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TENSOR PRODUCTS OF COMPLEX *L*-SPACES AND CONVOLUTION MEASURE ALGEBRAS

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1. Introduction

J. L. Taylor has defined in [12] an abstract convolution measure algebra as a complex L-space M equipped with a multiplication which makes M a Banach algebra and satisfies certain extra requirements. Complex L-spaces may be characterized as the preduals of commutative W^*-algebras. This is the point of view adopted in the present study of the projective tensor product $M \hat{\otimes} N$ of two complex L-spaces M and N. In Theorem 3.1 $M \hat{\otimes} N$ is identified as the predual of the W^*-tensor product $M^* \overline{\otimes} N^*$. The theory of tensor products set out in section 3 yields a lucid interpretation of the proof of a basic result needed in the construction of the structure semigroup of a commutative convolution measure algebra (Theorem 4.1). Theorems 4.2 and 4.3, which generalize some results of P. S. Chow [2], state that the projective tensor product $M \hat{\otimes} N$ of two convolution measure algebras is a convolution measure algebra whose structure semigroup, in case M and N are commutative, is topologically isomorphic to the product of the structure semigroups of M and N.

2. Preliminaries

In analogy with the classical notion of a real L-space (see e.g. [11, p. 457]), M.A. Rieffel has introduced in [8] the concept of a complex L-space, whose definition in J. L. Taylor's formulation [12] reads as follows: A (partially) ordered complex Banach space M is a *complex L-space* if (i) the real linear subspace M_r generated by the positive cone of M is a real L-space, (ii) for any $\mu \in M$ there exist unique elements $\mathrm{Re}\,\mu$, $\mathrm{Im}\,\mu \in M_r$ such that $\mu = \mathrm{Re}\,\mu + \mathrm{i}\mathrm{Im}\,\mu$, and (iii) $\|\mu\| = \||\mu|\|$ for all $\mu \in M$, where $|\mu| = \bigvee \{\mathrm{Re}\,(e^{i\Theta}\mu) \mid 0 \leq \Theta < 2\pi\}$. Using Kakutani's classical representation theorem for real L-spaces, Rieffel proved in [8, p. 37] that any complex L-space is isometrically linear and order isomorphic to $L^1(\Gamma, m)$ for some localizable measure space (Γ, m), so that its dual may be identified with $L^\infty(\Gamma, m)$. As any such space $L^1(\Gamma, m)$ is, conversely, an abstract complex L-space (see [8]), Proposition 1.18.1 in [9] yields the following characterization.

Theorem 2.1. *The complex L-spaces are precisely the preduals of commutative W^*-algebras.*

The Banach-Stone theorem shows that in the topological dual M^* of any complex L-space there is only one structure of a commutative C^*-algebra compatible with the norm and order determined by M. (In fact the word »commutative» in this statement may be omitted. This can be proved using Kadison's generalization [5] of the Banach-Stone theorem, since every C^*-algebra which is a dual Banach space has an identity.) It is this unique structure in M^* that will be referred to in the sequel.

A bounded linear map $T: M \to N$, with M and N complex L-spaces, is called an *L-homomorphism,* if $T\mu \geq 0$ and $\|T\mu\| = \|\mu\|$ for any $\mu \geq 0$ in M, and if $0 \leq \nu \leq T\mu$ with $\mu \geq 0$ implies that $\nu = T\omega$ for some $\omega \in M$, $0 \leq \omega \leq \mu$. The following basic result is due to J. L. Taylor [12, p. 153]. For an alternative proof, cf. [6, p. 664].

Theorem 2.2. *Let M and N be complex L-spaces. A bounded linear map $T: M \to N$ is an L-homomorphism, if and only if its transpose T^* : $N^* \to M^*$ is a *-homomorphism which preserves the identity.*

For any Banach spaces E and F, $E \hat{\otimes} F$ will denote their projective tensor product (i.e. the completion of the algebraic tensor product $E \otimes F$ with respect to the greatest cross-norm γ) and $E \hat{\hat{\otimes}} F$ their weak tensor product (i.e. the completion of $E \otimes F$ with respect to the least one λ of all cross-norms whose dual norms are also cross-norms). If E_1, E_2, F_1 and F_2 are Banach spaces and $\alpha : E_1 \to E_2$, $\beta : F_1 \to F_2$ bounded linear maps, there exist unique bounded linear maps $\alpha \hat{\otimes} \beta : E_1 \hat{\otimes} F_1 \to E_2 \hat{\otimes} F_2$ and $\alpha \hat{\hat{\otimes}} \beta : E_1 \hat{\hat{\otimes}} F_1 \to E_2 \hat{\hat{\otimes}} F_2$ which map $x \otimes y$ to $\alpha x \otimes \beta y$ for all $x \in E_1$, $y \in F_1$ [11, pp. 349 and 356]. In particular, we shall regard $E^* \otimes F^*$ as a subspace of both $(E \hat{\otimes} F)^*$ and $(E \hat{\hat{\otimes}} F)^*$ by defining $\langle x \otimes y, f \otimes g \rangle = \langle x, f \rangle \langle y, g \rangle$ for $x \in E$, $y \in F$, $f \in E^*$, $g \in F^*$.

If A and B are algebras, there is a unique algebra multiplication in $A \otimes B$ such that $(x \otimes y)(u \otimes v) = xu \otimes yv$ for x, $u \in A$, y, $v \in B$ [1, A III, p. 33]. For two C^*-algebras A and B, an involution is defined in $A \otimes B$ by setting $(x \otimes y)^* = x^* \otimes y^*$ for $x \in A$, $y \in B$, and making the extension by conjugate-linearity. If A and B are Banach algebras, the product described above extends uniquely to a Banach algebra product of $A \hat{\otimes} B$, namely to the mapping $(a, b) \mapsto ab = (\alpha \hat{\otimes} \beta)(\Theta(a \otimes b))$, where $\alpha : A \hat{\otimes} A \to A$ and $\beta : B \hat{\otimes} B \to B$ correspond to the multiplications in A and B, and Θ is the natural isometric isomorphism from $(A \hat{\otimes} B) \hat{\otimes} (A \hat{\otimes} B)$ onto $(A \hat{\otimes} A) \hat{\otimes} (B \hat{\otimes} B)$

[11 p. 358]. See also [3, p. 298] and [14, p. 148]. If A and B are commutative, so are $A \otimes B$ and $A \hat{\otimes} B$. For details on tensor products we refer to [1], [4], [10], [11] and [9].

3. The projective tensor product of two complex L-spaces

Throughout this section M and N denote complex L-spaces. For any compact Hausdorff space Z, $C(Z)$ is the C^*-algebra of all continuous complex functions on Z, and $M(Z) = C(Z)^*$, i.e. $M(Z)$ is the space of regular complex Borel measures on Z. We make the identifications $M^* = C(X)$, $N^* = C(Y)$, where X is the maximal ideal space of M^*, and Y that of N^*. Let μ_X (resp. ν_Y) denote the natural image of $\mu \in M$ in $M(X)$ (resp. $\nu \in N$ in $M(Y)$).

Clearly, $M \otimes N$ may be identified with a subspace of $M^{**} \otimes N^{**}$, and thus with a subspace of $(M^* \hat{\hat{\otimes}} N^*)^*$. Let E, equipped with the induced norm, denote the closure of $M \otimes N$ in $(M^* \hat{\hat{\otimes}} N^*)^*$. It is known that $M^* \hat{\hat{\otimes}} N^*$ is a commutative C^*-algebra having $M^* \otimes N^*$ with the product and involution defined in section 2 as a *-subalgebra. In fact, $C(X) \hat{\hat{\otimes}} C(Y) = C(X \times Y)$ [11, p. 357]. By some arguments used in [9, p. 66] it follows that E^* is a commutative W^*-algebra and the natural embedding of $M^* \hat{\hat{\otimes}} N^*$ into E^* is an isometric *-homomorphism. We denote $E^* = M^* \bar{\otimes} N^*$ and identify $M^* \hat{\hat{\otimes}} N^*$ with its image in $M^* \bar{\otimes} N^*$. Being a separating subspace of E^*, $M^* \otimes N^*$ and hence $M^* \hat{\hat{\otimes}} N^*$ is $\sigma(E^*, E)$-dense in $M^* \bar{\otimes} N^*$. As there is only one C^*-norm on $M^* \otimes N^*$ [9, p. 62], $M^* \bar{\otimes} N^*$ is actually the usual W^*-*tensor product* of M^* and N^* [9, p. 67].

Statements (i) and (iii) in the next theorem are essentially known. Indeed, (i) follows from the proof of Theorem III.4.4 in [2] and (iii) is stated in [6, pp. 664—665]. Our proof, however, is shorter than these measure theoretic considerations. The fact that $M \hat{\otimes} N$ is an L-space could also be proved by using [4, Cor. 4, p. 61] and the representation of a complex L-space as a space $L^1(\Gamma, m)$ [8, p. 37].

Theorem 3.1. (i) *There is an isometric isomorphism Θ from $M \hat{\otimes} N$ onto the closed subspace of $M(X \times Y)$ generated by the product measures $\mu_X \times \nu_Y$, where $\mu \in M$, $\nu \in N$, such that $\Theta(\mu \otimes \nu) = \mu_X \times \nu_Y$ for $\mu \in M$, $\nu \in N$.*

(ii) *With the order determined by the closed convex cone generated by the tensors $\mu \otimes \nu$, where $\mu \in M$, $\mu \geq 0$, $\nu \in N$, $\nu \geq 0$, $M \hat{\otimes} N$ is isometrically linear and order isomorphic to the predual of $M^* \bar{\otimes} N^*$.*

(iii) *With the order described in* (ii), *$M \hat{\otimes} N$ is a complex L-space.*

Proof. There is a linear injection of $M \otimes N$ into $M(X \times Y)$ which makes each $\mu \otimes \nu$ correspond to $\mu_X \times \nu_Y$. When $M \otimes N$ is identified with its image in $M(X \times Y)$, the greatest cross norm γ of $M \otimes N$ agrees with the usual norm of $M(X \times Y) = [C(X) \hat{\hat{\otimes}} C(Y)]^*$. This is a consequence of the remarks made at the end of page 59 in [9]. Thus (i) follows. To prove (ii), we first note that an element f of the space $E \subset (M^* \hat{\hat{\otimes}} N^*)^*$ appearing in the discussion before the theorem is positive as a functional on $M^* \hat{\hat{\otimes}} N^*$, if and only if f is positive as an element of the predual $(M^* \bar{\otimes} N^*)_* \subset (M^* \bar{\otimes} N^*)^*$ of $M^* \bar{\otimes} N^*$. Indeed, since $M^* \hat{\hat{\otimes}} N^*$ is weak* dense in the W^*-algebra $M^* \bar{\otimes} N^*$, and the multiplication in $M^* \bar{\otimes} N^*$ is separately weak* continuous, the identity of $M^* \hat{\hat{\otimes}} N^*$ coincides with that of $M^* \bar{\otimes} N^*$, and since by definition each $f \in E$ has the same norm in $(M^* \hat{\hat{\otimes}} N^*)^*$ and in $(M^* \bar{\otimes} N^*)_*$, Propositions 1.5.1 and 1.5.2 in [9, p. 9] can be applied. By virtue of (i) and the fact that $C(X) \hat{\hat{\otimes}} C(Y) = C(X \times Y)$, $M \hat{\otimes} N$ is isometrically isomorphic to E, the predual of $M^* \bar{\otimes} N^*$. To prove (ii) it is now enough to show that $P_1 = P_2$, where $P_1 = \{m \in M(X \times Y) \mid m \in \Theta(M \hat{\otimes} N),\ m \geq 0\}$ (with the map Θ of (i)) and P_2 is the closed convex cone in $M(X \times Y)$ generated by all $\mu_X \times \nu_Y$ with $\mu \geq 0$ in M and $\nu \geq 0$ in N. As the natural injections $M \to M(X)$ and $N \to M(Y)$ are bipositive, obviously $P_2 \subset P_1$, and the converse inclusion $P_1 \subset P_2$ can be proved by showing that any $m \geq 0$ in $\Theta(M \hat{\otimes} N)$ is the norm limit of linear combinations, with positive coefficients, of product measures $\mu_X \times \nu_Y$ with positive $\mu \in M$, $\nu \in N$. Indeed, since the natural image of M in $M(X)$ (resp. of N in $M(Y)$) is an L-subspace [12, pp. 151—152], a method used in [12, pp. 155—156] shows that each $m \in \Theta(M \hat{\otimes} N)$ is absolutely continuous with respect to some $\mu_X \times \nu_Y$ with $\mu \geq 0$ in M, $\nu \geq 0$ in N, and the Radon-Nikodym theorem yields the conclusion as e.g. in the proof of Theorem 2.2 in [7]. Finally, (iii) is an immediate consequence of (ii) and Theorem 2.1.

Convention. For any complex L-spaces M and N, $M \hat{\otimes} N$ will be regarded as a complex L-space with the order defined in the above theorem. In accordance with (ii) we write $(M \hat{\otimes} N)^* = M^* \bar{\otimes} N^*$.

Theorem 3.2. *Let M_1, M_2, N_1 and N_2 be complex L-spaces and $T_j : M_j \to N_j$ L-homomorphisms, $j = 1, 2$. Then*

$$T_1 \hat{\otimes} T_2 : M_1 \hat{\otimes} M_2 \to N_1 \hat{\otimes} N_2$$

is an L-homomorphism.

Proof. The following calculation shows that the restriction of $(T_1 \hat{\otimes} T_2)^*$ to $N_1^* \otimes N_2^* \subset (N_1 \hat{\otimes} N_2)^*$ is the ordinary algebraic tensor product mapping $T_1^* \otimes T_2^* : N_1^* \otimes N_2^* \to M_1^* \otimes M_2^*$.

For $\mu_1 \in M_1$, $\mu_2 \in M_2$ and $x_1 \in N_1^*$, $x_2 \in N_2^*$ we have

$\langle \mu_1 \otimes \mu_2 , \ (T_1 \hat{\otimes} T_2)^* (x_1 \otimes x_2) \rangle = \langle (T_1 \hat{\otimes} T_2)(\mu_1 \otimes \mu_2) , \ x_1 \otimes x_2 \rangle =$
$= \langle T_1 \mu_1 \otimes T_2 \mu_2 , \ x_1 \otimes x_2 \rangle = \langle T_1 \mu_1 , \ x_1 \rangle \langle T_2 \mu_2 , \ x_2 \rangle =$
$= \langle \mu_1 , \ T_1^* x_1 \rangle \langle \mu_2 , \ T_2^* x_2 \rangle = \langle \mu_1 \otimes \mu_2 , \ T_1^* x_1 \otimes T_2^* x_2 \rangle$. As the linear combinations of the tensors $\mu_1 \otimes \mu_2$ with $\mu_1 \in M_1$, $\mu_2 \in M_2$ are dense in $M_1 \hat{\otimes} M_2$, it follows that $(T_1 \hat{\otimes} T_2)^* (x_1 \otimes x_2) = T_1^* x_1 \otimes T_2^* x_2$. By Theorem 2.2 and the definition of the *-algebra structure in $M_1^* \otimes M_2^*$ and $N_1^* \otimes N_2^*$ the restriction $(T_1 \hat{\otimes} T_2)^* \mid N_1^* \otimes N_2^*$ is thus a *-homomorphism which maps the identity $1 \otimes 1$ of $N_1^* \otimes N_2^*$ to that of $M_1^* \otimes M_2^*$. Since $N_1^* \otimes N_2^*$ is weak* dense in $N_1^* \bar{\otimes} N_2^* = (N_1 \hat{\otimes} N_2)^*$ and the involution in $N_1^* \bar{\otimes} N_2^*$ is weak* continuous and multiplication separately weak* continuous (similar remarks holding for $M_1^* \bar{\otimes} M_2^*$), it follows that $(T_1 \hat{\otimes} T_2)^* : (N_1 \hat{\otimes} N_2)^* \to (M_1 \hat{\otimes} M_2)^*$ is a *-homomorphism preserving the identity. The proof is completed by appealing again to Theorem 2.2.

4. Applications to convolution measure algebras

A *convolution measure algebra* (or *CM-algebra* for short) is a complex L-space M with a Banach algebra product $(\mu , \nu) \mapsto \mu\nu$ such that the unique bounded linear map $\Theta : M \hat{\otimes} M \to M$ for which $\Theta(\mu \otimes \nu) = \mu\nu$, $\mu , \ \nu \in M$, is an L-homomorphism. This is the definition given in [13, p. 812], for the tensor product considered there is just the projective tensor product $M \hat{\otimes} N$ (see Theorem 3.1). For an equivalent definition, see [12].

The theory of commutative CM-algebras hinges on the notion of the *structure semigroup* [12]. An alternative construction of the structure semigroup may be sketched as follows. Let M be a commutative CM-algebra, and let Δ denote the set of the non-zero multiplicative linear functionals on M. The norm closed linear span D of Δ can be shown to be a C^*-subalgebra of M^* containing the identity of M^*. Let S denote the spectrum of D, i.e. the set of all non-zero multiplicative linear functionals on D endowed with the relative weak* topology. For any $F , \ G \in S$ there is a unique element FG of S such that $\langle \gamma , FG \rangle = \langle \gamma , F \rangle \langle \gamma , G \rangle$ whenever $\gamma \in \Delta$. With this multiplication S is a compact topological semigroup, which is topologically isomorphic to the structure semigroup of M in the sense of Taylor [12] (see e.g. [15, Theorem 2.1]). The crucial step in the above construction of S is to prove that $\Delta \cup \{0\}$ contains the identity of M^* and is closed with respect to multiplication and involution. We now proceed to give this result (Theorem 4.1) a proof based on the fact that $(M \hat{\otimes} M)^*$ contains $M^* \otimes M^*$ as a *-subalgebra and has $1 \otimes 1$ as its identity (section 3).

Lemma 4.1. *Let* A *be any Banach algebra and* $\Theta : A \hat{\otimes} A \to A$ *the bounded linear map corresponding to its multiplication. A functional* $f \in A^*$ *is multiplicative, if and only if* $\Theta^* f = f \otimes f$ *for the transpose*

$$\Theta^* : A^* \to (A \hat{\otimes} A)^*$$

of Θ .

Proof. If f is multiplicative and $x , y \in A$, then $\langle x \otimes y , \Theta^* f \rangle = \langle \Theta (x \otimes y) , f \rangle = \langle xy , f \rangle = \langle x , f \rangle \langle y , f \rangle = \langle x \otimes y , f \otimes f \rangle$. Since the linear combinations of the tensors $x \otimes y$ are dense in $A \hat{\otimes} A$, it follows that $\Theta^* f = f \otimes f$. Conversely, if $\Theta^* f = f \otimes f$, a similar calculation shows that $\langle xy , f \rangle = \langle x , f \rangle \langle y , f \rangle$.

Theorem 4.1. (J. L. Taylor [12, p. 157]). *Let* Δ *be the set of the non-zero multiplicative linear functionals on a CM-algebra* M .

(i) *The identity of the* W^**-algebra* M^* *is in* Δ .

(ii) *If* $f , g \in \Delta$, *then* $fg \in \Delta \cup \{0\}$.

(iii) *If* $f \in \Delta$, *then* $f^* \in \Delta$.

Proof. We use the preceding lemma. Since $\Theta^* : M^* \to (M \hat{\otimes} M)^*$ is a *-homomorphism, which preserves the identity, we have $\Theta^* 1 = 1 \otimes 1$, i.e. $1 \in \Delta$. If $f , g \in \Delta$, then $\Theta^* fg = \Theta^* f \Theta^* g = (f \otimes f)(g \otimes g) = fg \otimes fg$ and $\Theta^* f^* = (\Theta^* f)^* = (f \otimes f)^* = f^* \otimes f^*$, $f^* \neq 0$, i.e. $fg \in \Delta \cup \{0\}$ and $f^* \in \Delta$.

Theorem 4.2. *If* M_1 *and* M_2 *are CM-algebras, then* $M_1 \hat{\otimes} M_2$ *is a CM-algebra.*

Proof. By Theorem 3.1 $M_1 \hat{\otimes} M_2$ is a complex L-space. We must show that the bounded linear map $\Theta : (M_1 \hat{\otimes} M_2) \hat{\otimes} (M_1 \hat{\otimes} M_2) \to M_1 \hat{\otimes} M_2$ corresponding to the multiplication in $M_1 \hat{\otimes} M_2$ is an L-homomorphism. If $\Theta_1 : M_1 \hat{\otimes} M_1 \to M_1$ and $\Theta_2 : M_2 \hat{\otimes} M_2 \to M_2$ are the L-homomorphisms corresponding to the multiplications in M_1 and M_2 , then

$$\Theta_1 \hat{\otimes} \Theta_2 : (M_1 \hat{\otimes} M_1) \hat{\otimes} (M_2 \hat{\otimes} M_2) \to M_1 \hat{\otimes} M_2$$

is an L-homomorphism (Theorem 3.2), and so is Θ , for it is easily seen from the description of the positive cone of the projective tensor product of two complex L-spaces given in Theorem 3.1 that the natural isometric isomorphism from $(M_1 \hat{\otimes} M_2) \hat{\otimes} (M_1 \hat{\otimes} M_2)$ onto $(M_1 \hat{\otimes} M_1) \hat{\otimes} (M_2 \hat{\otimes} M_2)$ is also an order isomorphism.

The description of the spectrum of the tensor product of two commutative Banach algebras due to Gelbaum [3] and Tomiyama [14] is crucial for the next result.

Theorem 4.3. *Let* M_1 *and* M_2 *be commutative CM-algebras with struc-*

ture semigroups S_1 *and* S_2, *respectively. The structure semigroup of* $M_1 \hat{\otimes} M_2$ *is topologically isomorphic to the product* $S_1 \times S_2$.

Proof. Let Δ_j (resp. Δ) be the set of the non-zero multiplicative linear functionals on M_j, $j = 1, 2$ (resp. on $M_1 \hat{\otimes} M_2$). Denote by D_j (resp. D) the closed linear span of Δ_j in M_j^* (resp. of Δ in $(M_1 \hat{\otimes} M_2)^*$). We first show that the C^*-algebras $D_1 \hat{\hat{\otimes}} D_2$ and D are isometrically $*$-isomorphic. Let us regard $M_1^* \hat{\hat{\otimes}} M_2^*$ as a subspace of $(M_1 \hat{\otimes} M_2)^*$ in accordance with section 3. Let $\alpha_j : D_j \to M_j^*$, $j = 1, 2$, denote the inclusion maps. When $D_1 \otimes D_2$ and $M_1^* \otimes M_2^*$ are given the λ-norm, the $*$-homomorphism $\alpha_1 \otimes \alpha_2 : D_1 \otimes D_2 \to M_1^* \otimes M_2^*$ is isometric [10, Lemma 2.12, p. 35], and so is

$$\alpha_1 \hat{\hat{\otimes}} \alpha_2 : D_1 \hat{\hat{\otimes}} D_2 \to M_1^* \hat{\hat{\otimes}} M_2^* \subset (M_1 \hat{\otimes} M_2)^* .$$

Applying Lemma 1 in [14, p. 149] we see that a functional $f \in (M \hat{\otimes} N)^*$ belongs to Δ if and only if there exist $\varphi \in \Delta_1$ and $\psi \in \Delta_2$ such that $f = \varphi \otimes \psi$. It follows that the range of $\alpha_1 \hat{\hat{\otimes}} \alpha_2$, being closed, contains D. On the other hand, since $\varphi \otimes 1 \in \Delta$ for all $\varphi \in \Delta_1$, the range of the (continuous) $*$-homomorphism $x \mapsto x \otimes 1$, $x \in D_1$, is contained in D. Similarly, $1 \otimes y \in D$ for all $y \in D_2$. Hence $x \otimes y = (x \otimes 1)(1 \otimes y) \in D$ whenever $x \in D_1$, $y \in D_2$, and so $(\alpha_1 \hat{\hat{\otimes}} \alpha_2)(D_1 \hat{\hat{\otimes}} D_2) \subset D$. We conclude that $\alpha_1 \hat{\hat{\otimes}} \alpha_2$ defines an isometric $*$-isomorphism β from $D_1 \hat{\hat{\otimes}} D_2$ onto D. It is known [14, p. 150] that $S_1 \times S_2$ is homeomorphic to the spectrum of $D_1 \hat{\hat{\otimes}} D_2$ under the mapping $(F, G) \mapsto F \otimes G$, so that $(F, G) \mapsto (\beta^{-1})^* (F \otimes G)$ is a homeomorphism from $S_1 \times S_2$ onto S. From the definition of the multiplication in a structure semigroup it follows that this map is also a semigroup isomorphism. Indeed, identifying D with $D_1 \hat{\hat{\otimes}} D_2$ to simplify the notation so that

$$S = \{F_1 \otimes F_2 \mid F_1 \in S_1 ,\ F_2 \in S_2\} ,$$

and recalling the fact that $\Delta = \{\varphi \otimes \psi \mid \varphi \in \Delta_1 ,\ \psi \in \Delta_2\}$, we have for any F_1, $G_1 \in S_1$, F_2, $G_2 \in S_2$, $\varphi_1 \in \Delta_1$, and $\varphi_2 \in \Delta_2$,

$$\begin{aligned}
&\langle \varphi_1 \otimes \varphi_2 ,\ (F_1 \otimes F_2)(G_1 \otimes G_2)\rangle = \\
&= \langle \varphi_1 \otimes \varphi_2 ,\ F_1 \otimes F_2\rangle \langle \varphi_1 \otimes \varphi_2 ,\ G_1 \otimes G_2\rangle = \\
&= \langle \varphi_1 , F_1\rangle \langle \varphi_2 , F_2\rangle \langle \varphi_1 , G_1\rangle \langle \varphi_2 , G_2\rangle = \\
&= \langle \varphi_1 ,\ F_1G_1\rangle \langle \varphi_2 ,\ F_2G_2\rangle = \langle \varphi_1 \otimes \varphi_2 ,\ F_1G_1 \otimes F_2G_2\rangle , \quad \text{i.e.}
\end{aligned}$$

$(F_1, F_2)(G_1, G_2) = (F_1G_1, F_2G_2)$ is mapped to the product of the images of (F_1, F_2) and (G_1, G_2) in S.

Remark. P. S. Chow has proved (see [2, Theorems III.4.4 and III.4.5]) the following special case of Theorems 4.2 and 4.3: If M and N are L-subalgebras [13, p. 812] of the convolution algebras $M(X)$ and $M(Y)$, respectively, of bounded regular Borel measures on the locally compact semigroups X and Y with a separately continuous multiplication, then $M \hat{\otimes} N$ can be realized as an L-subalgebra of $M(X \times Y)$ (so that it is a CM-algebra), and its structure semigroup, in case X and Y are commutative, is topologically isomorphic to the product of those of M and N.

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