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GROUP ACTIONS AND EXTENSION PROBLEMS FOR MAPS OF BALLS

BY

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TIMO ERKAMA

Introduction

In a euclidean space, balls centered at the origin can be characterized as domains invariant under the action of the orthogonal group. Hence it is not surprising that some problems concerning maps of balls have a connection with symmetry. We shall study functions which commute with a pair of group actions, one action on the domain of the map and the other on its codomain. Such a condition imposed on the function often turns out to be equivalent with others, e.g. injectiveness or the existence of a boundary extension.

Some of our results depend on certain category theorems which are established in § 1. In § 2 we study interior maps in higher dimensional euclidean spaces; here some of the main tools come from algebraic topology. The two-dimensional case is slightly different and admits a more detailed discussion, as shown in §§ 3—4. In §§ 5—6 we specialize to conformal and quasiconformal actions and extensions, and in § 7 we interpret some of the previous theorems in terms of Schwarzian derivatives.

§ 1. Category theorems

1.1. Definitions. Let X be a topological space and $A \subset X$. We say that A is *nowhere dense in* X if its closure $\bar{A}$ contains no nonempty open subsets of X. Clearly, each subset of a nowhere dense set is nowhere dense. Moreover, the boundary ∂F of any closed set $F \subset X$ is nowhere dense in X.

If A is the union of a countable collection of nowhere dense subsets of X, then A is *meager in* X and its complement $X - A$ is *residual in* X. Meager sets are also called sets *of first category*. Baire's classical theorem states that no complete metric space is meager in itself.

Let $[-\infty, \infty]$ be the extended real line. Its topology has as a subbase the collection of all intervals $[-\infty, a)$ and $(a, \infty]$ where $a \in \mathbf{R}$. A map f from a topological space X to $[-\infty, \infty]$ is *upper semicontinuous* if $f^{-1}[-\infty, a)$ is open for each $a \in \mathbf{R}$. Similarly f is *lower semicontinuous* if $f^{-1}(a, \infty]$ is open for each $a \in \mathbf{R}$. A map is *semicontinuous*, if it is either upper or lower semicontinuous.

The following result appears in [6, p. 244]:

1.2. Theorem. *If* $f : X \to [-\infty, \infty]$ *is semicontinuous, then the set of points at which* f *is continuous is a residual* G_δ*-set in* X.

Proof. Replacing f by $-f$ if necessary we may assume that f is upper semicontinuous. Let $\mathbf{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers and $\mathbf{Q}$ the set of rational numbers. Then the set of points at which f is *not* continuous coincides with

$$\bigcup_{(q,n) \in \mathbf{Q} \times \mathbf{N}} \partial f^{-1}[q, \infty] \cap f^{-1}[q + 2^{-n}, \infty]$$

where each member of the union is closed and nowhere dense. □

1.3. Theorem. *Let* X *and* Y *be topological spaces and* (Z, d) *a pseudometric space. Let* y_0 *be a limit point of* Y *having a countable local basis of neighborhoods, and let* $F : X \times Y \to Z$ *be a map with the following properties:*

(i) *the map* $x \mapsto F(x, y)$ *from* X *to* Z *is continuous for each* $y \in Y - \{y_0\}$.

(ii) *the map* $y \mapsto F(x, y)$ *from* Y *to* Z *is continuous at* y_0 *for each* $x \in X$.

Then there exists a residual G_δ*-set* $C \subset X$ *such that* F *is continuous at* (x, y_0) *for each* $x \in C$.

Proof. Let $\{V_n\}_{n \in \mathbf{N}}$ be a countable local basis of neighborhoods at y_0. Given $n \in \mathbf{N}$ define a map $d_n : X \to [-\infty, \infty]$ so that $d_n(x)$ equals the diameter of $F(\{x\} \times V_n)$ for each $x \in X$. To show that d_n is lower semicontinuous, assume that $x \in d_n^{-1}(a, \infty]$ where $a \in \mathbf{R}$. Then there exist $y_1, y_2 \in V_n$ such that $d[F(x, y_1), F(x, y_2)] > a$. By (ii) we may assume that y_1 and y_2 are in $V_n - \{y_0\}$ because y_0 is a limit point of Y. By (i) there is a neighborhood U of x such that $d[F(x', y_1), F(x', y_2)] > a$ for each $x' \in U$. Consequently $U \subset d_n^{-1}(a, \infty]$ and d_n is semicontinuous.

By theorem 1.2 the set C_n of points at which d_n is continuous is a residual G_δ-set in X. Then $C = \bigcap_{n \in \mathbf{N}} C_n$ is also a residual G_δ-set in X and we need only show that F is continuous at every point of $C \times \{y_0\}$. Given $x \in C$ and $\varepsilon > 0$ there exists $n \in \mathbf{N}$ such that $d_n(x) < \varepsilon/6$, by (ii). If $y \in V_n - \{y_0\}$, then by continuity of d_n at x and by (i) there exists a neighborhood U of x such that $d_n(x') < \varepsilon/3$ and $d[F(x', y), F(x, y)] < \varepsilon/2$ for each $x' \in U$. For $(x', y') \in U \times V_n$ it follows that

$$d[F(x', y'), F(x, y_0)] \leqq d[F(x', y'), F(x', y)] + d[F(x', y), F(x, y)] \\ + d[F(x, y), F(x, y_0)] < d_n(x') + \varepsilon/2 + d_n(x) < \varepsilon,$$

which proves the continuity of F at (x, y_0). □

Theorem 1.3 is related to a result of Weston [19, Theorem 2]. It contains some classical propositions as special cases:

1.4. Corollary. *Let $\{f_n\}_{n \in \mathbf{N}}$ be a convergent sequence of continuous maps from a topological space X to a pseudometric space Z. Then there exists a residual G_δ-set $C \subset X$ such that $f = \lim f_n$ is continuous at each point of C.*

Proof. Let Y be the subspace of $[-\infty, \infty]$ consisting of all natural numbers and the point $y_0 = \infty$. Apply theorem 1.3 to the map $F : X \times Y \to Z$ defined by $F(x, n) = f_n(x)$ and $F(x, y_0) = f(x)$. □

For real functions corollary 1.4 is due to Baire. In the general case a proof similar to the above has been given by Fort [7, p. 278]. The next corollary also extends one of Baire's classical results.

1.5. Corollary. *Let X and Y be first countable topological spaces and Z a pseudometric space. Suppose that $F : X \times Y \to Z$ is continuous in each variable separately. Then for each $y_0 \in Y$ there exists a residual G_δ-set $C \subset X$ such that F is continuous at (x, y_0) for each $x \in C$.*

Proof. If y_0 is an isolated point of Y, there is nothing to prove. If y_0 is a limit point of Y, apply theorem 1.3. □

Our application of theorem 1.3 depends on the following result which is related to a theorem of Collingwood [5, p. 76]:

1.6. Corollary. *Let B be the open unit ball of a normed linear space, and let f be a continuous map from B to a pseudometric space Z. Suppose that the radial limit $\lim_{r \to 1} f(rw)$ exists for each $w \in \partial B$. Then there exists a residual G_δ-set $C \subset \partial B$ such that $\lim_{r \to 1} f(rw) = \lim_{x \to w} f(x)$ for each $w \in C$.*

Proof. The map $F : \partial B \times [0, 1] \to Z$ defined by $F(w, t) = \lim_{r \to 1} f(rtw)$ satisfies the hypotheses of theorem 1.3 for $y_0 = 1$. Thus there exists a residual G_δ-set $C \subset \partial B$ such that F is continuous at $(w, 1)$ for each $w \in C$. Since $f(x) = F(x/\|x\|, \|x\|)$ for $x \in B - \{0\}$, it follows that $F(w, 1) = \lim_{x \to w} f(x)$ for each $w \in C$. □

§ 2. Interior mappings of a ball

2.1. Definitions. An *action of a group* G *on a set* S is a homomorphism from G to the group $\operatorname{Aut} S$ of bijections of S onto itself. We say that a subgroup H of $\operatorname{Aut} S$ *acts transitively on* a set $E \subset S$, if $E = \{h(x)\,;\, h \in H\}$ for each $x \in E$.

Let $O(n)$ be the multiplicative group of real orthogonal $n \times n$ matrices with determinant 1. Regarding the elements of $O(n)$ as Hilbert space automorphisms of the n-dimensional euclidean space R^n we obtain a natural action of $O(n)$ on R^n. In particular, each $g \in O(n)$ maps the open unit ball $B^n = \{x \in R^n\,;\, \|x\| < 1\}$ as well as the unit sphere $S^{n-1} = \partial B^n$ onto itself. Unless otherwise specified, we shall assume $n \geqq 2$.

2.2. Lemma. $O(n)$ *acts transitively on* S^{n-1}.

Proof. Let $e = (1\,, 0\,, \ldots\,, 0)$ and $y \in S^{n-1}$. Since y is contained in a complete orthonormal set, there exists $g = (a_{ij}) \in O(n)$ such that $y = (a_{11}\,, \ldots\,, a_{n1})$. Hence $y \in \{g(e)\,;\, g \in O(n)\}$. □

2.3. Lemma. *Let* $x\,, y$ *and* z *be points of* S^{n-1} *where* $n \geqq 3$, *and suppose that* z *is orthogonal to* $x - y$. *Then there exists* $g \in O(n)$ *such that* $g(x) = y$ *and* $g(z) = z$.

Proof. By lemma 2.2 we may assume that $z = e$. Then if $x = (x_1\,, \ldots\,, x_n)$ and $y = (y_1\,, \ldots\,, y_n)$, we have $x_1 = y_1$ and consequently $x_2^2 + \ldots + x_n^2 = y_2^2 + \ldots + y_n^2$. By lemma 2.2 there exists $h \in O(n-1)$ such that $h(x_2\,, \ldots\,, x_n) = (y_2\,, \ldots\,, y_n)$. Then $g = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$ has the required properties. □

2.4. Theorem. *Let* $f : S^{n-1} \to S^n$ *be a nonconstant continuous map where* $n \geqq 3$. *Then the following conditions are equivalent:*

(a) f *is injective.*

(b) *there is an action* β *of* $O(n)$ *on* fS^{n-1} *such that* $f(g(x)) = \beta(g)f(x)$ *for each* $g \in O(n)$ *and* $x \in S^{n-1}$.

Proof. If (a) holds and $g \in O(n)$, there is a unique map $\beta(g) : fS^{n-1} \to fS^{n-1}$ such that $\beta(g)f(x) = f(g(x))$ for each $x \in S^{n-1}$. It is clear that $\beta(g') \circ \beta(g) = \beta(g'g)$ for $g\,, g' \in O(n)$, and (b) follows.

Conversely assume that (b) holds. We first consider the case when $f(x) \neq f(y)$ if $x \neq y$ and $x \neq -y$. Since the $(n-1)$-dimensional real projective space cannot be imbedded in S^n [9, p. 172], there exists $x \in S^{n-1}$ such that $f(x) \neq f(-x)$. For each $g \in O(n)$ we have

$$f(g(x)) = \beta(g)f(x) \neq \beta(g)f(-x) = f(g(-x)) = f(-g(x))\,;$$

hence f is injective by lemma 2.2.

In the remaining case there exist $x, y \in S^{n-1}$ such that $f(x) = f(y)$ and $x \neq \pm y$. Since $f(g(x)) = f(g(y))$ for each $g \in O(n)$, we may assume that $y = e$. Then if $x = (x_1, \ldots, x_n)$ and $t = \sqrt{1 - x_1^2}$, we have $t > 0$ because $x \neq \pm e$. The next step is to prove that f is constant in $U = \{z \in S^{n-1}\,;\, \|z - x\| < 2t\}$.

Define $\phi : S^{n-2} \to S^{n-1}$ by $\phi(\xi_1, \ldots, \xi_{n-1}) = (x_1, t\xi_1, \ldots, t\xi_{n-1})$. Then $\phi(S^{n-2})$ is a path connected set whose diameter is at least $2t$, the distance between $x = \phi(x_2/t, \ldots, x_n/t)$ and

$$2x_1 e - x = \phi(-x_2/t, \ldots, -x_n/t).$$

Hence given $z \in U$ there exists $\xi \in S^{n-2}$ such that $\|z - x\| = \|\phi(\xi) - x\|$. Since x is orthogonal to $\phi(\xi) - z$, by lemma 2.3 there exists $g \in O(n)$ such that $g(\phi(\xi)) = z$ and $g(x) = x$. Moreover, since e is orthogonal to $x - \phi(\xi)$, there exists $h \in O(n)$ such that $h(x) = \phi(\xi)$ and $h(e) = e$. Therefore

$$f(\phi(\xi)) = f(h(x)) = \beta(h)f(x) = \beta(h)f(e) = f(h(e)) = f(e) = f(x)$$

and consequently

$$f(z) = f(g(\phi(\xi))) = \beta(g)f(\phi(\xi)) = \beta(g)f(x) = f(g(x)) = f(x).$$

This shows that f is constant in U.

Since each $g \in O(n)$ maps S^{n-1} topologically onto itself, we see from $fgU = \beta(g)fU$ and lemma 2.2 that f is locally constant in S^{n-1}. Thus f is constant because S^{n-1} is connected. This contradicts the assumption, and the proof is complete. □

2.5. Definitions. Let X and Y be topological spaces and let U be open in X. A map $f : U \to Y$ is *open* if it maps each open subset of U onto an open subset of Y. A map f^* is an *extension* of f, if the domain of f^* contains U and $f^*(x) = f(x)$ for each $x \in U$.

2.6. Lemma. *Let $f : U \to Y$ be an open map having a continuous extension $f^* : \bar{U} \to Y$.*

(a) *If X is hausdorff and f is injective, then $f^*\partial U \subset \partial fU$.*

(b) *If Y is hausdorff and $\bar{U}$ is compact, then $\partial fU \subset f^*\partial U$.*

Proof. (a) Since $f^*\partial U \subset \overline{fU}$ by continuity, it suffices to show that $fU \cap f^*\partial U = \emptyset$. Given $x \in U$ and $y \in \partial U$ there exist disjoint open sets D and V in X such that $x \in D \subset U$ and $y \in V$. Since f is open and injective, $f(D)$ and $f(U \cap V)$ are disjoint open sets. By continuity $f(D)$ and $f^*(\bar{U} \cap V)$ are also disjoint and therefore $f(x) \neq f^*(y)$.

(b) By continuity $f^*(\bar{U})$ is a compact subset of $\overline{fU}$ containing fU. Since Y is hausdorff, $f^*(\bar{U})$ is closed and consequently $f^*(\bar{U}) = \overline{fU}$. Hence $\partial fU \subset f^*(\bar{U})$ and because f^*U is open, $\partial fU \subset f^*\partial U$. □

2.7. Theorem. *Let $f : B^n \to S^n$ be an open continuous map whose restriction to the sphere $S^{n-1}(\varrho) = \{x \in R^n ; \|x\| = \varrho\}$ is injective for each $\varrho \in (0\,,1)$. Suppose that for $\varrho\,, \varrho' \in (0\,,1)$ the sets $fS^{n-1}(\varrho)$ and $fS^{n-1}(\varrho')$ are either equal or disjoint. Then f is injective.*

Proof. We prove first that $(0\,,1)$ can be covered by open intervals I such that the restriction of f to the ring $R(I) = \{x \in B^n ; \|x\| \in I\}$ is injective. Given $\varrho \in (0\,,1)$ there exist points $\zeta_1\,, \zeta_2 \in S^n$ which are separated by $fS^{n-1}(\varrho)$, by the Jordan—Brouwer separation theorem. Let $I \subset (0\,,1)$ be an open interval containing ϱ such that $fR(I) \subset S^n - (\{\zeta_1\} \cup \{\zeta_2\})$, and for $r \in I$ define $f_r : S^{n-1} \to S^n - (\{\zeta_1\} \cup \{\zeta_2\})$ by $f_r(x) = f(rx)$. Since each f_r is homotopic to f_ϱ and $f_\varrho S^{n-1}$ separates ζ_1 and ζ_2, it follows from the Borsuk separation theorem [8, p. 275] that none of the maps f_r is homotopic to a constant map and that $fS^{n-1}(r)$ separates ζ_1 and ζ_2 for each $r \in I$.

If $f|R(I)$ (i.e. the restriction of f to $R(I)$) is not injective, there exist $a\,, b \in I$ such that $a < b$ and $fS^{n-1}(a) = fS^{n-1}(b)$. By lemma 2.6 (b) we have $\partial fR(a\,, b) \subset fS^{n-1}(a)$; consequently $fR(a\,, b)$ must contain one of the components of $S^n - fS^{n-1}(a)$. But this is impossible because $fS^{n-1}(a)$ separates ζ_1 and ζ_2. Hence $f|R(I)$ is injective.

Suppose that f is not injective. Then in view of lemma 2.6 (a) $f|B^n - \{0\}$ is not injective and there exist $a\,, b \in (0\,,1)$ such that $a < b$ and $fS^{n-1}(a) = fS^{n-1}(b)$. Let ϱ be the greatest number in $(a\,, b]$ such that $f|R(a\,, \varrho)$ is injective; we show next that $fS^{n-1}(a) = fS^{n-1}(\varrho)$.

Choose $\alpha\,, \beta \in (0\,,1)$ such that $\alpha < \varrho < \beta$ and $f|R(\alpha\,, \beta)$ is injective. Then $\alpha > a$ by definition of ϱ. Moreover, by compactness there exist convergent sequences $\{x_n\}_{n \in \mathbf{N}}$ and $\{y_n\}_{n \in \mathbf{N}}$ in $R(a\,, \beta)$ such that $\lim \|x_n\| \leq \alpha$, $\lim \|y_n\| = \varrho$ and $f(x_n) = f(y_n)$ for each $n \in \mathbf{N}$. Let $x = \lim x_n$ and $y = \lim y_n$; then by continuity $f(x) = f(y)$ and by lemma 2.6 (a) $f(y) \in \partial fR(a\,, \varrho)$. Since f is open, it follows that $\|x\| = a$ and consequently $fS^{n-1}(a) = fS^{n-1}(\varrho)$.

Applying lemma 2.6 we now see that f maps $R(a\,, \varrho)$ topologically onto one of the components of $S^n - fS^{n-1}(\varrho)$. But this is impossible because $R(a, \varrho)$ is homotopically equivalent to S^{n-1} while the $(n-1)$-dimensional homology of $S^n - fS^{n-1}(\varrho)$ is zero [8, p. 361]. The contradiction shows that f is injective. $\square$

2.8. Definition. Let X and Y be topological spaces and $A \subset X$. We say that A is *totally disconnected*, if each of its components consists of a single point. A map $f: X \to Y$ is an *interior map*, if f is continuous and open and $f^{-1}\{y\}$ is totally disconnected for each $y \in Y$.

2.9. Theorem. *Let $f: B^n \to S^n$ be an interior map where $n \geqq 3$. Then the following conditions are equivalent:*

(a) *f is injective.*

(b) *there is an action β of $O(n)$ on fB^n such that $f(g(x)) = \beta(g)f(x)$ for each $g \in O(n)$ and $x \in B^n$.*

Proof. The proof that (a) implies (b) is trivial as in theorem 2.4. Therefore assume that (b) holds. Given $\varrho \in (0, 1)$ theorem 2.4 shows that $f|S^{n-1}(\varrho)$ is injective or constant. Since $S^{n-1}(\varrho)$ is not totally disconnected, $f|S^{n-1}(\varrho)$ is injective.

Let $\varrho' \in (0, 1)$ be such that $fS^{n-1}(\varrho)$ and $fS^{n-1}(\varrho')$ have a common point y. Then $fS^{n-1}(\varrho)$ and $fS^{n-1}(\varrho')$ coincide with the orbit $\{\beta(g)(y) ; g \in O(n)\}$, and f is injective by theorem 2.7. $\square$

2.10. Definitions. We recall some basic facts from the theory of quasiconformal and quasiregular mappings; these will be needed in theorem 2.11. The general references are [12] and [18].

Let D be a domain in R^n. A *path* in D is a continuous map from an interval of $\mathbf{R}$ into D. We define an outer measure M, called the *modulus*, in the space $P(D)$ of all paths in D as follows. Given $\Gamma \subset P(D)$ denote by $F(\Gamma)$ the class of all Borel measurable functions $\varrho: R^n \to [0, \infty]$ such that for each rectifiable $\gamma \in \Gamma$ the integral $\int_\gamma \varrho ds$ with respect to arc length of γ has value $\geqq 1$. Then

$$M(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{R^n} \varrho^n dm_n ,$$

where m_n is the n-dimensional Lebesgue measure. If Γ contains no rectifiable paths, then $M(\Gamma) = 0$.

Let f be a sense-preserving homeomorphism from D onto a domain $D' \subset R^n$. Given $\Gamma \subset P(D)$ let $\Gamma^f = \{f \circ \gamma ; \gamma \in \Gamma\}$ be the image family in $P(D')$. We say that f is *quasiconformal*, if there exists a constant K

with $1 \leqq K < \infty$ such that $K^{-1}M(\Gamma) \leqq M(\Gamma^f) \leqq KM(\Gamma)$ for each $\Gamma \subset P(D)$ [18, Definition 13.1].

Let f be a continuous map from D to R^n. We say that f is *quasiregular,* if there exist a constant K with $1 \leqq K < \infty$ and a set $E \subset D$ with $m_n(E) = 0$ such that the following two conditions are fulfilled:

(i) each $x \in D$ has an open neighborhood $U \subset D$ such that the components of $f|U$ have generalized derivatives (in the Sobolev sense) of class L^n in U.

(ii) if $x \in D - E$ and $h \in S^{n-1}$, then f is differentiable at x and satisfies $\|Df(x)(h)\|^n \leqq KJf(x)$, where $Df(x)$ and $Jf(x)$ are the derivative and the Jacobian of f at x, respectively.

The above definition is a generalization of the analytic definition of quasiconformality [18, Theorem 34.6]. More precisely, an injective map $f: D \to R^n$ is quasiregular if and only if it agrees with a quasiconformal homeomorphism of D. On the other hand, a nonconstant quasiregular map is an interior map [12, Lemma 2.26].

Let β be an action of a group G on a topological space X (cf. Definition 2.1). We say that β is *continuous* if $\beta(g)$ is continuous for each $g \in G$.

2.11. Theorem. *Let $f: B^n \to R^n$ be a nonconstant quasiregular map where $n \geqq 3$. Then the following conditions are equivalent:*

(a) *f is quasiconformal and has a homeomorphic extension to the closure $\bar{B}^n$ of B^n.*

(b) *there is a continuous action β of $O(n)$ on the closure of fB^n such that $f(g(x)) = \beta(g)f(x)$ for each $g \in O(n)$ and $x \in B^n$.*

Proof. Suppose first that (a) holds and let f^* be the extension of f; then in view of lemma 2.6 f^* is a homeomorphism from $\bar{B}^n$ onto the closure of fB^n. For $g \in O(n)$ the map $\beta(g) = f^*g(f^*)^{-1}$ is a homeomorphism of the closure of fB^n onto itself, and we thus obtain a continuous action β satisfying (b).

Suppose conversely that (b) holds. Since f is an interior map, it is quasiconformal by theorem 2.9. To define the extension of f we first show that f has radial limits.

Choose $\varrho \in (0, 1)$ and for each $x \in S^{n-1}$ define $\gamma_x : (\varrho, 1) \to B^n$ by $\gamma_x(t) = tx$. Then the path family $\Gamma = \{\gamma_x ; x \in S^{n-1}\}$ has positive modulus [18, Example 7.5]. Since f is quasiconformal, the modulus of $\Gamma' = \{f \circ \gamma_x ; x \in S^{n-1}\}$ is also positive and it follows that $f \circ \gamma_x$ is rectifiable for some $x \in S^{n-1}$ (cf. 2.10). Thus the radial limit $\zeta = \lim_{r \to 1} f(rx)$ exists and belongs to the closure of fB^n. For $g \in O(n)$ we have

$$\beta(g)(\zeta) = \lim_{r \to 1} \beta(g)f(rx) = \lim_{r \to 1} f(g(rx)) = \lim_{r \to 1} f(rg(x))$$

and hence by lemma 2.2 f has a radial limit at each point of S^{n-1}.

We define the extension f^* by $f^*(x) = \lim_{r \to 1} f(rx)$. Then $f^*(g(x)) = \beta(g)f^*(x)$ for each $g \in O(n)$ and $x \in \bar{B}^n$. Moreover, by corollary 1.6 there exists a residual G_δ-set $C \subset \partial B^n$ such that $f^*(w) = \lim_{x \to w} f(x)$ for each $w \in C$, and by Baire's theorem C is not empty (cf. 1.1). Given $w \in C$ and $g \in O(n)$ we have

$$f^*(g(w)) = \beta(g)f^*(w) = \lim_{x \to w} \beta(g)f(x) = \lim_{x \to w} f(g(x)) \, .$$

Since g maps $\bar{B}^n$ topologically onto itself, it follows that $f^*(g(w)) = \lim_{x \to g(w)} f(x)$; hence f^* is continuous at $g(w)$. Because $g \in O(n)$ is arbitrary, by lemma 2.2 f^* is continuous in the whole of $\bar{B}^n$.

By theorem 2.4, $f^*|S^{n-1}$ is either injective or constant. On the other hand $\partial f B^n$ contains more than one point and is contained in f^*S^{n-1} by lemma 2.6 (b). Hence $f^*|S^{n-1}$ is injective, and in view of lemma 2.6 (a) the same is true of f^*. □

2.12. Corollary. *Let f be a quasiconformal homeomorphism from B^n onto a domain $D \subset R^n$ where $n \geqq 3$. Then f has a homeomorphic extension to the closure of B^n, if and only if every quasiconformal automorphism of D has a continuous extension to the closure of D.*

Proof. Suppose first that f has a homeomorphic extension $f^* : \bar{B}^n \to \bar{D}$, and let ϕ be a quasiconformal automorphism of D. Then $\alpha = f^{-1}\phi f$ is a quasiconformal automorphism of B^n and has therefore a homeomorphic extension $\alpha^* : \bar{B}^n \to \bar{B}^n$ [18, Theorem 17.20]. It is clear that $\phi^* = f^*\alpha^*(f^*)^{-1}$ is a continuous extension of ϕ.

Suppose conversely that every quasiconformal automorphism ϕ of D has a continuous extension ϕ^* to $\bar{D}$. Define an action β of $O(n)$ on $\bar{D}$ such that $\beta(g)(y) = (fgf^{-1})^*(y)$ for each $g \in O(n)$ and $y \in \bar{D}$. Then β is continuous and $f(g(x)) = \beta(g)f(x)$ for each $g \in O(n)$ and $x \in B^n$. By theorem 2.11, f has a homeomorphic extension to the closure of B^n. □

§ 3. Groups of projective transformations

3.1. Introduction. Let $\mathbf{P}$ be the complex projective line with the topology coinduced by the canonical projection $\pi : \mathbf{C}^2 - \{0\} \to \mathbf{P}$. The image of $\mathbf{R}^2 - \{0\}$ under π is the real projective line P; clearly $\mathbf{P}$ and P are homeomorphic with S^2 and S^1, respectively.

Let $SL(2\,, \mathbf{R})$ be the multiplicative group of real 2×2 matrices with determinant 1. There is a continuous action β of $SL(2\,, \mathbf{R})$ on $\mathbf{P}$ such that $\beta(A)\pi(x\,, y) = \pi(ax + by\,, cx + dy)$ whenever $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in

$SL(2, \mathbf{R})$ and $(x, y) \in \mathbf{C}^2 - \{0\}$. We also use the notation $\beta(A) = \{a, b, c, d\}$. The image of β is the group Γ of *real proper projective transformations* of $\mathbf{P}$, and the kernel of β is a group of order 2 generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It is clear that $g(P) = P$ for each $g \in \Gamma$; we shall be interested in those subgroups of Γ which act transitively on P.

The eigenvalues of a matrix $A \in SL(2, \mathbf{R})$ are the solutions of the equation $\lambda^2 - \lambda \operatorname{tr} A + 1 = 0$ where $\operatorname{tr} A$ is the trace of A. The eigenvectors correspond to fixed points of $\beta(A)$ in $\mathbf{P}$. If $\beta(A)$ is not the identity of Γ, the number of fixed points of $\beta(A)$ in P equals the number of real eigenvalues of A. This number can be 0, 1 or 2, and accordingly we call $\beta(A)$ *elliptic, parabolic* or *hyperbolic*. Since $\operatorname{tr} A$ is invariant under conjugation, it follows that the classes of elliptic, parabolic and hyperbolic elements are invariant under inner automorphisms of Γ.

The isotropy subgroup of a point $\zeta \in \mathbf{P}$ is $\Gamma_\zeta = \{g \in \Gamma ; g(\zeta) = \zeta\}$, and we use the special notations $\Gamma^0 = \Gamma_{\pi(i, 1)}$ and $\Gamma' = \Gamma_{\pi(1, 0)}$. Then Γ^0 and Γ' are images under β of the orthogonal and upper triangular subgroups of $SL(2, \mathbf{R})$, respectively. By elementary theory of linear fractional transformations, Γ_ζ is conjugate to Γ^0 (resp. Γ') if and only if $\zeta \in \mathbf{P} - P$ (resp. $\zeta \in P$). It follows that Γ_ζ acts transitively on P if and only if $\zeta \in \mathbf{P} - P$.

3.2. Lemma. *Let g and h be in Γ and let $[g, h] = ghg^{-1}h^{-1}$.*

(a) *If g is elliptic and has no common fixed points with h, then $[g, h]$ is hyperbolic.*

(b) *If g and h are hyperbolic and have precisely one common fixed point, then $[g, h]$ is parabolic.*

Proof. (a) By passing into suitable conjugates we may assume that $g \in \Gamma^0$. Choose $\alpha \in (0, \pi)$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ so that $h = \beta(B)$ and $g = \beta(A)$ where $A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$. Then $\operatorname{tr}[A, B] = 2 + [(a-d)^2 + (b+c)^2] \sin^2\alpha$ is greater than 2 because h is not in Γ^0 by assumption. Hence $[g, h] = \beta[A, B]$ is hyperbolic.

(b) We may assume that $g = \{r, 0, 0, r^{-1}\}$ and $h = \{s, b, 0, s^{-1}\}$ where $|r| \neq 1$ and $b \neq 0$. Then $[g, h] = \{1, sb(r^2 - 1), 0, 1\}$ is parabolic. □

3.3. Lemma. *Each Γ_ζ is a maximal subgroup of Γ.*

Proof. Suppose that Γ_ζ is a proper subgroup of a subgroup $G \subset \Gamma$;

we have to show that $G = \Gamma$. Since the conjugates of a maximal subgroup are maximal, we need only consider the cases of Γ^0 and Γ^t.

In the case of Γ^t it is clear that G acts transitively on P and hence contains Γ_ζ for each $\zeta \in P$. If $g = \{a, b, c, d\}$ is in Γ and $a \neq 0$, then

$$g = \{1, 0, a^{-1}c, 1\}\{a, b, 0, a^{-1}\}$$

where the factors are in $\Gamma_{\pi(0,1)}$ and Γ^t, respectively. If $a = 0$ but $d \neq 0$, then

$$g = \{d^{-1}, b, 0, d\}\{1, 0, cd^{-1}, 1\}$$

and again $g \in G$. In the final case $a = d = 0$ we can write

$$g = \{-b, 2b, 0, c\}\{2, -1, 1, 0\}$$

where the last factor belongs to $\Gamma_{\pi(1,1)}$. The assertion $G = \Gamma$ follows.

In the case of Γ^0 we have $\Gamma_{g\pi(i,1)} \subset G$ for each $g \in G$. On the other hand we can choose $g \in G - \Gamma^0$ so that $g\pi(i, 1) = \pi(\gamma i, 1)$ for some $\gamma > 1$. Given such a γ it suffices to show that Γ^0 and $\Gamma_{\pi(\gamma i,1)}$ generate Γ.

For each $\alpha \in (0, \pi)$ write $k_\alpha = (1 + \gamma^2 \cot^2 \alpha)^{-1/2}$,

$$\begin{aligned} g_\alpha &= \{\gamma k_\alpha \cot \alpha, -k_\alpha, k_\alpha, \gamma k_\alpha \cot \alpha\}, \\ h_\alpha &= \{\cos \alpha, \gamma \sin \alpha, -\gamma^{-1} \sin \alpha, \cos \alpha\}. \end{aligned}$$

Then $g_\alpha \in \Gamma^0$ and $h_\alpha \in \Gamma_{\pi(\gamma i,1)}$ so that $g_\alpha h_\alpha \in G$. Computing

$$g_\alpha h_\alpha = \left\{\frac{\sin \alpha}{\gamma k_\alpha}, (\gamma^2 - 1)k_\alpha \cos \alpha, 0, \frac{\gamma k_\alpha}{\sin \alpha}\right\}$$

and

$$[g_{\pi/2}h_{\pi/2}, g_\alpha h_\alpha] = \left\{1, -\frac{(\gamma^2-1)^2}{2\gamma^3} \sin 2\alpha, 0, 1\right\}$$

we see that G contains each parabolic element of Γ^t. Hence G also contains the elements

$$\left\{1, -\frac{\gamma^2-1}{2\gamma} \sin 2\alpha, 0, 1\right\} g_\alpha h_\alpha = \left\{\frac{\sin \alpha}{\gamma k_\alpha}, 0, 0, \frac{\gamma k_\alpha}{\sin \alpha}\right\}$$

which together with the parabolic transformations generate the whole of Γ^t. Since Γ^t is a maximal subgroup of Γ, we conclude that $G = \Gamma$. □

3.4. Convention. Let $SL(2, \mathbf{R})$ have the topology induced by the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a\,,\,b\,,\,c\,,\,d)$$

from $SL(2\,,\,\mathbf{R})$ to R^4, and let Γ have the topology coinduced by β. Then $SL(2\,,\,\mathbf{R})$ and Γ are connected topological groups and β determines a two-sheeted covering projection from $SL(2\,,\,\mathbf{R})$ onto Γ. Also for $\zeta \in \mathbf{P}$, the map $g \mapsto g(\zeta)$ from Γ to $\mathbf{P}$ is continuous.

3.5. Theorem. *Let G be a closed subgroup of Γ acting transitively on P. Then either $G = \Gamma$ or there exists $\zeta \in \mathbf{P} - P$ such that $G = \Gamma_\zeta$.*

Proof. Suppose that G is not contained in Γ_ζ for any $\zeta \in \mathbf{P}$; we have to show that $G = \Gamma$. Since G acts transitively on P, it contains uncountably many elements. Hence G cannot be discrete, and a well-known result of Siegel [17] implies that each neighborhood of the identity e of G contains elliptic elements of G.

Let $\{g_n\}_{n\in\mathbf{N}}$ be a sequence of elliptic elements of G converging to e. Since G is not contained in any Γ_ζ, lemma 3.2 (a) shows that G contains a hyperbolic element h and that $\{[g_n\,,\,h]\}_{n\in\mathbf{N}}$ is a sequence of hyperbolic elements of G converging to e.

Since G acts transitively on P, each $[g_n\,,\,h]$ has a conjugate $h_n = \{s_n\,,\,b_n\,,\,0\,,\,s_n^{-1}\}$ in $G \cap \Gamma^t$. Then $|s_n| \neq 1$ because h_n is hyperbolic. Moreover $\lim_{n\to\infty}|s_n + s_n^{-1}| = 2$ because the trace of a matrix is invariant under conjugation and $\lim_{n\to\infty}[g_n\,,\,h] = e$.

In order to show that $\Gamma^t \subset G$ we first consider the case when $\pi(1\,,\,0)$ is the only common fixed point for all h_n. Then by lemma 3.2 (b) $G \cap \Gamma^t$ contains a parabolic element $p = \{1\,,\,b\,,\,0\,,\,1\}$ where $b \neq 0$. Considering $h_n^k p h_n^{-k} = \{1\,,\,s_n^{2k}b\,,\,0\,,\,1\}$ for different integers k we see that b can be chosen arbitrarily small. Consequently, because G is a closed subgroup of Γ, it contains each parabolic transformation of Γ^t. Combining these parabolic transformations with the elements of $\{h_n\}_{n\in\mathbf{N}}$ we obtain, because $\lim_{n\to\infty}|s_n| = 1$, a system which generates a dense subgroup of Γ^t. Since G is closed, it follows that $\Gamma^t \subset G$.

In the remaining case each h_n has the same fixed points. By the above reasoning the assertion $\Gamma^t \subset G$ will follow if $G \cap \Gamma^t$ contains at least one parabolic element. Replacing G by a conjugate subgroup if necessary, we may assume that the points $\pi(0\,,\,1)$ and $\pi(1\,,\,0)$ are fixed points for each h_n. Then $b_n = 0$ for $n \in \mathbf{N}$ and we see that the elements of $\{h_n\}_{n\in\mathbf{N}}$ generate a dense subgroup of $\Gamma_{\pi(0,\,1)} \cap \Gamma^t$. Since G is closed, it follows that $\eta_r = \{r\,,\,0\,,\,0\,,\,r^{-1}\}$ is in G for each $r \neq 0$.

Choose $n \in \mathbf{N}$ so that g_n is of the form $\{a\,,\,b\,,\,c\,,\,d\}$ where $a \neq 0$; this is possible because $\lim g_n = e$. Then $\eta_r g_n$ is elliptic for $r = 1$ and hyperbolic for sufficiently large values of r. By continuity there exists $\varrho > 1$ such that $\eta_\varrho g_n$ is parabolic, and the assertion $\Gamma' \subset G$ follows.

Since Γ' is a proper subgroup of G, we conclude by lemma 3.3 that $G = \Gamma$. $\square$

§ 4. Interior mappings of a disc

4.1. Introduction. We want to establish 2-dimensional analogues for the results of section 2. It is convenient to use complex notation and replace B^2 by the open unit disc $U = \{z \in \mathbf{C} \,;\, |z| < 1\}$ and S^1 by the unit circle $T = \partial U$.

We identify the complex plane $\mathbf{C}$ with a subset of $\mathbf{P}$ by means of the imbedding $z \mapsto \pi(i + iz \,,\, 1 - z)$ from $\mathbf{C}$ to $\mathbf{P}$. Then T corresponds to P and each $g \in \Gamma$ maps $\bar{U}$ onto itself. In fact, the set of restrictions $g|U$ with $g \in \Gamma$ coincides with the set of holomorphic automorphisms of U. The complement of $\mathbf{C}$ with respect to $\mathbf{P}$ consists of $\pi(1 \,,\, i)$, »the point at infinity», and accordingly we shall regard $\mathbf{P}$ as the Riemann sphere $\mathbf{C} \cup \{\infty\}$. In particular, the holomorphic automorphisms of $\mathbf{P}$ are linear fractional transformations.

Let G be a group of bijections of a set S onto itself, and let f be a map from S to a set S'. We then write

$$R(S \,,\, G) = \{(x \,,\, g(x)) \,;\, x \in S \;\text{ and }\; g \in G\}$$

$$R(S \,,f) = \{(x \,,\, y) \in S \times S \,;\, f(x) = f(y)\}$$

It is clear that $R(S \,,\, G)$ and $R(S \,,f)$ are both equivalence relations in S.

4.2. Theorem. *Let $f : T \to \mathbf{P}$ be a nonconstant continuous map, and let G be a subgroup of Γ acting transitively on T. Then the following conditions are equivalent:*

(a) *there is a finite normal subgroup F of G such that $R(T, F) = R(T, f)$.*

(b) *there is an action β of G on fT such that $f(g(x)) = \beta(g)f(x)$ for each $g \in G$ and $x \in T$.*

Moreover, if G is not contained in any Γ_ζ, then (a) *and* (b) *hold if and only if f is injective.*

Proof. Assume first that (a) holds and let $(x \,,\, y) \in R(T \,,f)$. Then there exists $h \in F$ such that $y = h(x)$. Given $g \in G$ we have $g(y) = (ghg^{-1})g(x)$ and consequently $(g(x) \,,\, g(y)) \in R(T \,,\, F)$ because F is normal in G. Hence $f(g(x)) = f(g(y))$ and we can define a map $\beta(g) : fT \to fT$ such that $\beta(g)f(x) = f(g(x))$ for $x \in T$. Clearly $\beta(g') \circ \beta(g) = \beta(g'g)$ for $g, g' \in G$, and (b) follows.

Conversely assume that (b) holds. We first consider the case $G = \Gamma_\zeta$ where $\zeta \in U$ and denote by F the stabilizer $\{h \in \Gamma_\zeta ; f(1) = f(h(1))\}$ of $f(1)$ in G. Because f is nonconstant, the orbit $\{h(1) ; h \in F\}$ cannot be dense in T; hence F is finite. On the other hand, given $x, y \in T$ there exist unique elements $g, h \in \Gamma_\zeta$ such that $g(1) = x$ and $h(x) = y$. Since (b) holds and Γ_ζ is abelian, we have $f(g(1)) = f(h(g(1)))$ if and only if $f(1) = f(h(1))$. This is equivalent to $h \in F$, and the proof of the case $G = \Gamma_\zeta$ is complete.

In the remaining case G is not contained in any Γ_ζ. If $(x, y) \in R(T, f)$ and $g \in G$, then $f(g(x)) = f(g(y))$ by (b). Since G is dense in Γ by theorem 3.5, it follows that $f(g(x)) = f(g(y))$ for each $g \in \Gamma$. This is possible only if $x = y$ because f is nonconstant. Hence f is injective and (a) holds if F is the trivial subgroup of G. □

4.3. Theorem. *Let $f : U \to \mathbf{P}$ be an interior map, and let G be a subgroup of Γ acting transitively on T. Then the following conditions are equivalent:*

(a) *there is a finite normal subgroup F of G such that $R(U, F) = R(U, f)$.*

(b) *there is an action β of G on fU such that $f(g(z)) = \beta(g)f(z)$ for each $g \in G$ and $z \in U$.*

Moreover, if G is not contained in any Γ_ζ, then (a) *and* (b) *hold if and only if f is injective.*

Proof. The proof that (a) implies (b) is trivial as in theorem 4.2. To prove the converse we again start with the case $G = \Gamma_\zeta$ where $\zeta \in U$. Precomposing f with a suitable holomorphic automorphism of U we may assume that $\zeta = 0$.

Given $\varrho \in (0, 1)$ let f_ϱ be the restriction of f to $S^1(\varrho)$ (cf. 2.7). Then by theorem 4.2 there is a subgroup F_ϱ of Γ_0 of finite order n_ϱ such that $R(S^1(\varrho), F_\varrho) = R(S^1(\varrho), f_\varrho)$. In particular, $fS^1(\varrho)$ is a Jordan curve. Choose two points ζ_1 and ζ_2 from different components of $\mathbf{P} - fS^1(\varrho)$ and a homeomorphism h from $\mathbf{P} - (\{\zeta_1\} \cup \{\zeta_2\})$ onto $\mathbf{C} - \{0\}$, and define $\gamma_\varrho : [0, 2\pi] \to \mathbf{C} - \{0\}$ by $\gamma_\varrho(t) = h(f(\varrho e^{it}))$. Then n_ϱ coincides with the absolute value of the index $n(\gamma_\varrho, 0)$ of γ_ϱ with respect to 0 [1, Section I.10]. Since $n(\gamma_\varrho, 0)$ depends continuously on ϱ, it follows that the map $\varrho \mapsto n_\varrho$ from $(0, 1)$ to $\mathbf{N}$ is constant.

Let n be the value of n_ϱ for $\varrho \in (0, 1)$, and define $f_0 : U \to \mathbf{P}$ so that $f_0(z^n) = f(z)$ for $z \in U$. Then f_0 is an interior map whose restriction to $S^1(\varrho)$ is injective for each $\varrho \in (0, 1)$. Moreover, if $f_0S^1(\varrho)$ and $f_0S^1(\varrho')$

have a common point y, they coincide with the orbit $\{\beta(g)(y) ; g \in \Gamma_0\}$. Thus by theorem 2.7 f_0 is injective and (a) holds if F is the subgroup of Γ_0 of order n.

In the remaining case G is not contained in any Γ_ζ. If $(z, w) \in R(U, f)$ and $g \in G$, then $f(g(z)) = f(g(w))$ by (b). Since G is dense in Γ by theorem 3.5, it follows that $f(g(z)) = f(g(w))$ for each $g \in \Gamma$. This is possible only if $z = w$ because f is an interior map. Hence f is injective and (a) holds if F is the trivial subgroup of G. □

Our next goal is to prove an analogue of theorem 2.11. The appropriate mappings are now quasiconformal functions (for the definition and basic properties see [11, Chapter VI]).

4.4. Theorem. *Let $f : U \to \mathbf{P}$ be a nonconstant quasiconformal function, and let G be a subgroup of Γ acting transitively on T. Then the following conditions are equivalent:*

(a) *there is a finite normal subgroup F of G and a continuous extension $f^* : \bar{U} \to \mathbf{P}$ of f such that $R(\bar{U}, F) = R(\bar{U}, f^*)$.*

(b) *there is a continuous action β of G on the closure of fU such that $f(g(z)) = \beta(g)f(z)$ for each $g \in G$ and $z \in U$.*

Moreover, if G is not contained in any Γ_ζ, then (a) *and* (b) *hold if and only if f is injective and has a homeomorphic extension to the closure of U.*

Proof. Suppose first that (a) holds. Then F is contained in Γ_ζ for some $\zeta \in U$ and we may obviously assume that $\zeta = 0$. If F is the trivial subgroup of Γ_0, then f^* is injective and we can define $\beta(g) = f^*g(f^*)^{-1}$ for each $g \in G$. If the order of F is $n > 1$, by lemma 3.2 (a) G is contained in Γ_0 because F is normal in G. Moreover, there exists a unique homeomorphism $f_0 : \bar{U} \to \overline{fU}$ such that $f^*(z) = f_0(z^n)$ for each $z \in \bar{U}$. We can then define an action β satisfying (b) by $\beta(g) = f_0 g^n f_0^{-1}$.

Suppose conversely that (b) holds. We first consider the case when f is injective and thus agrees with a quasiconformal homeomorphism of U. Then there exists $x \in T$ such that f has an angular limit ζ at x [13, p. 73]. Let $\Delta = \{z \in U ; |\arg(1 - z/x)| < \alpha\}$ be a Stolz region at x with $0 < \alpha < \pi/2$. Given $g \in G$ we have

$$\beta(g)(\zeta) = \lim_{\Delta \ni z \to x} \beta(g)f(z) = \lim_{\Delta \ni z \to x} f(g(z))$$

which shows that f has an angular limit at $g(x)$, too. Since G acts transitively on T, it follows that f has an angular limit at every point of T.

We define the extension f^* by $f^*(z) = \lim_{r \to 1} f(rz)$; then $f^*(g(z)) = \beta(g)f^*(z)$ for each $g \in G$ and $z \in \bar{U}$. The continuity of f^* now follows from (b) and corollary 1.6 just as in the proof of theorem 2.11. Hence in view of lemma 2.6 we need only show that $f^*|T$ is injective.

First of all, $f^*|T$ is not constant because $f^*T = \partial fU$ and U is not quasiconformally equivalent with a punctured sphere [11, p. 44]. Next, by theorem 4.2 there is a finite normal subgroup F of G such that $R(T, F) = R(T, f^*|T)$. Hence f^*T is a Jordan curve. Choose two points ζ_1 and ζ_2 from different components of $\mathbf{P} - f^*T$ and a positive number $r < 1$ such that $f^*S^1(\varrho) \subset \mathbf{P} - (\{\zeta_1\} \cup \{\zeta_2\})$ for each $\varrho \in (r, 1)$. Let h be a homeomorphism from $\mathbf{P} - (\{\zeta_1\} \cup \{\zeta_2\})$ onto $\mathbf{C} - \{0\}$ and for $\varrho \in (r, 1]$ define $\gamma_\varrho : [0, 2\pi] \to \mathbf{C} - \{0\}$ by $\gamma_\varrho(t) = h(f^*(\varrho e^{it}))$. Since f is injective, we then have $|n(\gamma_\varrho, 0)| = 1$ for $\varrho < 1$ and consequently $|n(\gamma_1, 0)| = 1$ by continuity. Hence the order of F is one and $f^*|T$ is injective.

In the remaining case f is not injective. However, since f is an interior map, by theorem 4.3 there exist $\zeta \in U$ such that $G = \Gamma_\zeta$ and a finite normal subgroup F of G such that $R(U, F) = R(U, f)$. Precomposing f with a suitable holomorphic automorphism of U we may assume that $\zeta = 0$.

Let n be the order of F and define $f_0 : U \to \mathbf{P}$ so that $f_0(z^n) = f(z)$ for $z \in U$. Then f_0 is a sense-preserving topological map which satisfies $f_0(g^n(z^n)) = \beta(g)f_0(z^n)$ for each $g \in \Gamma_0$ and $z \in U$. Moreover, in view of the geometric characterization of quasiconformal functions [11, Section VI.1] we see that $f_0|U - \{0\}$ is quasiconformal. The same is then true of f_0 by the removability of a point [11, p. 43]. On the other hand, because $\beta(g) = \beta(hg)$ for $g \in \Gamma_0$ and $h \in F$, there is a continuous quotient action β_0 of Γ_0 on $\overline{fU}$ such that $\beta(g) = \beta_0(g^n)$ for each $g \in \Gamma_0$. Thus $f_0(g^n(z^n)) = \beta_0(g^n)f_0(z^n)$ for $g \in \Gamma_0$ and $z \in U$, and the proof of the injective case shows that f_0 has a homeomorphic extension $f_0^* : \bar{U} \to \mathbf{P}$. We can now define the extension of f by $f^*(z) = f_0^*(z^n)$; then it is clear that $R(\bar{U}, F) = R(\bar{U}, f^*)$.

Finally, if G is not contained in any Γ_ζ, then in view of theorem 4.3 (a) and (b) hold if and only if f is injective and has a homeomorphic extension to the closure of U. □

4.5. Corollary. *Let D be a simply connected domain in $\mathbf{P}$ with at least two boundary points. Then D is a Jordan domain if and only if every holomorphic automorphism of D has a continuous extension to the closure of D.*

Proof. Suppose first that D is a Jordan domain, and let ϕ be a holomorphic automorphism of D. By the Riemann mapping theorem there exists a conformal map $f : U \to D$, and because D is a Jordan domain, f has a homeomorphic extension $f^* : \bar{U} \to \bar{D}$. Then $\alpha = f^{-1}\phi f$ is a holomorphic automorphism of U and hence agrees in U with some $g \in \Gamma$ (cf. 4.1). It follows that $f^* g (f^*)^{-1}$ is a continuous extension of ϕ.

Suppose conversely that every holomorphic automorphism ϕ of D has a continuous extension ϕ^* to $\bar{D}$. Define an action β of Γ on $\bar{D}$ such that $\beta(g) = (fgf^{-1})^*$ for each $g \in \Gamma$. Then β is continuous and $f(g(z)) = \beta(g) f(z)$ for each $g \in \Gamma$ and $z \in U$. Hence by theorem 4.4, f has a homeomorphic extension to the closure of U. □

§ 5. Conformal extensions

5.1. Definition. We define a metric k in $\mathbf{P}$ by the formula

$$k(\pi(z_1, z_2), \pi(w_1, w_2)) = \overline{\text{arc}} \tan \left| \frac{z_1 w_2 - z_2 w_1}{z_1 \bar{w}_1 + z_2 \bar{w}_2} \right|$$

where the principal value of arc tan lies between 0 and $\pi/2$. This *spherical metric* is a complete Riemannian metric compatible with the complex structure of $\mathbf{P}$.

In this section and the next some proofs depend on a normal family argument contained in the following lemma.

5.2. Lemma. *Let D be a domain in $\mathbf{P}$, and for each $n \in \mathbf{N}$ let f_n be a K-quasiconformal map from D onto another domain of $\mathbf{P}$. Suppose that the sequence $\{f_n\}_{n \in \mathrm{N}}$ converges at three distinct points $\zeta_i \in D$ where $i \in \{1, 2, 3\}$, and that the limits $\zeta_i' = \lim f_n(\zeta_i)$ are also distinct. Then there exists a subsequence $\{f_{n_k}\}_{k \in \mathrm{N}}$ which converges to a K-quasiconformal map, uniformly on compact subsets of D.*

Proof. By assumption there exist $n_0 \in \mathbf{N}$ and $d > 0$ such that $k(f_n(\zeta_i), f_n(\zeta_j)) > d$ whenever $i \neq j$ and $n > n_0$. On the other hand, the numbers $k(f_n(\zeta_i), f_n(\zeta_j))$ with $i \neq j$ and $n \leqq n_0$ have a positive lower bound. Hence $\{f_n\}_{n \in \mathrm{N}}$ is a normal family [11, Theorem II.5.1]. The assertion then follows by [11, Theorem II.5.3]. □

5.3. Definition. Let β be an action of a group G on a set $E \subset \mathbf{P}$. We say that β is *conformal* (resp. *quasiconformal*), if for each $g \in G$ the map $\beta(g)$ has a conformal (resp. quasiconformal) extension to a domain of $\mathbf{P}$ containing E.

5.4. Theorem. *Let f be a conformal map from U onto a domain of $\mathbf{P}$, and let G be a subgroup of Γ acting transitively on T. Then the following conditions are equivalent:*

(a) *f has a conformal extension to* $\mathbf{P}$.

(b) *there is a conformal action β of G on* $\mathbf{P}$ *such that $f(g(z)) = \beta(g)f(z)$ for each $g \in G$ and $z \in U$.*

Proof. If (a) holds and f^* is a conformal extension of f to $\mathbf{P}$, then f^* maps $\mathbf{P}$ onto itself and we can define β by $\beta(g) = f^*g(f^*)^{-1}$. Suppose conversely that (b) holds. We again start with the case $G = \Gamma_\zeta$ and may assume that $\zeta = 0$.

The *Schwarzian derivative* Sf of f is a holomorphic function of U defined by

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

where f', f'' and f''' are ordinary derivatives of f. (Note that f maps at most one point of U to $\mathbf{P} - \mathbf{C}$; such a point is a removable singularity for Sf.) Given $g \in \Gamma_0$ and $z \in U$ the composition rules for Sf [2, p. 130] yield $Sf(g(z)) \cdot g'(z)^2 = Sf(z)$, because g and $\beta(g)$ are both linear fractional transformations. On the other hand, the elements of Γ_0 are of the form $z \mapsto xz$ with $x \in T$, and consequently $Sf(xz) \cdot x^2 = Sf(z)$ for each $x \in T$ and $z \in U$.

For a fixed $z \in U$, the map $x \mapsto Sf(xz) \cdot x^2$ is holomorphic in the disc $\{x \in \mathbf{C} ; |x| < 1/|z|\}$ and takes the constant value $Sf(z)$ on T. Thus $Sf(xz) \cdot x^2 = Sf(z)$ holds for each $x \in U$, proving that $Sf \equiv 0$. Hence f agrees in U with a linear fractional transformation and (a) follows.

In the remaining case G is not contained in any Γ_ζ so that G is dense in Γ. We now extend the definition of β to the whole of Γ in the following way. Given $g \in \Gamma$ choose a sequence $\{g_n\}_{n \in \mathbf{N}}$ in G so that $\lim g_n = g$; then $\lim \beta(g_n)f(z) = f(g(z))$ for each $z \in U$. By lemma 5.2 a subsequence $\{\beta(g_{n_k})\}_{k \in \mathbf{N}}$ converges uniformly to a linear fractional transformation, and we set $\beta(g) = \lim_{k\to\infty} \beta(g_{n_k})$. Then $f(g(z)) = \beta(g)f(z)$ for each $g \in \Gamma_0$ and $z \in U$, and (a) follows by the previous part of the proof. □

5.5. Theorem. *Let f be a conformal map from U onto a domain of* $\mathbf{P}$. *Then the following conditions are equivalent:*

(a) *f has a conformal extension to a domain containing $\bar{U}$.*

(b) *there exist $\zeta \in U$ and a conformal action β of Γ_ζ on a domain D of* $\mathbf{P}$ *containing $\overline{fU}$ such that $f(g(z)) = \beta(g)f(z)$ for each $g \in \Gamma_\zeta$ and $z \in U$.*

Proof. Suppose first that (a) holds and let f^* be a conformal extension of f to a disc $\Delta_r = \{z \in \mathbf{C} ; |z| < r\}$ where $r > 1$. Then f maps Δ_r

onto a domain D containing $\overline{fU}$, and (b) is fulfilled if we choose $\zeta = 0$ and define $\beta(g) = f^*g(f^*)^{-1}$ for each $g \in \Gamma_0$.

Suppose conversely that (b) holds; we may again assume that $\zeta = 0$. Then each $\beta(g)$ determines a holomorphic automorphism of $D - \{f(0)\}$. Since every orbit $\{\beta(g)f(z) ; g \in \Gamma_0\}$ with $z \neq 0$ has limit points in D, it follows that D is simply connected [14, p. 257]. Thus there exists a conformal map $\phi : \Delta \to D$ where Δ is either U, $\mathbf{C}$ or $\mathbf{P}$.

If $\Delta = \mathbf{P}$, then also $D = \mathbf{P}$ and (a) holds by theorem 5.4. If $\Delta = \mathbf{C}$, then D is $\mathbf{P}$ minus a point and each $\beta(g)$ can be extended to a holomorphic automorphism of $\mathbf{P}$. Therefore theorem 5.4 implies (a) in this case too. It remains to study the case $\Delta = U$.

Clearly we can choose ϕ so that $\phi(0) = f(0)$. Then $\{\phi^{-1}\beta(g)\phi ; g \in \Gamma_0\}$ is an infinite group of rotations around 0, each mapping $\phi^{-1}fU$ onto itself. Consequently $\phi^{-1}fU$ is a disc Δ_r where $r < 1$. Moreover, there exists $x \in T$ such that $f(z) = \phi(rxz)$ for each $z \in U$, and we can define an extension $f^* : \Delta_{1/r} \to D$ of f by $f^*(z) = \phi(rxz)$. □

Combining lemma 5.2 with a category argument we can replace condition (b) of theorem 5.5 by an apparently weaker one:

5.6. Theorem. *Let f be a conformal map from U onto a domain of $\mathbf{P}$. Then the following conditions are equivalent:*

(a) *f has a conformal extension to a domain containing $\bar{U}$.*

(b) *there exist $\zeta \in U$ and a conformal action β of Γ_ζ on $\overline{fU}$ such that $f(g(z)) = \beta(g)f(z)$ for each $g \in \Gamma_\zeta$ and $z \in U$.*

Proof. We need only show that (b) implies (a) and may again assume that $\zeta = 0$. For each $g \in \Gamma_0$ let a_g be the greatest number in the interval $(0 , 1]$ such that $\beta(g)$ has a conformal extension $\beta(g)^*$ to the domain

$$D(a_g) = \bigcup_{z \in U} \{\zeta \in \mathbf{P} ; k(\zeta , f(z)) < a_g\} .$$

The first step is to prove that the map $\delta : \Gamma_0 \to [-\infty , \infty]$ defined by $\delta(g) = a_g$ is upper semicontinuous.

Suppose that $\{g_n\}_{n \in \mathbf{N}}$ is a convergent sequence of points of $\delta^{-1}[a , \infty]$ where $a > 0$, and let $g = \lim g_n$. For each $n \in \mathbf{N}$ denote by ϕ_n the restriction of $\beta(g_n)^*$ to $D(a)$. Then $\lim_{n\to\infty} \phi_n(f(z)) = f(g(z))$ for each $z \in U$, and hence by lemma 5.2 there exists a subsequence $\{\phi_{n_k}\}_{k \in \mathbf{N}}$ of $\{\phi_n\}_{n \in \mathbf{N}}$ which converges to a conformal map, uniformly on compact subsets of $D(a)$. Since the limit is a conformal extension of $\beta(g)$, we have $\delta(g) \geqq a$. Thus $\delta^{-1}[a , \infty]$ is closed and δ is semicontinuous.

The next step is to show that δ has a positive lower bound. By compactness of Γ_0 we need prove this only locally. Choose $g \in \Gamma_0$ and let $g_0 \in \Gamma_0$ be a point of continuity of δ (cf. theorem 1.2). Then there is a neighborhood N of the identity of Γ_0 such that $\delta(g'g_0) \geqq \frac{1}{2}\delta(g_0)$ for $g' \in N$. Moreover, there is $\alpha_g \in (0\,,\,\delta(gg_0^{-1})]$ such that $\beta(gg_0^{-1})^*$ maps $D(\alpha_g)$ into $D(\frac{1}{2}\delta(g_0))$. If $g' \in N$, then $g'g = (g'g_0)(gg_0^{-1})$ because Γ_0 is abelian. It follows that $\beta(g'g_0)^* \circ (\beta(gg_0^{-1})^* | D(\alpha_g))$ is a conformal extension of $\beta(g'g)$ to $D(\alpha_g)$; hence δ has the lower bound α_g in Ng.

Let $a > 0$ be a lower bound of δ in Γ_0. Then the family $\{\beta(g)^* | D(a)\}_{g \in \Gamma_0}$ satisfies a uniform Lipschitz condition in every compact subset of $D(a)$ [11, p. 73]. In particular, there exists $\alpha \in (0\,,\,a]$ such that $\beta(g)^* D(\alpha) \subset D(a)$ for each $g \in \Gamma_0$. It follows that $D = \bigcup_{h \in \Gamma_0} \beta(h)^* D(\alpha)$ is a domain in $D(a)$ containing $\overline{fU}$. Moreover, $\beta(gh)^* | D(\alpha) = \beta(g)^* \circ (\beta(h)^* | D(\alpha))$ for $g\,,\,h \in \Gamma_0$.

Each point of D has an expression $\beta(h)^*(y)$ with $h \in \Gamma_0$ and $y \in D(\alpha)$. If $\beta(h)^*(y) = \beta(h')^*(y')$, then $\beta(gh)^*(y) = \beta(gh')^*(y')$ for each $g \in \Gamma_0$. Thus we can define an action β^* of Γ_0 on D such that $\beta^*(g)\beta(h)^*(y) = \beta(gh)^*(y)$ for $g\,,\,h \in \Gamma_0$ and $y \in D(\alpha)$. Now (a) follows by theorem 5.5 because $f(g(z)) = \beta^*(g)f(z)$ for each $g \in \Gamma_0$ and $z \in U$. □

5.7. Corollary. *Let* $C \subset \mathbf{P}$ *be a Jordan curve and let* D *be one of the components of* $\mathbf{P} - C$. *Then* C *is an analytic curve if and only if every holomorphic automorphism of* D *has a conformal extension to a domain containing* $D \cup C$.

Proof. Let f be a conformal map from U onto D and suppose first that C is analytic. Then f has a conformal extension f^* to a disc Δ_r with $r > 1$. If ϕ is a holomorphic automorphism of D, then $f^{-1}\phi f$ agrees in U with some $g \in \Gamma$. Moreover, the inverse image $g^{-1}\Delta_r$ of Δ_r under g is a domain containing $\bar{U}$ and the image of $\Delta_r \cap g^{-1}\Delta_r$ under f^* is a domain D' containing $D \cup C$. It is clear that $f^* g (f^*)^{-1} | D'$ is a conformal extension of ϕ.

Suppose conversely that every holomorphic automorphism ϕ of D has a conformal extension ϕ^* to a domain containing $D \cup C$. Define an action β of Γ on $D \cup C$ such that $\beta(g) = (fgf^{-1})^* | D \cup C$ for each $g \in \Gamma$. Then β is conformal and $f(g(z)) = \beta(g)f(z)$ for each $g \in \Gamma$ and $z \in U$. Hence by theorem 5.6, f has a conformal extension to a domain containing $\bar{U}$. □

§ 6. Quasiconformal extensions

6.1. Theorem. *Let* f *be a conformal map from* U *onto a domain of* $\mathbf{P}$. *Then the following conditions are equivalent:*

(a) *f has a quasiconformal extension to* $\mathbf{P}$.

(b) *there is a quasiconformal action β of Γ on the closure of fU such that $f(g(z)) = \beta(g)f(z)$ for each $g \in \Gamma$ and $z \in U$.*

This theorem is a generalization of a result of Rickman [16, Theorem 2]. We start the proof with two lemmas.

Lemma 1. *Suppose that* (b) *holds, C is a compact subset of Γ and $p \in \mathbf{P} - \overline{fU}$. Then there exists $K \geqq 1$ such that for each $g \in C$ the map $\beta(g)$ has a K-quasiconformal extension $\beta(g)^* : \mathbf{P} \to \mathbf{P}$ with $\beta(g)^*(p) = p$.*

Proof. By a well-known extension theorem for quasiconformal mappings [11, p. 100], each $\beta(g)$ possesses quasiconformal extensions to $\mathbf{P}$ having p as a fixed point. Among these extensions there exists one, say $\beta(g)^*$, with the smallest possible maximal dilatation $K(g)$, in view of lemma 5.2. We prove that the map $\delta : g \mapsto K(g)$ from Γ to $[-\infty, \infty]$ is lower semicontinuous.

Suppose that $\{g_n\}_{n \in \mathbf{N}}$ is a convergent sequence of points of $\delta^{-1}[-\infty, \varrho]$ where $\varrho > 0$, and let $g = \lim g_n$. Then $\lim_{n\to\infty} \beta(g_n)^* f(z) = \beta(g)f(z)$ for each $z \in U$; thus by lemma 5.2 there exists a subsequence $\{\beta(g_{n_k})^*\}_{k \in \mathbf{N}}$ of $\{\beta(g_n)^*\}_{n \in \mathbf{N}}$ which converges to a ϱ-quasiconformal map, uniformly on $\mathbf{P}$. Since the limit is an extension of $\beta(g)$, we have $\delta(g) \leqq \varrho$. This shows that $\delta^{-1}[-\infty, \varrho]$ is closed and that δ is semicontinuous.

Since $SL(2, \mathbf{R})$ is a complete metric space and the canonical map $SL(2, \mathbf{R}) \to \Gamma$ is a two-sheeted covering projection (cf. 3.4), the topology of Γ is induced by a complete metric too. Thus there exists $g_0 \in \Gamma$ such that δ is continuous at g_0, by Baire's theorem and theorem 1.2. Let N be an open neighborhood of the identity of Γ such that $\delta(g' g_0) \leqq 2\delta(g_0)$ for $g' \in N$. Since $\delta(gh) \leqq \delta(g)\delta(h)$ and $\delta(g^{-1}) = \delta(g)$ for $g, h \in \Gamma$, it follows that $\delta(g') \leqq \delta(g' g_0)\delta(g_0^{-1}) \leqq 2\delta(g_0)^2$ for each $g' \in N$.

Since Γ is connected, the family $\{N^n\}_{n \in \mathbf{N}}$ is an open covering of Γ [15, p. 148]. Because C is compact, there exists $m \in \mathbf{N}$ such that $C \subset N^m$. It follows that $K = (2\delta(g_0)^2)^m$ is an upper bound for δ in C. □

Lemma 2. *There exists a compact set $C \subset \Gamma$ with the following property: if x, y and z are points of T with*

$$0 < |x - y| \leqq |y - z| \leqq |x - z| \leqq 1$$

and if $g \in \Gamma_z \cap \Gamma_{-z}$ satisfies $g(x) = y$, then $g \in C$.

Proof. Let

$$\varrho = \frac{(x-y)^2 z^2}{(x^2-z^2)(y^2-z^2)} = \frac{\left(\dfrac{x}{y}-1\right)\left(1-\dfrac{y}{x}\right)}{\left(\dfrac{x}{z}-\dfrac{z}{x}\right)\left(\dfrac{y}{z}-\dfrac{z}{y}\right)} = \frac{\left|\dfrac{x}{y}-1\right|^2}{4\,\mathrm{Im}\,\dfrac{x}{z}\,\mathrm{Im}\,\dfrac{y}{z}}\,;$$

then it is clear that

$$\varrho = |\varrho| \leqq |(x+z)(y+z)|^{-1} \leqq 1/3\,.$$

Moreover, taking into account the identification $z = \pi(i + iz\,, 1 - z)$ (cf. 4.1), a computation shows that

$$g = \{\sqrt{1+\varrho} + \sqrt{\varrho}\,\mathrm{Re}\,z,\ -\sqrt{\varrho}\,\mathrm{Im}\,z,\ -\sqrt{\varrho}\,\mathrm{Im}\,z,\ \sqrt{1+\varrho} - \sqrt{\varrho}\,\mathrm{Re}\,z\}\,.$$

Thus g belongs to the compact set C of elements $\{a\,, b\,, c\,, d\}$ with $a^2 + b^2 + c^2 + d^2 \leqq 10/3$. □

Proof of theorem 6.1. It is again trivial that (a) implies (b). To prove the converse we assume that (b) holds; then by theorem 4.4 f has a homeomorphic extension f^* to the closure of U. We need only show that the boundary curve f^*T of fU is a quasiconformal curve [11, Theorem II.8.3]. Composing f with a suitable rotation of $\mathbf{P}$ we may assume that $f^*\bar{U} \subset \mathbf{C}$.

Let $x\,, y$ and z be points of T with

$$0 < |x-y| \leqq |y-z| \leqq |x-z| \leqq 1\,,$$

and let $\xi = f^*(x)$, $\eta = f^*(y)$ and $\zeta = f^*(z)$. In view of [11, Theorem II.8.7] it suffices to find a constant $M > 0$, independent of the choice of $x\,, y$ and z, such that $|\eta - \zeta| \leqq Md$ where $d = |\xi - \zeta|$.

Let $C \subset \Gamma$ be the compact set described in lemma 2. By lemma 1 there exists $K \geqq 1$ such that for each $g \in C$ the map $\beta(g)$ has a K-quasiconformal extension $\beta(g)^* : \mathbf{C} \to \mathbf{C}$. Choose $g \in \Gamma_z \cap \Gamma_{-z}$ so that $g(x) = y$; then $\beta(g)^*(\zeta) = \zeta$ and $\beta(g)^*(\xi) = \eta$. Since z is the attractive fixed point of g, we have

$$m = \min_{|w-\zeta| = d} |\beta(g)^*(w) - \zeta| < d\,.$$

Applying a lemma of Mori [13, Lemma 4] it follows that

$$|\eta - \zeta| \leqq \max_{|w-\zeta|=d} |\beta(g)^*(w) - \zeta| \leqq e^{7K} m < e^{7K} d\,;$$

hence we can choose $M = e^{7K}$. □

It is natural to ask if Γ can be replaced by some Γ_ζ in theorem 6.1. The following result gives a partial answer in this direction. It also contains theorem 5.4 for $G = \Gamma_\zeta$ as a special case.

6.2. Theorem. *Let f be a conformal map from U onto a domain of* $\mathbf{P}$. *Suppose that there exist $\zeta \in U$ and $K < 49/47$ such that for each $g \in \Gamma_\zeta$ the map fgf^{-1} has a K-quasiconformal extension to* $\mathbf{P}$. *Then f has a K'-quasiconformal extension to $\mathbf{P}$ where $K' = (49K - 47)/(49 - 47K)$.*

Proof. We may obviously assume that $\zeta = 0$ and $fU \subset \mathbf{C}$. Moreover, in view of a well-known result of Ahlfors and Weill [2, Theorem 5] we need only show that

$$\sup_{z \in U} |Sf(z)|(1 - |z|^2)^2 \leqq 96 \frac{K-1}{K+1} .$$

Given $z \in U$ let h be a holomorphic (affine) automorphism of $\mathbf{C}$ such that $h(0) = f(z)$, $hU \subset fU$ and hT meets the boundary of fU. For each $x \in T$ let g_x be the element of Γ_0 with constant derivative x, and let ϕ_x be a K-quasiconformal extension of fg_xf^{-1} to $\mathbf{P}$. If $k = h^{-1} \circ f$, we have $f \circ (g_x | U) = \phi_x \circ f = \phi_x \circ h \circ k$ and hence

$$Sf(xz) \cdot x^2 = S(\phi_x \circ h)(0) \cdot k'(z)^2 + Sf(z)$$

by the transformation rules for Sf.

Since ϕ_x is K-quasiconformal,

$$|S(\phi_x \circ h)(0)| \leqq 6 \frac{K-1}{K+1}$$

[10, Corollary 2]. Moreover, $|k'(z)|(1 - |z|^2) \leqq 4$ by Koebe's one-quarter theorem. Therefore

$$|Sf(xz) \cdot x^2 - Sf(z)| \leqq 96 \frac{K-1}{K+1} (1 - |z^2|)^{-2}$$

for $x \in T$, and by the maximum principle the same is true for each $x \in U$. The substitution $x = 0$ yields

$$|Sf(z)|(1 - |z|^2)^2 \leqq 96 \frac{K-1}{K+1} ,$$

the desired estimate. □

§ 7. Schwarzian derivatives

7.1. Introduction. Let Σ be the set of Schwarzian derivatives of univalent functions. Each element of Σ is then of the form $\phi = Sf$ where f is a conformal map from U onto a domain of $\mathbf{P}$. We define a metric d in Σ by the formula

$$d(\phi , \psi) = \sup_{z \in U} |\phi(z) - \psi(z)|(1 - |z|^2)^2 .$$

It is well-known that (Σ , d) is a complete metric space and that $d(\phi , 0) \leqq 6$ for each $\phi \in \Sigma$ [3, Section 1].

There is a continuous action β of the opposite group of Γ on Σ such that $\beta(g)(\phi)(z) = \phi(g(z))g'(z)^2$ for $g \in \Gamma , \phi \in \Sigma$ and $z \in U$. Each $\beta(g)$ is in fact an isometry of Σ. We say that a set $S \subset \Sigma$ is *invariant* if $\beta(g)S = S$ for each $g \in \Gamma$.

Let $f : U \to D$ and $g : U \to D'$ be conformal maps onto domains of $\mathbf{P}$. There is an equivalence relation H in Σ such that $(Sf , Sg) \in H$ if and only if the map $g \circ f^{-1}$ has a homeomorphic extension to a domain containing the closure of D. Replacing the word »homeomorphic» by »quasiconformal» and »conformal», we obtain two additional equivalence relations in Σ, denoted by Q and C. Each $\phi \in \Sigma$ then belongs to three different equivalence classes $H(\phi)$, $Q(\phi)$ and $C(\phi)$. In particular, $Q(0)$ is the universal Teichmüller space defined in [3].

7.2. Theorem. *Let $\phi \in \Sigma$ and let L denote one of the letters H, Q and C. Then $\phi \in L(0)$ if and only if $L(\phi)$ is invariant.*

Proof. Let $f : U \to D$ be a conformal map such that $\phi = Sf$. Then $H(\phi)$ is invariant if and only if for each $g \in \Gamma$ the map fgf^{-1} has a homeomorphic extension to a domain containing the closure of D. In view of corollary 4.5 this is equivalent to $\phi \in H(0)$. If $L = Q$ or $L = C$, we obtain the result by appealing to theorem 6.1 or theorem 5.6 instead of corollary 4.5. □

The condition of theorem 7.2 can be weakened if ϕ is restricted to the set Σ_0 of elements of Σ for which the map $t_\phi : \Gamma \to \Sigma$ defined by $t_\phi(g) = \beta(g)(\phi)$ is continuous.

7.3. Theorem. *Let $\phi \in \Sigma_0$ and let L denote one of the letters H, Q and C. Then $\phi \in L(0)$ if and only if $L(\phi)$ contains a nonempty open subset of an invariant subset of Σ_0.*

Proof. Since Σ_0 is invariant (see proposition 7.5), we need only prove the sufficiency. Let $O \subset L(\phi)$ be a nonempty open subset of an invariant subset $S \subset \Sigma_0$, and let $f : U \to D$ be a conformal map such that $Sf \in O$. Since t_{Sf} is continuous and S is invariant, there exists a neighborhood N of the identity of Γ such that $t_{Sf}(g) = S[f \circ (g | U)]$ is in O for each $g \in N$. It follows that for $g \in N$ the map fgf^{-1} has a homeomorphic, quasiconformal or conformal extension to a domain containing the closure of D, and since N generates Γ, the same is true for each $g \in \Gamma$. Hence $L(\phi)$ is invariant and $\phi \in L(0)$ by theorem 7.2. □

7.4. Remark. It is well known that $Q(0)$ is open in Σ, and it has also been conjectured that $Q(0)$ is dense in Σ (see [3, 1.6 and 1.7] and [4, p. 598]). If this conjecture were true, $Q(0)$ would consist of those $\phi \in \Sigma$ for which $Q(\phi)$ contains a nonempty open subset of Σ. This problem was the original motivation for theorem 7.3.

The set Σ_0 does not seem to appear in the literature. The following is a list of some of its properties.

7.5. Proposition. *Σ_0 is a closed and invariant proper subset of Σ which contains all ϕ with $\lim_{|z|\to 1} |\phi(z)|(1 - |z|^2)^2 = 0$.*

Proof. We first prove that Σ_0 is closed. Suppose that $\{\phi_k\}_{k\in\mathbf{N}}$ is a convergent sequence of elements of Σ_0, and let $\phi = \lim \phi_k$. To show that t_ϕ is continuous, let $\{g_n\}_{n\in\mathbf{N}}$ be a convergent sequence in Γ with $g = \lim g_n$. For $k , n \in \mathbf{N}$ we have

$$d[t_\phi(g_n) , t_\phi(g)] \leqq d[t_\phi(g_n) , t_{\phi_k}(g_n)] + d[t_{\phi_k}(g_n) , t_{\phi_k}(g)] +$$
$$d[t_{\phi_k}(g) , t_\phi(g)] = 2d(\phi_k , \phi) + d[t_{\phi_k}(g_n) , t_{\phi_k}(g)]$$

because $\beta(g_n)$ as well as $\beta(g)$ is an isometry. Moreover, given $\varepsilon > 0$ there exist $k , m \in \mathbf{N}$ such that $d(\phi_k , \phi) < \varepsilon/4$ and $d[t_{\phi_k}(g_n) , t_{\phi_k}(g)] < \varepsilon/2$ for $n \geqq m$. Then $d[t_\phi(g_n) , t_\phi(g)] < \varepsilon$ for $n \geqq m$ and we see that $\phi \in \Sigma_0$.

Next assume that $\phi \in \Sigma$ satisfies $\lim_{|z|\to 1} |\phi(z)|(1 - |z|^2)^2 = 0$. To show that t_ϕ is continuous, again let $\{g_n\}_{n\in\mathbf{N}}$ be a convergent sequence in Γ with $g = \lim g_n$. For $z \in U$ and $n \in \mathbf{N}$ write

$$M(z , n) = |\phi(g_n(z))g_n'(z)^2 - \phi(g(z))g'(z)^2|(1 - |z|^2)^2 ;$$

we then have to prove that $\lim_{n\to\infty} \sup_{z\in U} M(z , n) = 0$.

Given $\varepsilon > 0$ choose $r \in (0 , 1)$ so that $|\phi(z)|(1 - |z|^2)^2 < \varepsilon/2$ for all $z \in U$ with $|z| \geqq r$. Since $g_n \to g$ uniformly in U, there exists $\varrho \in (0 , 1)$ such that $|g_n(z)| \geqq r$ for each $n \in \mathbf{N}$ and $z \in U$ with $|z| \geqq \varrho$. Then

$$\begin{aligned} M(z , n) &\leqq \big(|\phi(g_n(z))||g_n'(z)|^2 + |\phi(g(z))||g'(z)|^2\big)(1 - |z|^2)^2 \\ &= |\phi(g_n(z))|(1 - |g_n(z)|^2)^2 + |\phi(g(z))|(1 - |g(z)|^2)^2 < \varepsilon \end{aligned}$$

for each $n \in \mathbf{N}$ and $z \in U$ with $|z| \geqq \varrho$.

For $|z| \leqq \varrho$ we obtain

$$\begin{aligned} M(z\,,\,n) &\leqq \big(|\phi(g_n(z)) - \phi(g(z))|\,|g_n'(z)|^2 + |g_n'(z)^2 - g'(z)^2|\,|\phi(g(z))|\big)(1-|z|^2)^2 \\ &= |\phi(g_n(z)) - \phi(g(z))|(1-|g_n(z)|^2)^2 + \\ &\quad + \left|\frac{g_n'(z)^2}{g'(z)^2} - 1\right| |\phi(g(z))|(1-|g(z)|^2)^2 \\ &\leqq |\phi(g_n(z)) - \phi(g(z))| + 6\left|\frac{g_n'(z)^2}{g'(z)^2} - 1\right| . \end{aligned}$$

Since $\phi \in \Sigma$ and $g_n \to g$ uniformly in U, the family $\{\phi \circ (g_n|U)\}_{n\in\mathbf{N}}$ is uniformly bounded in every compact subset of U, hence equicontinuous. Thus $\phi(g_n(z)) \to \phi(g(z))$ uniformly for $|z| \leqq \varrho$. On the other hand $g_n'(z)/g'(z) \to 1$ uniformly in U, and it follows that

$$\lim_{n\to\infty} \sup_{|z|\leqq\varrho} M(z\,,\,n) = 0\ .$$

Together with the estimate for $|z| \geqq \varrho$ this proves that $\phi \in \Sigma_0$.

To show that Σ_0 is a proper subset of Σ, we consider the Koebe function

$$f(z) = \frac{z}{(1-z)^2} = \sum_{n\in\mathbf{N}} nz^n$$

and its Schwarzian derivative

$$\phi(z) = Sf(z) = \frac{-6}{(1-z^2)^2}\ .$$

For each $x \in T$ let g_x be the (unique) element of Γ such that $g_x(z) = xz$ for $z \in U$. Then $\lim_{x\to 1} g_x = e$, the identity of Γ, while

$$d[t_\phi(g_x)\,,\,t_\phi(e)] = 6\sup_{z\in U}\left|\frac{x^2}{(1-x^2z^2)^2} - \frac{1}{(1-z^2)^2}\right|(1-|z|^2)^2 \geqq 6$$

for $x \neq e$. Hence t_ϕ is not continuous at e.

It remains to prove that $\beta(h)\Sigma_0 \subset \Sigma_0$ for each $h \in \Gamma$. Suppose that $g\,,h \in \Gamma$ and $\phi \in \Sigma_0$, and let $\{g_n\}_{n\in\mathbf{N}}$ be a sequence in Γ converging to g. Then $t_\phi(hg) = \lim_{n\to\infty} t_\phi(hg_n)$ which is equivalent to

$$\beta(g)\beta(h)(\phi) = \lim_{n\to\infty} \beta(g_n)\beta(h)(\phi)\ .$$

This shows that $t_{\beta(h)(\phi)}$ is continuous at g. □

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