 4.20, either $M_p(\Gamma_i) = 0$ or $\int\limits_{S_i} (f^p + g^p) dm = \infty$. Thus (5.9) is true even in this case. From (5.9) and 5.1 we infer that

$$\sum_{i=1}^{k} a_i M_p(\Gamma_i) \leq \int\limits_{\Gamma} u^p dM_p + \int\limits_{\Gamma} v^p dM_p \,.$$

Since $a_i = \inf (u^p + v^p)(\Gamma_i)$, we obtain the desired inequality and the theorem is proved.

5.10. Remark. 5.5 can be proved as above if we consider functions $f_1, \ldots, f_k$ instead of two functions f and g. However, the result may not be directly generalized by induction, for $u^p + v^p$ is not necessarily an integral.

5.11. Next we shall consider the transformation of the modulus integral under quasiconformal mappings in R^n. For this purpose we define the modulus and the modulus integral for all path families, not only for rectifiable paths (cf. [6] section 6). We consider only the case $p = n$. Let Γ be a path family in R^n. We denote by Γ_r the family of all rectifiable paths $\gamma \in \Gamma$ and define

$$M_n(\Gamma) = M_n(\Gamma_r)$$

and

$$\int\limits_{\Gamma} u dM_n = \int\limits_{\Gamma_r} u dM_n \,,$$

if $\int\limits_{\Gamma_r} u dM_n$ is defined. Then the theory of the modulus integral is valid also in this generalized case.

Let D and D' be domains in R^n, $K \geq 1$ and f a K-quasiconformal mapping $D \to D'$. This means that f is a homeomorphism and

$$K^{-1}M_n(\Gamma) \leq M_n(\Gamma') \leq KM_n(\Gamma)$$

for every path family Γ in D. By Γ' we mean the image family $\{f \circ \gamma \mid \gamma \in \Gamma\}$. We generalize this double inequality for the modulus integral.

Let Γ be a path family in D. We denote by Γ_0 the family of all paths $\gamma \in \Gamma$ such that γ is not rectifiable or f is not absolutely continuous on γ or f^{-1} on $f \circ \gamma$. Then $M_n(\Gamma_0) = M_n(\Gamma_0') = 0$ (see [6] 28.2). We define a function $\bar{f} : \Gamma \to \Gamma'$ by $\bar{f}(\gamma) = f \circ \gamma$. Suppose that $S \subset D$ is a Borel set and $\gamma \in \Gamma \setminus \Gamma_0$. Then f is absolutely continuous on γ and f^{-1} on $\gamma' = f \circ \gamma$. By the usual transformation formula for integrals, we see that $\int_\gamma \chi_S ds = 0$ if and only if $\int_{\gamma'} \chi_{f(S)} ds = 0$, which means that $\mu_\gamma(S) = 0$ if and only if $\mu_{\gamma'}(f(S)) = 0$. This implies that $\{\Gamma_1, \ldots, \Gamma_k\}$ is a separate partition of $\Gamma \setminus \Gamma_0$ if and only if $\{\Gamma_1', \ldots, \Gamma_k'\}$ is a separate partition of $\Gamma' \setminus \Gamma_0'$. Let $u : \Gamma' \to \bar{R}$ be a non-negative function. By the quasiconformality of f, we obtain

$$K^{-1} \inf u \circ \bar{f}(\Gamma_i) M_n(\Gamma_i) \leq \inf u(\Gamma_i') M_n(\Gamma_i') \leq K \inf u \circ \bar{f}(\Gamma_i) M_n(\Gamma_i)$$

for any separate partition $\{\Gamma_1, \ldots, \Gamma_k)$ of Γ_0. Summing over i, taking suprema over all separate partitions of $\Gamma \setminus \Gamma_0$ and $\Gamma' \setminus \Gamma_0'$ and using 3.12, we obtain the following theorem:

5.12. Theorem. *Suppose that $K \geq 1$ and f is a K-quasiconformal mapping from a domain $D \subset R^n$ onto a domain $D' \subset R^n$.* If *Γ is a path family in D and $u : \Gamma' \to \bar{R}$ is a non-negative function, then*

$$K^{-1} \int_\Gamma u \circ \bar{f} dM_n \leq \int_{\Gamma'} u dM_n \leq K \int_\Gamma u \circ \bar{f} dM_n .$$

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