ON IRREDUCIBLE MODULES OF A LIE ALGEBRA WHICH ARE COMPOSED OF FINITE-DIMENSIONAL MODULES OF A SUBALGEBRA

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1. Introduction

Let $G$ be a Lie algebra and $K$ a subalgebra of $G$. If $K$ is semisimple (or at least reductive) then the finite-dimensional $K$-modules are well-known. We can then pose the following question: What are the irreducible $G$-modules which, when regarded as a $K$-module, are direct sums of irreducible finite-dimensional $K$-modules? We call such modules $K$-finite.

This problem has been extensively studied in the following special case (see e.g. [1]—[3], [5], [7]): Let $G'$ be a non-compact semisimple Lie group and let $K'$ be the maximal compact subgroup of $G'$. Let $G$ (resp. $K$) be the Lie algebra of $G'$ (resp. $K'$). As was shown by Harish-Chandra, study of unitary irreducible representations of $G'$ in a Hilbert space leads in a natural way to a study of irreducible $K$-finite $G$-modules.

In this paper $G$ is an arbitrary (finite-dimensional) complex Lie algebra and $K$ is a semi-simple (or reductive) subalgebra of $G$. The work is divided into two parts. In section 3 we study irreducible $G$-modules admitting a vector of maximal weight $\lambda$ with respect to a Cartan subalgebra $H_T$ of $G$ such that $H = K \cap H_T$ is a Cartan subalgebra of $K$. We prove that for a «special» subalgebra $K$ (Definition 3.7) and for any weight $\lambda$ such that the restriction $\lambda|_H$ is a dominant integral weight of $K$ there exists a unique equivalence class of $K$-finite $G$-modules which have the maximal weight $\lambda$.

In section 4 we study irreducible $G$-modules $V$ with the help of the minimal component $V_{\min}$ of $V$; if $\alpha$ is a dominant integral weight of $K$ we denote by $V_\alpha$ the sum of all irreducible finite-dimensional $K$-modules in $V$ which have $\alpha$ as their maximal weight; by definition $V_{\min} = V_\alpha$ if $V_\alpha \neq 0$ and $V_\beta = 0$ for all $\beta < \alpha$. Let rank $G = \text{rank } K$. We prove that if $\alpha$ is «large enough» (see Definition 4.1) then there exists a unique equivalence class $[V]$ of irreducible $K$-finite $G$-modules $V$ such that $V_{\min} = V_\alpha$. Such $G$-modules are called discrete because they are completely characterized by the weight $\alpha$ i.e. by a sequence of integers consisting of the components of $\alpha$.

We take profit at crucial steps (Theorem 3.9 and Lemma 4.7) of the results of J. Lepowsky and G. W. McCollum, [6]: If $V$ is a $G$-module
such that $V_\alpha \neq 0$ for some dominant integral weight $\alpha$ and $V$ is generated by $V_\alpha (V = \mathcal{E}(G)V_\alpha$ where $\mathcal{E}(G)$ is the enveloping algebra of $G$) then $V$ is $K$-finite. In addition $V$ is completely determined by the action of $\mathcal{E}(K)C$ on $V_\alpha$ where $C$ is the centralizer of $K$ in $\mathcal{E}(G)$. These results have earlier been obtained by Harish-Chandra, [1] and [2], in case $G$ is semi-simple.

See also for related recent results by van den Hombergh in »A note on Mickelsson’s step algebra» and »On some Harish-Chandra modules» (to appear in Indagationes Mathematicae).

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2. Notation

In this paper all Lie algebras are finite-dimensional. All algebras and vector spaces are over $\mathbb{C}$, the field of complex numbers. If $A$ is an algebra, $V$ an $A$-module, $S$ a subset of $A$ and $X$ a subset of $V$ we denote by $SX$ the subset $\{sx \mid s \in S, x \in X\}$ of $V$.

We denote by $V^*$ the dual space of a vector space $V$.

Let $K$ be a semi-simple Lie algebra, $H$ a Cartan subalgebra of $K$ and $\langle , \rangle$ the Killing form on $K$. Because $\langle , \rangle$ is non-degenerate, there exists (for each $x \in H^*$) an element $h_x \in H$ such that $\langle h_x, h \rangle = \alpha(h)$ for all $h \in H$. Let $( , )$ be the symmetric non-degenerate bilinear form on $H^*$ defined by $(x, \beta) = \langle h_x, h_\beta \rangle$.

For any Lie algebra $G$, we denote by $x \mapsto \text{ad} x$ the adjoint representation of $G$, $\text{ad} x(y) = [x, y]$.

If $L$ is any Lie algebra then $\mathcal{E}(L)$ is the universal enveloping algebra of $L$. If $L'$ is a subalgebra of $L$ then $\mathcal{E}(L')$ can be identified in a natural way with a subalgebra of $\mathcal{E}(L)$.

If $\{x_1, x_2, \ldots, x_n\}$ is a basis of $L$ then the monomials $x_{i(1)}x_{i(2)}\ldots x_{i(k)}$ ($i(1) \leq i(2) \leq \ldots \leq i(k)$) along with 1 give a basis of $\mathcal{E}(L)$ (Poincaré-Birkhoff-Witt theorem).

For any subset $S$ of $\mathcal{E}(L)$ we denote by $\mathcal{E}(S)$ the left ideal generated by $S$, $\mathcal{E}(S) = \mathcal{E}(L)S$.

Let $\mathcal{B}$ be an associative algebra, $\mathcal{A}_1$ an ideal of $\mathcal{B}$ and $V$ a $\mathcal{B}/\mathcal{A}_1$-module. If $\mathcal{A}_2$ is any ideal of $\mathcal{B}$ such that $\mathcal{A}_2 \subset \mathcal{A}_1$ then by the extension of $V$ into a $\mathcal{B}/\mathcal{A}_2$-module we mean the $\mathcal{B}/\mathcal{A}_2$-module $V$, where the action of $\mathcal{B}/\mathcal{A}_2$ is defined by

$$(b + \mathcal{A}_2)v = i(b)v; \ b \in \mathcal{B}, \ v \in V,$$

where $i: \mathcal{B} \to \mathcal{B}/\mathcal{A}_1$ is the canonical projection.
3. Modules with maximal weight

If $L$ is any Lie algebra and $H_L$ a Cartan subalgebra of $L$ then we can write

$$L = H_L \oplus \bigoplus_{\alpha \neq 0} L_{\alpha}$$

where $L_{\alpha}$ is the root subspace corresponding to the non-zero root $\alpha$. Because we assume $L$ to be finite-dimensional, the sum is finite. By definition $L_{\alpha}$ consists of all elements $x \in L$ such that

$$(\text{ad } h - \alpha(h))x = 0 \text{ for some positive integer } n.$$  

Let $H_L^*$ be the dual of $H_L$ and \{h_1, h_2, \ldots, h_l\} a fixed basis of $H_L$. Let $\lambda, \mu \in H_L^*$ with $\lambda \neq \mu$. We say that $\lambda$ is bigger than $\mu (\lambda > \mu)$ if the first non-zero number in the sequence

$$\lambda(h_1) - \mu(h_1), \ldots, \lambda(h_l) - \mu(h_l)$$

is of the form $x + iy$ with $x > 0$ or $x = 0$ and $y > 0$. We denote by $L_+(L_-)$ the solvable subalgebra of $L$ generated by the subspaces $L_{\alpha}$ with $\alpha > 0$ ($\alpha < 0$).

**Definition 3.1.** Let $V$ be an $L$-module. We denote by $V^+$ the subspace of $V$ consisting of all vectors $v$ with the property $xv = 0$ for all $x \in L_+$. We say that $V$ is bounded above if $1 \leq \dim V^+ < \infty$.

**Lemma 3.2.** If $V$ is an irreducible $L$-module which is bounded above then $\dim V^+ = 1$ and there exists $\lambda \in H_L^*$ such that $hv = \lambda(h)v$ for each $v \in V^+$ and $h \in H_L$. (We say that $\lambda$ is the maximal weight of $V$ and $v$ is a maximal vector.)

**Proof.** Because $1 \leq \dim V^+ < \infty$ and $H_L$ is nilpotent there is a common eigenvector $v \in V^+$ of all $h \in H_L$ ($V^+$ is clearly $H_L$-invariant). Because of the irreducibility of $V$ and of the Poincaré-Birkhoff-Witt theorem we have $V = \mathcal{E}(L_-)v$. If $v' = vw$, $u \in \mathcal{E}(L_-)L_-$, is another vector in $V^+$ then

$$v' = \mathcal{E}(L)v' = \mathcal{E}(L_-)\mathcal{E}(H_L)v'$$

is a non-trivial ($v \notin V'$) invariant subspace if $v' \neq 0$. Thus $v' = 0$ and \{v\} is a basis of $V^+$.

**Definition 3.3.** Let $V$ be an $L$-module. For each $\alpha \in H_L^*$ the weight subspace $V_{\alpha}$ consists of all vectors $v \in V$ for which

$$(h - \alpha(h))^n v = 0 \text{ for all } h \in H_L \text{ and for some positive integer } n.$$  

**Theorem 3.4.** Let $L$ be a Lie algebra and let $H_L$ be a Cartan subalgebra of $L$. For each $\lambda \in H_L^*$ such that $\lambda|_{[H_L, H_L]} = 0$ there exists a unique equivalence class of irreducible $L$-modules which are bounded above and have maximal weight $\lambda$. Any such an $L$-module is a direct sum of weight subspaces
of finite dimension.

Proof. We define \( W^\lambda = \mathcal{E}(L)[\mathcal{L}^\lambda] \) where \( \mathcal{L}^\lambda \) is an ideal,

\[ \mathcal{L}^\lambda = \mathcal{L}(L^\alpha) + \mathcal{L}([h - \lambda(h) \cdot 1 | h \in H_L]) \]

\( W^\lambda \) is an \( L^- \) (and \( \mathcal{E}(L) \)) module in a natural way. Let \( \nu_0 = 1 + \mathcal{L}^\lambda \).

Then

\[ L^-\nu_0 = 0, \quad \h_n = \lambda(h)e_0 \quad (h \in H_L), \quad W^\lambda = \mathcal{E}(L^-)\nu_0. \]

It follows that a basis of \( W^\lambda \) is given by vectors of the type

\[ (*) \quad e_{\beta_1}e_{\beta_2} \ldots e_{\beta_k}v_0 \quad (k = 0, 1, 2, \ldots) \]

where \( e_{\beta_i} \) \((i = 1, 2, \ldots, k)\) is any element of some fixed basis of \( L_{\beta_i} \) and \( 0 > \beta_1 \geq \beta_2 \geq \ldots \geq \beta_k \) are negative roots of \( L \). Let \( W^\lambda_\alpha \) be the subspace of \( W^\lambda \) spanned by the vectors \((*)\) for which

\[ \alpha = \beta_1 + \ldots + \beta_k + \lambda. \]

We show by induction on \( k \) that \( W^\lambda_\alpha \) is a weight subspace with weight \( \alpha \). Assume that the vector \( v \) has weight \( \beta \),

\[ (h - \beta(h))v = 0 \quad \text{for all} \quad h \in H_L. \]

Let \( \gamma \) be a root and \( e_\gamma \in L_\gamma \),

\[ (\text{ad } h - \gamma(h))e_\gamma = 0 \quad \text{for all} \quad h \in H_L. \]

Then

\[ (h - (\beta + \gamma(h))^n v = \sum_{k=0}^{n+m} (h - \gamma(h))^k e_\gamma \cdot \binom{n + m}{k}. \]

Thus the weight of \( e_\gamma v \) is \( \beta + \gamma \). It follows that each \( v \in W^\lambda_\alpha \) is of weight \( \alpha \leq \lambda \). It is clear that \( \dim W^\lambda_\alpha < \infty \) and each vector of weight \( \alpha \) belongs to \( W^\lambda_\alpha \). Note that \( W^\lambda_\alpha \) is spanned by the vector \( v_0 = 1 + \mathcal{L}^\lambda \). Let \( N^\lambda \) be the sum of all invariant subspaces in \( W^\lambda \) which do not contain \( v_0 \). Then \( v_0 \notin N^\lambda \) and we define

\[ V^\lambda = W^\lambda / N^\lambda. \]

The \( L \)-module \( V^\lambda \) is irreducible, has \( v_0 + N^\lambda \) as the maximal vector and \( \lambda \) is the maximal weight. The uniqueness part of the proof goes as in the case of a semisimple Lie algebra (see [4, p. 109]).

Let \( G \) be a Lie algebra, \( K \) a semi-simple subalgebra of \( G \) and \( H \) a Cartan subalgebra of \( K \). Because \( K \) is semi-simple, there exists a subspace \( T \) in \( G \) such that \( G = K \oplus T \) and \([K, T] \subset T\). We denote by \( T_0 \) the null component of \( H \) in \( T \),

\[ \ldots \]
Lemma 3.5. Let $H_T$ be a Cartan subalgebra of the Lie algebra $H + T_0 \subset G$. Then $H \subset H_T$ and $H_T$ is even a Cartan subalgebra of $G$.

Proof. There exists $x \in H + T_0$ such that

$$H_T = \{ y \in H + T_0 \mid (\text{ad } x)^n y = 0 \text{ for some } n \in \mathbb{N} \},$$

[4, pp.79-80]. Now $[x, H] = 0$ for all $x \in H + T_0$, thus $H \subset H_T$. Next let $S$ be the normalizer of $H_T$ in $G$. From $[S, H_T] \subset H_T$ it follows that $[S, H] = 0$ and therefore $S \subset H + T_0$. Because $H_T$ is a Cartan subalgebra of $H + T_0$ it follows that $S = H_T$ and we can conclude that $H_T$ is a Cartan subalgebra of $G$.

Let $\Phi$ be the set of roots of $K$ relative to $H$, $\Delta \subset \Phi$ is a set of simple roots and $\Phi^+$ (resp. $\Phi^-$) is the set of positive (resp. negative) roots with respect to $\Delta$. Next we divide $T$ into weight subspaces,

$$T_\lambda = \{ x \in T \mid [h, x] = \lambda(h) x, \forall h \in H \}.$$

We denote by $\Psi$ the set of weights of $K$ in $T$, $\Psi^+$ (resp. $\Psi^-$) is the set of positive (resp. negative) weights in $\Psi$ relative to an ordered basis $\{h_1, \ldots, h_l\}$ of $H$ which is dual to the basis $\{h_{\alpha_1}, \ldots, h_{\alpha_l}\}$,

$$\alpha_i(h_j) = \langle h_{\alpha_i}, h_j \rangle = \delta_{ij}.$$

Here $\alpha_1, \ldots, \alpha_l$ are the distinct simple roots of $K$.

Definition 3.6. The semi-simple subalgebra $K$ of $G$ is a special subalgebra if

$$N(\{x\}) \cap N(\Psi^+) = \{0\}$$

for all $x \in \Delta$. If $\Omega \subset H^*$ is any subset, we denote by $N(\Omega)$ the linear span of $\Omega$ with non-negative integral coefficients.

Example 3.7. Let $G = gl(n, \mathbb{C})$, the Lie algebra with basis $\{e_{ij}\}_{i,j=1}^n$ and commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}.$$

Let $K$ be the subalgebra spanned by the vectors $(2 \leq p \leq n - 2)$

$$e_{ij}, i \neq j, 1 \leq i, j \leq p; e_{ii} - e_{i-1,i+1}, i = 1, 2, \ldots, p - 1;$$

and

$$e_{ij}, i \neq j, p + 1 \leq i, j \leq n; e_{ii} - e_{i+1,i+1}, i = p + 1, p + 2, \ldots, n - 1.$$

Note that $K$ is isomorphic to $A_{p-1} \oplus A_{n-p-1}$. As $H$ we can take the subalgebra of $K$ spanned by the vectors $e_{ii} - e_{i+1,i+1}, 1 \leq i \leq n - 1, i \neq p$. It is easily seen that in this case
\[ T_0 = \{ a \sum_{i=1}^{p} e_{ii} + b \sum_{i=p+1}^{n} e_{ii} \mid a, b \in \mathbf{C} \} \]

and \( H_T = H + T_0 \). Using the properties of the roots of the classical simple Lie algebras \( A_i \), it is not difficult to verify that \( K \) is special.

We return to the general case. Let \( K \) be a special subalgebra of \( G \) and \( l = \text{rank} \ K, \ p = \text{rank} \ G \). We fix an ordered basis \( \{ h_1, h_2, \ldots, h_p \} \) of \( H_T \) such that \( \{ h_1, h_2, \ldots, h_l \} \) is the basis of \( H \) described above.

We define the following subalgebras of \( G \):

\[
\begin{align*}
G_+ &= K_+ + \sum_{i \geq 0} T_i + S_+, \\
G_- &= K_- + \sum_{i < 0} T_i + S_-
\end{align*}
\]

where \( K_+ \) (resp. \( K_- \)) is the subalgebra of \( K \) spanned by the vectors belonging to positive (resp. negative) roots of \( K \). We define

\[
H + T_0 = S_+ + S_- + H_T
\]

to be the corresponding decomposition for \( H + T_0 \). Because of our choice of basis of \( H_T \) (see also Lemma 3.5) it is clear that \( G = G_+ + G_- + H_T \) is a similar decomposition for \( G \) relative to the Cartan subalgebra \( H_T \).

**Definition 3.8.** A \( G \)-module \( V \) is \( K \)-finite if it is a sum of finite-dimensional \( K \)-modules when considered as a \( K \)-module by restriction to \( K \).

Let \( \Delta \) be the set of dominant integral elements in \( H^* \):

\[
\Lambda = \{ \lambda \in H^* \mid (\lambda, \alpha) \text{ is a non-negative integer for all } \alpha \in \Delta \}.
\]

**Theorem 3.9.** Let \( K \) be a special subalgebra of \( G \). Then for each \( \lambda \in H^*_T \) such that \( \lambda|_H \in \Lambda \) and \( \lambda|_{H_T} = 0 \), there exists a unique equivalence class of \( K \)-finite irreducible \( G \)-modules which are bounded above and have \( \lambda \) as the maximal weight.

**Proof.** The uniqueness follows from Theorem 3.4. We have to prove the existence. We define an ideal

\[
\gamma\lambda = \gamma(G_+) + \{ h - \lambda(h) \cdot 1 \mid h \in H_T \}
\]

and \( W^\lambda = \mathcal{E}(G)/\gamma\lambda \). Consider the subset \( S_\lambda \) of \( W^\lambda \),

\[
S_\lambda = \{ e_{\alpha}^{n_{\alpha}+1} + \gamma\lambda \mid \alpha \in \Lambda \}
\]

where \( e_{-\alpha} \) belongs to the root \( -\alpha \) and

\[
n_{\alpha} = 2 \cdot \frac{(\lambda|_H, \alpha)}{(\alpha, \alpha)}, \ \alpha \in \Delta.
\]

Let \( U^\lambda = \mathcal{E}(G)S_\lambda \) be the submodule of \( W^\lambda \) generated by \( S_\lambda \). We claim that \( U^\lambda \) does not contain the vector \( 1 + \gamma\lambda \). It is well-known that \( S_\lambda \)
is annihilated by $K_+$ (see [4, p. 115]). Now $W^j$ is a direct sum of weight subspaces, $W^j_\lambda$ is spanned by the vector $1 + \gamma_\lambda$ and $\lambda$ is the highest weight in $W^j$ (compare the proof of Theorem 3.4). Suppose that $1 + \gamma_\lambda \in U^j$; using the Poincaré-Birkhoff-Witt theorem it is easily seen that then there exists $\beta_1, \beta_2, \ldots, \beta_k \in \Psi^+$ such that
\[
\beta_1 + \beta_2 + \ldots + \beta_k - (n_\alpha + 1) \cdot \alpha = 0
\]
for at least one weight $\alpha \in \Lambda$. But this is impossible because $K$ is a special subalgebra of $G$.

Let again $N^j$ be the sum of all invariant subspaces of $W^j$ not containing the vector $1 + \gamma_\lambda$. It is clear that $U^j \subseteq N^j$. We define
\[
V^j = W^j/N^j.
\]
The $G$-module $V^j$ is irreducible and has a maximal vector $v = 1 + \gamma_\lambda + N^j$ of weight $\lambda$. Furthermore, $V^j$ contains a finite-dimensional $K$-module, namely $\mathcal{E}(K)v$ ([4, p. 115]). It follows from proposition 4.2, [6], that $V^j$ is $K$-finite. (See also [1, Theorem 1.1])

4. Discrete $G$-modules

If not otherwise stated, the notation of the previous sections is in force also in this section.

Let $C$ be the centralizer of $K$ in $\mathcal{E}(G)$. The algebra $C$ is a finitely generated subalgebra of $\mathcal{E}(G)$ (see [8, p. 162, Theorem 2.3.1.4]).

Let an ordered basis $\{t_1, t_2, \ldots, t_r\}$ be given for the subspace $T$ of $G$, such that

\[ [h, t_i] = \lambda_i(h)t_i, \ h \in H, \ i = 1, 2, \ldots, r, \]

where $\lambda_i \in \Psi$ ($i = 1, 2, \ldots, r$) and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. We complete this to an ordered basis of $G$,

\[ \{t_i, e_{\alpha_1}, h_i, e_{\beta_i}\} \]

where the order is defined through the ordering of roots,

\[ \alpha_1 < \alpha_2 < \ldots < \alpha_7 < 0 < \beta_1 < \beta_2 < \ldots < \beta_4, \]

and through a labelling of the basis elements $h_i$ of $H$. According to the Poincaré-Birkhoff-Witt theorem this ordering induces a basis for $\mathcal{E}(G)$ by ordered monomials in the basis elements of $G$. If $u \in \mathcal{E}(G)$ is such a basis vector we denote by $\deg(u)$ the number of vectors $t_i$ contained in $u$. If $v \in \mathcal{E}(G)$ is an arbitrary (finite) linear combination of ordered monomials,
\[ v = \sum_{k=1}^{n} a_k u_k \quad (a_k \in \mathbb{C}), \]

we define \( \deg (v) = \max_{k=1,\ldots,n} \deg (u_k). \)

**Definition 4.1.** Let \( c_1, c_2, \ldots, c_z \) be a generating sequence of \( C. \) We define
\[
\nu_c = \max_{k=1,2,\ldots,z} \deg (c_k).
\]

We call an element \( \lambda \) of \( A, \) the set of dominant integral weights of \( K, \) large if
\[
\lambda + \omega_1 + \omega_2 + \ldots + \omega_k \in A \quad \text{for all} \quad \omega_k \in \mathcal{P}, \quad k = 1, 2, \ldots, n_c.
\]

Let \( V \) be any \( G \)-module. Consider \( V \) as a \( K \)-module by restriction. For any \( \lambda \in A \) we denote by \( V_{\lambda} \) the sum of all irreducible finite-dimensional \( K \)-submodules of \( V \) with maximal weight \( \lambda. \) We define
\[
V_{\lambda}^+ = \{ x \in V_{\lambda} \mid e_{\beta} x = 0 \quad \forall \beta \in \mathcal{P}^+ \},
\]
the subspace of vectors with maximal weight in \( V_{\lambda}; \) in other words,
\[
V_{\lambda}^- = \{ x \in V_{\lambda} \mid hx = \lambda(h)x \quad \forall h \in H \}.
\]
We denote by \( \mathfrak{g}_{\lambda} \) the annihilator in \( ^{\mathcal{C}}(K) \) of the maximal vector in an irreducible finite-dimensional \( K \)-module with maximal weight \( \lambda; \) according to [4, p. 115],
\[
\mathfrak{g}_{\lambda} = \mathfrak{g}(K_+) + \mathfrak{g}(\{ h - \lambda(h) \cdot 1 \mid h \in H \}) + \mathfrak{g}(\{ e_{\alpha}^{n_\alpha+1} \mid \alpha \in A \})
\]
where \( K_+ \) and the numbers \( n_\alpha \) are defined as in section 3.

For all \( \beta \) and \( \alpha \) in \( A \) we define \( A_{\beta,\alpha} \) to be the subset of \( ^{\mathcal{C}}(G) \) for which
\[
A_{\beta,\alpha} V_{\alpha}^+ \subseteq V_{\beta}^+.
\]

**Lemma 4.2.** \( A_{\beta,\alpha} = \{ u \in ^{\mathcal{C}}(G) \mid \mathfrak{g}_{\beta} u \subseteq ^{\mathcal{C}}(G) \mathfrak{g}_{\alpha} \}. \)

*Proof.* Let \( V \) be a \( G \)-module such that \( V_{\alpha} \neq 0. \) Any such \( G \)-module is a factor module of the left-module \( ^{\mathcal{C}}(G) \mathfrak{g}(G)_{\alpha}. \) It follows that
\[
A_{\beta,\alpha} = \{ u \in ^{\mathcal{C}}(G) \mid u V_{\alpha}^+ \subseteq V_{\beta}^+ \}
\]
where \( V = ^{\mathcal{C}}(G) \mathfrak{g}(G)_{\alpha}. \) Let now \( u \in A_{\beta,\alpha}. \) If \( x = 1 + ^{\mathcal{C}}(G)_{\alpha} \) then \( x \in V_{\alpha}^+ \) and
\[
x u x = u + ^{\mathcal{C}}(G)_{\alpha} \in V_{\beta}^+
\]
and therefore \( \mathfrak{g}_{\beta} u \subseteq ^{\mathcal{C}}(G) \mathfrak{g}_{\alpha}. \) To prove the converse, assume that \( \mathfrak{g}_{\beta} u \subseteq ^{\mathcal{C}}(G) \mathfrak{g}_{\alpha}. \) Let \( x \in V_{\alpha}^+. \) Then
It follows that \( E(K)ux \) is a finite-dimensional \( K \)-module with \( ux \) as the vector of maximal weight (which is \( \beta \)) and thus \( ux \in V^+_\beta \).

**Lemma 4.3.** Let \( \omega \) be an element of \( \Lambda \) such that \( \omega + \lambda \in \Lambda \) for any \( \lambda \in \Psi \). Then for each \( t_i \in T \) there exists \( u \in A_{\omega + \lambda_1} \) of the form

\[
u = t_i + \sum_{j:j > i} t_j v_j\]

where \( v_j \in E(K_\lambda) \).

**Proof.** We can write

\[
T = \bigoplus_{r} T^{(r)}
\]

where \( T^{(r)} \) is the irreducible component of \( T \) under the adjoint action of \( K \), with maximal weight \( r \). We can assume that the basis \( \{ t_j \}_{j=1}^{r} \) of \( T \) is chosen in such a way that it is compatible with the decomposition \( (*) \); thus we may assume that \( t_i \in T^{(r)} \) for some weight \( r \).

Put \( D_\omega = E(K)\omega \) and consider the tensor product \( T^{(r)} \otimes D_\omega \), which is a \( K \)-module under the diagonal action:

\[
k(x \otimes y) = [k, x] \otimes y + x \otimes ky; \quad k \in K, \quad x \in T^{(r)}, \quad y \in D_\omega.
\]

It is known that the module \( T^{(r)} \otimes D_\omega \) contains an irreducible submodule with maximal weight \( \omega + \lambda \) (note that \( \omega + \lambda \in \Lambda \) for any weight \( \lambda \) in \( T^{(r)} \)) with a multiplicity which is equal to the multiplicity \( m(\lambda_i, r) \) of the weight \( \lambda_i \) in \( T^{(r)} \); in other words there are \( m(\lambda_i, r) \) linearly independent vectors in \( T^{(r)} \otimes D_\omega \) which are annihilated by \( \gamma_{\omega + \lambda_i} \) (see e.g. \([4, \text{pp. } 141-142]\)). It follows that for each \( t_i \in T^{(r)} = T^{(r)} \otimes T^{(r)} \) there exists a nonzero element \( u_0 \) of \( T^{(r)} \otimes D_\omega \) of the form

\[
u_0 = t_i \otimes (a \cdot 1 + \gamma_\omega) + \sum_{j:j > i} t_j \otimes (v_j + \gamma_\omega),
\]

\[
a \in \mathcal{C}, \quad v_j \in E(K_\lambda),
\]

such that \( u_0 \) is annihilated by \( \gamma_{\omega + \lambda_i} \). We define \( v_i = a \cdot 1 \) and let \( k \) be the smallest value of the index \( j \) for which \( v_j \in \gamma_\omega \); because of

\[
e_\alpha u_0 = 0 \quad \text{for all } \alpha \in \Phi^+, \quad u_0 \in \gamma_\omega,
\]

we have \( e_\alpha v_k \in \gamma_\omega \) for all \( \alpha \in \Phi^+ \). Now any vector in \( D_\omega \) which is annihilated by \( K_\lambda \) is a multiple of \( 1 + \gamma_\omega \); thus \( k = i \) and \( a \neq 0 \). We may assume that \( a = 1 \) (multiply \( u_0 \) by \( a^{-1} \)).

Consider the linear mapping
\( \varphi : T^{(e)} \otimes D_{\omega} \rightarrow \mathcal{E}(G)_e \mathcal{E}(G) \mathcal{E}_{\omega} \)

induced by the multiplication map \( T^{(e)} \otimes \mathcal{E}(K) \rightarrow \mathcal{E}(G) \). This mapping is a \( K \)-module homomorphism; in fact,

\[
\varphi (k (t \otimes (v + \mathcal{E}_{\omega}))) = \varphi ((k, t) \otimes (v + \mathcal{E}_{\omega}) + t \otimes (kv + \mathcal{E}_{\omega})) = [k, t]v + \mathcal{E}(G) \mathcal{E}_{\omega} + tkv + \mathcal{E}(G) \mathcal{E}_{\omega} = ktv + \mathcal{E}(G) \mathcal{E}_{\omega}
\]

for all \( k \in K \), \( t \in T^{(e)} \) and \( v \in \mathcal{E}(K) \). Let \( u \in \mathcal{E}(G) \),

\[ u = t_i + \sum_{j \neq i} t_j v_j \]

Then \( \varphi(u_0) = u + \mathcal{E}(G) \mathcal{E}_{\omega} \) and therefore \( \mathcal{E}_{\omega+i} u \in \mathcal{E}(G) \mathcal{E}_{\omega} \). In other words (Lemma 4.2), \( u \in A_{\omega+i,\omega} \).

We denote by \( P \) the projection \( P : \mathcal{E}(G) \rightarrow \mathcal{E}(G) \) such that \( \text{Ker} \ P = \mathcal{E}(G) \mathcal{E}_{\alpha} + U_1 \mathcal{E}(K_-) \) and \( P(\mathcal{E}(G)) = U_1 \) where \( U_1 \) consists of the elements

\[ b \cdot 1 + \sum a_{i_1 \ldots i_k} t_i \ldots t_k \]

where \( b, a_{i_1 \ldots i_k} \in \mathbb{C} \) and \( i_1 \leq \ldots \leq i_k \).

**Lemma 4.4.** Let \( u_1, u_2 \in A_{\beta,\alpha} \) such that \( P(u_1) = P(u_2) \). Then

\[ u_1 - u_2 \in \mathcal{E}(G) \mathcal{E}_{\alpha} \]

**Proof.** We shall again use the fact that any vector in \( \mathcal{E}(K) \mathcal{E}_{\alpha} \) which is annihilated by \( K_+ \) is a multiple of \( 1 + \mathcal{E}_{\alpha} \). First we write

\[ u_1 - u_2 = w + t_{i_1} \ldots t_{i_k} v + \sum t_{i_1} \ldots t_{i_m} v_{i_1 \ldots i_m} \]

where each term is a sum of ordered monomials and \( w \in \mathcal{E}(G) \mathcal{E}_{\alpha} \), \( v \) and \( v_{i_1 \ldots i_m} \in \mathcal{E}(K_-) \) and

\[ \lambda_{i_1} + \ldots + \lambda_{i_m} \geq \lambda_{j_1} + \ldots + \lambda_{j_k} \]

If \( m = k \) then \( j_s \neq i_s \) for at least one value of the index \( v \). From \( K_+(u_1 - u_2) \in \mathcal{E}(G) \mathcal{E}_{\alpha} \) it follows that

\[ K_+ v \subset \mathcal{E}_{\alpha} \]

and thus \( v \in \mathcal{E}_{\alpha} \) (\( v \in a \cdot 1 + \mathcal{E}_{\alpha} \) for any \( a \neq 0 \) because of \( P(u_1 - u_2) = 0 \)). By induction it follows that the coefficient of any \( t_{i_1} \ldots t_{i_m} \) belongs to \( \mathcal{E}_{\alpha} \) and therefore \( u_1 - u_2 \in \mathcal{E}(G) \mathcal{E}_{\alpha} \).

It is clear that Lemma 4.4 is valid also if we replace \( A_{\beta,\alpha} \) by

\[ A_{\alpha} = \sum_{\beta} A_{\beta,\alpha} \]
Lemma 4.5. Let $\alpha \in A$ be large. Then any $u \in A_\alpha$ such that $\deg(u) \leq n_e$ can be written in the form

$$u = v + a \cdot 1 + \sum_{k \leq \deg(u)} u_{i_1} u_{i_2} \ldots u_{i_k} (a \in C, \quad v \in \mathcal{E}(G) \mathcal{O}_\alpha, \quad u_{ir} \in \mathcal{E}(G); \quad r = 1, 2, \ldots, k)$$

where $k \leq \deg(u), \quad i_1 \leq i_2 \leq \ldots \leq i_k$ and

\[ (*) \quad u_{i_1} u_{i_r+1} \ldots u_{i_k} \in A_{\delta_r, n}; \quad \delta_v = \alpha + \lambda_i + \ldots + \lambda_k (v = 1, 2, \ldots, k). \]

Proof. (1) Let $S$ be the set consisting of finite sequences $i = (i_1, i_2, \ldots, i_k)$ where $k \leq n_e$ and the integers $i_v$ satisfy the inequalities

$$0 < i_1 \leq i_2 \leq \ldots \leq i_k \leq r = \dim T.$$

We denote by $e$ the empty sequence. We define an order in $S$ by putting

$$(i_1, i_2, \ldots, i_k) < (j_1, j_2, \ldots, j_m) \text{ if } k < m \text{ or } k = m \text{ and }$$

the first non-zero number in the sequence $i_1 - j_1, \ i_2 - j_2, \ldots$ is positive. In addition, for each $i \in S$ we define

$$t_i = t_{i_1} \ldots t_{i_k} \in \mathcal{E}(G)$$

and $t_e = 1 \in \mathcal{E}(G)$. Let $V$ be the subspace of $\mathcal{E}(G)$ which has the set \{ $t_i \mid i \in S$ \} as an ordered basis (the order is defined through the ordering of $S$).

(2) We put $u^e = 1 \in A_\alpha$. From the fact that $\alpha$ is large and from Lemma 4.3, it follows that for each $i = (i_1, i_2, \ldots, i_k) \in S$ there exists

$$u^i = u_{i_1} u_{i_2} \ldots u_{i_k} \in A_\alpha (u_{ir} \in \mathcal{E}(G); r = 1, 2, \ldots, k)$$

where each $u_{ir}$ is of the type described in Lemma 4.3, $P(u^i) = t_i$ and $u_i$ satisfies the relations \((*)\). We denote by $U$ the subspace of $A_\alpha$ which has the set

\{ $u^i \mid i \in S$ \}

as an ordered basis.

(3) It is clear that the operator $P$ induces a linear mapping from $U$ into $V$. Furthermore,

$$P(u^i) = t_i + \text{lower terms}$$

as follows easily from the properties of the $u^i : s$ (see Lemma 4.3). Thus the matrix representing $P$ is triangular in the ordered basis described above, the diagonal elements being equal to 1. It follows that the inverse of $P$ exists and therefore for each $u \in A_\alpha$, $\deg(u) \leq n_e$, there exists $u' \in U$ such that
P(u) = P(u').

(Note that $P(u) \in V$. From Lemma 4.4 it follows that there exists $v \in \mathcal{E}(G)^{\gamma}_x$ such that $u = v + u'$.

Lemma 4.6. Let $\alpha, \beta \in A$, and let $V$ be an irreducible $G$-module such that $V_\alpha \neq 0$; then $V_\beta^+ = A_{\beta, \alpha} V_\alpha^+$.

Proof. It is sufficient to prove the statement for $V = \mathcal{E}(G)^{\gamma}_x$ (compare with the proof of Lemma 4.2). Then

$$V_\beta^+ = \{ u + \mathcal{E}(G)^{\gamma}_x | \gamma_x u \subset \mathcal{E}(G)^{\gamma}_x \} = A_{\beta, \alpha} + \mathcal{E}(G)^{\gamma}_x = A_{\beta, \alpha} (1 + \mathcal{E}(G)^{\gamma}_x)$$

$$\subset A_{\beta, \alpha} V_\alpha^+.$$

The relation $A_{\beta, \alpha} V_\alpha^+ \subset V_\beta^+$ follows from the definition of $A_{\beta, \alpha}$.

Let $C$ be the centralizer of $K$ in $\mathcal{E}(G)$. If $V$ is any $G$-module then $V_\alpha$ and $V_\alpha^+$ are $C$-modules by restriction of $\mathcal{E}(G)$ to the subalgebra $C$; in fact $V_\alpha$ is even a $\mathcal{E}(K)C$-module.

Lemma 4.7. Let $V$ be an irreducible $G$-module, $V_\alpha \neq 0$. Then the equivalence class $[V]$ of $V$ is completely determined by the equivalence class of the $C$-module $V_\alpha^+$. $V_\alpha^+$ is an irreducible $C$-module.

Proof. This is an easy consequence of Theorem 5.5, [6]. (Note that the action of $\mathcal{E}(K)C$ on $V_\alpha$ is completely determined by the action of $C$ on $V_\alpha^+$.)

Let $G'_\alpha$ be the set of all equivalence classes $[V]$ of irreducible $G$-modules $V$ such that $V_\alpha \neq 0$ and $V_\alpha = 0$ for each $\beta < \alpha$. We call $V_\alpha$ the minimal component of $V$. Now an irreducible $G$-module $V$ is $K$-finite if and only if $V_\alpha = 0$ for some weight $\gamma \in A$, [6, proposition 4.2]. It follows that $V$ is $K$-finite if and only if $V$ has a minimal component. Thus the set $G'$ of all equivalence classes of irreducible $K$-finite $G$-modules is equal to

$$\bigcup_{\alpha \in A} G'_\alpha.$$

Of course $G'_\alpha \cap G'_\beta = \emptyset$ when $\alpha \neq \beta$.

Let $M_\alpha = \sum_{\beta < \alpha} A_{\beta, \alpha}$. If $[V] \in G'_\alpha$ then $V_\alpha^+$ is in a natural way a $C/C \cap \mathcal{E}(G)M_\alpha$-module. We denote by $C'_\alpha$ the set of all equivalence classes of irreducible $C/C \cap \mathcal{E}(G)M_\alpha$-modules.

Theorem 4.8. The mapping $V \rightarrow V_\alpha^+$ induces a bijection between $G'_\alpha$ and $C'_\alpha$.

Proof. If $[V], [W] \in G'_\alpha$ then it is clear that $V_\alpha^+$ and $W_\alpha^+$ are equivalent as $C$-modules if and only if they are equivalent as $C/C \cap \mathcal{E}(G)M_\alpha$-modules. The injectivity of the mapping follows now from Lemma 4.7.

Let next $[W] \in C'_\alpha$. We have to show that there exists $[V] \in G'_\alpha$ such that $V_\alpha^+ \cong W$ as $C/C \cap \mathcal{E}(G)M_\alpha$-modules. First we extend $W$ to a $C$-
module. Let $x$ be a non-zero element of $W$, and let $\mathcal{L}$ be the annihilator of $x$ in $C$ so that $W = C \mathcal{L}$. We define a left ideal of $\mathcal{E}(G)$ by

$$\mathcal{M} = \left\{ u \in \mathcal{E}(G) \mid \mathcal{E}(G)u \cap C \subseteq \mathcal{L} \right\}.$$ 

Consider the $G$-module $V = \mathcal{E}(G) \mathcal{M}$. First we show that $V$ is irreducible i.e. the left ideal $\mathcal{M}$ is maximal. Let $\mathcal{M} \subseteq \mathcal{E}(G)$ be a left ideal such that $1 \in \mathcal{M}$ and $\mathcal{M} \subset \mathcal{M}$. Then

$$\mathcal{L} = C \cap \mathcal{M} \subset C \cap \mathcal{E}(G) \mathcal{M}.$$ 

Because of the irreducibility of $W$, $\mathcal{L}$ is a maximal left ideal in $C$. Now $1 \in C \cap \mathcal{E}(G) \mathcal{M}$ and therefore $\mathcal{L} = C \cap \mathcal{E}(G) \mathcal{M}$. From the definition of $\mathcal{M}$ it follows that $\mathcal{M} \subset \mathcal{M}$; thus $\mathcal{M} = \mathcal{M}$ and $\mathcal{M}$ is maximal.

Since $\mathcal{M}$ is the annihilator of $x$ in $C$, we observe that $C \cap \mathcal{M} = \mathcal{L}$ and therefore $1 + \mathcal{M} \in V^+$. From Lemma 4.7 we conclude that $V^+$ consists of vectors $c + \mathcal{M}$, $c \in C$. From $C \cap \mathcal{M} = \mathcal{L}$ it then follows that the mapping

$$\varphi : V^+ \to C[\mathcal{L}], \varphi(c + \mathcal{M}) = c + \mathcal{L}$$

is a $C$-linear isomorphism. Thus $V^+ \cong W$ as $C$-modules. Next we observe that $C \cap \mathcal{E}(G) \mathcal{M}$ is a $C$-module, so $\mathcal{M} \subset \mathcal{M}$ and therefore $V^+_\beta = A_{\beta, x} V^+_\alpha = 0$ for $\beta < x$. It follows that $[V] \in G'_x$.

By Lemma 3.5, rank $K = \text{rank } G$ if and only if $T_0 = 0$.

**Theorem 4.9.** Let rank $G = \text{rank } K$. Then for any large weight $x \in A$ the set $G'_x$ contains exactly one element $[V]$ and $\dim V^+_x = 1$.

**Proof.** Let $c$ be one of the generators $c_1, \ldots, c_\alpha$ of $C$ (see Definition 4.1). Then $c \in A_x$, $\deg (c) \leq n_x$ and $[H, c] = 0$. Then $c$ can be written in the form described in Lemma 4.5. Since $[H, c] = 0$, $\lambda_i + \lambda_j + \ldots + \lambda_k = 0$ for each of the products $u_i u_j \ldots u_k$. Now $\lambda_i \geq \lambda_j \geq \ldots \geq \lambda_k$ and $\lambda_i \neq 0$ ($v = 1, 2, \ldots, k$) ($T_0 = 0$); thus $\lambda_i < 0$ and $u_i u_j \ldots u_k \in \mathcal{M}_\alpha$. It follows that the generators $c$ belong to the subalgebra $C \cdot 1 + C \cap \mathcal{E}(G) \mathcal{M}_\alpha$ of $C$; hence this is true for all $c \in C$.

We conclude that the algebra $C/C \cap \mathcal{E}(G) \mathcal{M}_\alpha$ is isomorphic (when $\alpha$ is large) to the algebra $C$ of complex numbers and therefore there exists exactly one equivalence class of irreducible (non-zero) $C/C \cap \mathcal{E}(G) \mathcal{M}_\alpha$-modules and the dimension of such a module is equal to one. Theorem 4.8 completes the job.

**Remark 4.10.** The results of this section can be easily extended to the case in which $K$ is a reductive subalgebra of $G$.

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