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ON THE ALGEBRAIC INDEPENDENCE OF THE VALUES OF SOME E-FUNCTIONS

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In the following $\mathbf{Q}$ denotes the field of rational numbers, $\mathbf{C}$ the field of complex numbers and $\mathbf{C}(z)$ the field of rational functions of z with coefficients in $\mathbf{C}$.

1. In this paper we shall consider the functions

$$(1)\quad u_{0,i}(z,\nu,\mu) = (1/i!)\frac{\partial^i}{\partial\mu^i}K_{\nu,\mu}(z), \quad u_{i,0}(z,\nu,\mu) = (1/i!)\frac{\partial^i}{\partial\nu^i}K_{\nu,\mu}(z), \quad i = 0,1,2,\dots,$$

where $K_{\nu,\mu}$ is the Kummer function

$$(2)\quad K_{\nu,\mu}(z) = \sum_{n=0}^{\infty}(\mu(\mu+1)\dots(\mu+n-1)/n!\nu(\nu+1)(\nu+n-1))z^n, \quad \nu,\mu \neq 0,-1,-2,\dots.$$

The function values $K_{\nu,\mu}(\alpha)$, $K'_{\nu,\mu}(\alpha)$, where $\alpha \neq 0$ is an algebraic number and ν, $\mu \in \mathbf{Q}$, have been considered in many papers (see [1], [3]). We shall now prove the following theorems. (Throughout this paper we denote differentiation with respect to z by a dash.)

Theorem 1. *Let $\alpha \neq 0$ be an algebraic number, and let ν and μ be rational numbers satisfying the conditions*

$$\nu \neq 0,-1,-2,\dots; \quad \mu,\ \mu-\nu \neq 0,\pm 1,\pm 2,\dots.$$

Then the numbers of each of the two sets

$$u_{0,k}(\alpha,\nu,\mu),\quad u'_{0,k}(\alpha,\nu,\mu),\quad e^{\alpha},\quad k = 0,1,\dots,n;$$
$$u_{i,0}(\alpha,\nu,\mu),\quad u'_{i,0}(\alpha,\nu,\mu),\quad e^{\alpha},\quad i = 0,1,\dots,m,$$

are algebraically independent.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Further, let $\nu \neq 1,2,\dots$. Then the $2(m+n)+3$ numbers*

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$$(3)\qquad u_{i,0}(\alpha, \nu, \mu), \quad u'_{i,0}(\alpha, \nu, \mu), \quad u_{0,k}(\alpha, \nu, \mu), \quad u'_{0,k}(\alpha, \nu, \mu), \quad e^{\alpha},$$
$$i = 0,1,\ldots, m, \qquad k = 1,2,\ldots, n,$$

are algebraically independent.

2. In the proofs of our theorems we shall use the method of Siegel and Šidlovskiĭ. Thus we shall prove the following lemmas.

Lemma 1. *Let* ν *and* μ *satisfy the conditions of Theorem* 1. *Further, let* $\gamma \neq 0$ *be a rational number. Then the* $2n + 3$ *functions*

$$u_{0,k}(z, \nu, \mu), \qquad u'_{0,k}(z, \nu, \mu), \qquad e^{\gamma z}, \qquad k = 0,1,\ldots, n,$$

are algebraically independent over $\mathbf{C}(z)$.

Lemma 2. *Let the conditions of Lemma* 1 *be satisfied. Then the* $2m + 3$ *functions*

$$u_{i,0}(z, \nu, \mu), \qquad u'_{i,0}(z, \nu, \mu), \qquad e^{\gamma z}, \qquad i = 0,1,\ldots, m,$$

are algebraically independent over $\mathbf{C}(z)$.

Lemma 3. *Let* ν *and* μ *satisfy the conditions of Theorem* 2. *Further, let* $\gamma \neq 0$ *be a rational number. Then the* $2(m + n) + 3$ *functions*

$$u_{i,0}(z, \nu, \mu), \quad u'_{i,0}(z, \nu, \mu), \quad u_{0,k}(z, \nu, \mu), \quad u'_{0,k}(z, \nu, \mu), \quad e^{\gamma z},$$
$$i = 0,1,\ldots, m, \qquad k = 1,2,\ldots, n,$$

are algebraically independent over $\mathbf{C}(z)$.

3. We begin by proving Lemma 1. For the sake of simplicity we put $u_{0,k}(z, \nu, \mu) = u_k$, $k = 0,1,\ldots$, and use in the following the notations $u(k)$ and $u'(k)$ to denote respectively the functions $u_0, u_1, \ldots, u_k$ and $u'_0, u'_1, \ldots, u'_k$.

The function $u_0 = K_{\nu,\mu}(z)$ satisfies the differential equation

$$(4)\qquad u''_0 + (\nu/z - 1)u'_0 - (\mu/z)\, u_0 = 0.$$

Thus we can deduce that the functions $u(n)$ satisfy the following system of differential equations,

$$(5)\quad u''_k + (\nu/z - 1)u'_k - (\mu/z)\, u_k - (1/z)\, u_{k-1} = 0, \quad u_{-1} \equiv 0, \quad k = 0,1,\ldots, n.$$

It follows that if $P = P(z, u(n), u'(n), e^{\gamma z})$ is a polynomial in z, $u(n)$, $u'(n)$, $e^{\gamma z}$, then zP' will also be a polynomial in the same variables.

In the proof of Lemma 1, which will be performed by induction, we need the following lemma.

Lemma 4. *Let us assume that we have an irreducible polynomial* $P = P(z, u(n), u'(n), e^{\gamma z})$ *satisfying*

$$P = \sum_{i=0}^{m} P_i e^{i\gamma z} = 0, \tag{6}$$

where P_i, $i = 0,1,\ldots,m$, *are polynomials in* z, $u(n)$, $u'(n)$ *such that* $P_m \not\equiv 0$. *If the functions* $u(n)$, $u'(n)$ *are algebraically independent over* $\mathbf{C}(z)$, *then* P *must satisfy, identically in* z, $u(n)$, $u'(n)$, $e^{\gamma z}$, *the equation*

$$zP' = (az + b)P, \qquad a, \ b \in \mathbf{Q}, \tag{7}$$

and, further, P *must be of the form*

$$P = P_m e^{m\gamma z} + P_0, \qquad m \geqq 1,$$

where the polynomials P_m *and* P_0 *are homogeneous with respect to* $u(n)$, $u'(n)$ *and one of them is a polynomial in* z *alone.*

For the proof of this lemma we refer to [7] (pp. 5—6).

First let $n = 0$. The algebraic independence of the functions u_0, u_0' is proved in [3]. If the functions u_0, u_0', $e^{\gamma z}$ were algebraically dependent over $\mathbf{C}(z)$, then we should have an equation of the form (6). By Lemma 4, this yields

$$P = P_m e^{m\gamma z} + P_0 = 0, \tag{8}$$

where P satisfies (7), P_m and P_0 are homogeneous polynomials with respect to u_0, u_0' such that $P_m = c_m z^b$ or $P_0 = c_0 z^b$ with non-zero constants c_m, c_0. Further, we can assume that $\gamma > 0$.

1°. Let $P_m = c_m z^b$. Then $a = m\gamma$, so that P_0 satisfies, by (7), the differential equation

$$z P_0' = (m\gamma z + b) P_0. \tag{9}$$

If we denote

$$P_0 = \sum_{i=0}^{l} A_i u_0^{l-i} u_0'^{i},$$

where A_i are polynomials in z, then (9) and the algebraic independence of the functions u_0, u_0' implies that

$$\begin{gathered} zA_i' + (l-i+1) z A_{i-1} + (i+1)\mu A_{i+1} = ((m\gamma - i)z + b + i\nu) A_i, \\ i = 0,1,\ldots,l, \qquad A_{-1} \equiv A_{l+1} \equiv 0. \end{gathered} \tag{10}$$

Let k_i be the degree of the polynomial A_i, $i = 0,1,\ldots,l$. If $m\gamma - i \neq 0$ $\forall\, i = 0,1,\ldots,l$, then (10) with $i = 0,1,\ldots,l-1$ gives

$$k_i = k_0 + i, \qquad i = 0,1,\ldots,l.$$

Thus $k_{l-1} + 1 = k_l$. But if $i = l$, then (10) gives $k_{l-1} = k_l$. This contradiction means that there exists one i, say i_0, such that $m\gamma = i_0$. Thus (8) must be of the form

(11)
$$c_m z^b e^{i_0 z} + \sum_{i=0}^{l} A_i u_0^{l-i} u_0'^{i} = 0 .$$

We now denote

$$B_{\nu,\mu}(z) = e^{-z/2} u_0 , \qquad U_{\nu,\mu}(z) = z^{(\nu-1)/2} B_{\nu,\mu}(z) .$$

From the proof of Lemma 7 of [1] it follows that the functions $U_{\nu,\mu}(z)$, $U'_{\nu,\mu}(z)$, $e^{z/2}$ are algebraically independent over $\mathbf{C}(z)$. Since $\nu \in \mathbf{Q}$, this implies the algebraic independence of the functions $B_{\nu,\mu}(z)$, $B'_{\nu,\mu}(z)$, $e^{z/2}$. From (11) it follows that

$$c_m z^b e^{i_0 z} + e^{lz/2} \sum_{i=0}^{l} A_i B_{\nu,\mu}^{l-i} (B_{\nu,\mu}/2 + B'_{\nu,\mu})^i = 0 .$$

Since $B_{\nu,\mu}(z)$, $B'_{\nu,\mu}(z)$, $e^{z/2}$ are algebraically independent over $\mathbf{C}(z)$, it follows that $2i_0 = l$. Further, we must have $l \geqq 1$. This leads to a contradiction.

2°. If $P_0 = c_0 z^b$, then $a = 0$, and P_m satisfies the differential equation

$$z P'_m = (-m\gamma z + b) P_m .$$

If we repeat the reasonings used in the early steps of the first case, we get

$$-m\gamma = i_0 , \qquad i_0 \in \{0,1,\ldots,l\} .$$

This is impossible, since $-m\gamma < 0$. Thus Lemma 1 is true when $n = 0$.

Now let us assume that this lemma holds when $n = k$. Next we shall prove that this implies the truth of Lemma 1 with $n = k+1$. We split the proof into three steps, called in the following A, B and C.

4. *Step* A. We define

$$t = u_0 u'_{k+1} - u'_0 u_{k+1}$$

and prove that the functions $u(k)$, $u'(k)$, t are algebraically independent over $\mathbf{C}(z)$. If we assume, against this, that these functions are algebraically dependent, then we have

(12)
$$P = \sum_{i=0}^{m} P_i t^i = 0 ,$$

where P is an irreducible polynomial in z, $u(k)$, $u'(k)$, t, and P_i, $i = 0,1,\ldots,m$, are polynomials in z, $u(k)$, $u'(k)$ such that $P_m \not\equiv 0$. By induction hypothesis, $m \geqq 1$.

By (5), we obtain

$$z t' = (z - \nu) t + u_0 u_k .$$

From this and (5) it follows that zP' is also a polynomial in z, $u(k)$, $u'(k)$, t. Further, by (12), $zP' = 0$. Thus the induction hypothesis implies that the polynomial zP' must be divisible by the polynomial P. Since the degree of zP' with respect to t does not exceed that of P, there is a polynomial S in z, $u(k)$, $u'(k)$ such that

$$zP' = SP \tag{13}$$

identically in z, $u(k)$, $u'(k)$, t. If we compare the coefficients of t^m in (13), we obtain an identity

$$zP'_m + m(z - \nu)P_m = SP_m .$$

Since the degree of zP'_m with respect to $u(k)$, $u'(k)$ does not exceed that of P_m, and the degree of zP'_m with respect to z exceeds that of P_m at most by one, it follows that $S = az + b$ with constants a and b. Thus the identity (13) takes the form

$$zP' = (az + b)P . \tag{14}$$

Here we can assume, by Lemma 10 of [4], that a, $b \in \mathbf{Q}$.

If $\bar{u}_0$, $\bar{u}_1, \ldots, \bar{u}_{k+1}$ is another solution of (5) with $n = k + 1$, and $\bar{t} = \bar{u}_0\bar{u}'_{k+1} - \bar{u}'_0\bar{u}_{k+1}$, then

$$z\bar{t}' = (z - \nu)\,\bar{t} + \bar{u}_0\bar{u}_k .$$

Thus (14) holds even when we replace the solution $u(k+1)$ by any other solution $\bar{u}(k+1)$. Now denote $\bar{u}(k) = u(k)$, $\bar{u}_{k+1} = u_{k+1} + \sigma y$, where y is a solution of (4), linearly independent of u_0, and σ is an arbitrary constant. By integrating (14) we then obtain

$$\sum_{i=0}^{m} P_i(t + \sigma(u_0y' - u'_0y))^i = c(\sigma)\, z^b e^{az} .$$

When we differentiate this equation with respect to σ and put $\sigma = 0$, we obtain

$$(u_0y' - u'_0y) \sum_{i=1}^{m} i\,P_i t^{i-1} = c'(0)\, z^b e^{az} .$$

Now $u_0y' - u'_0y = c\,z^{-\nu}e^z$, $c \neq 0$, and this gives the following equation

$$\sum_{i=1}^{m} i\,P_i t^{i-1} = c'(0)\, c^{-1} z^{b+\nu} e^{(a-1)z} . \tag{15}$$

First let $m > 1$. If $c'(0) = 0$, we have $\sum_{i=1}^{m} i\,P_i t^{i-1} = 0$. This is impossible, since P is irreducible. If $c'(0) \neq 0$, then $b + \nu$ must be a non-negative integer. If $a = 1$, then we have the same contradiction as before. If $a \neq 1$, then we can eliminate t from the equations (12) and

(15), and thus obtain an algebraic relation over $\mathbf{C}(z)$ between the functions $u(k)$, $u'(k)$, $e^{(a-1)z}$. This conflicts with the induction hypothesis. Thus $m = 1$, and (15) has the form

$$P_1 = c'(0)\, c^{-1} z^{b+\nu} e^{(a-1)z}.$$

Thus $b + \nu$ must again be a non-negative integer. Further, it follows from the induction hypothesis that $a = 1$. This implies that

$$P_1 = c_0 z^{b+\nu}, \qquad c_0 \neq 0.$$

We have now proved that the equation (12) has the form

$$P = c_0 z^{b+\nu} t + P_0 = 0. \tag{16}$$

Now we shall analyze the polynomial P_0. Denote

$$P_0 = \sum_{i=0}^{l} H_i,$$

where each H_i is a homogeneous polynomial of degree i with respect to $u(k)$, $u'(k)$. Replace now in (14) u_i by $\bar{u}_i = \sigma u_i$, $i = 0, 1, \ldots, k+1$, with an arbitrary constant σ. By integrating (14) we then obtain the equation

$$c_0 z^{b+\nu} \sigma^2 t + \sum_{i=0}^{l} \sigma^i H_i = c(\sigma)\, z^b e^z.$$

When we differentiate this equation twice with respect to σ, and put $\sigma = 0$, we have

$$2\, c_0 z^{b+\nu} t + 2\, H_2 = c''(0)\, z^b e^z.$$

Thus $H_2 - P_0 = c''(0)\, z^b e^z/2$, and this together with the induction hypothesis gives the result $P_0 \equiv H_2$. Thus P_0 is a homogeneous polynomial of degree two with respect to $u(k)$, $u'(k)$.

Let $s = [k/2]$. If $s > 0$, put $\bar{u}\,(k-1) = u\,(k-1)$, $\bar{u}_k = u_k + \sigma\, u_0$, $\bar{u}_{k+1} = u_{k+1} + \sigma\, u_1$ with an arbitrary constant σ. By putting

$$P_0 = A_1 + P_{10} u_k + P_{01} u_k' + P_{20} u_k^2 + P_{11} u_k u_k' + P_{02} u_k'^2,$$

where A_1 and P_{ij} do not contain u_k or u_k', and integrating (14), we obtain

$$c_0 z^{b+\nu}(t + \sigma(u_0 u_1' - u_0' u_1)) + A_1 + P_{10}(u_k + \sigma u_0) + P_{01}(u_k' + \sigma\, u_0') +$$
$$+ P_{20}(u_k + \sigma\, u_0)^2 + P_{11}(u_k + \sigma\, u_0)\,(u_k' + \sigma\, u_0') + P_{02}(u_k' + \sigma\, u_0')^2 = c(\sigma)\, z^b e^z.$$

When we differentiate this equation with respect to σ, put $\sigma = 0$ and again invoke the induction hypothesis, we obtain the result

$$P = c_0 z^{b+\nu}(t + (u_1 u_k' - u_1' u_k)) + a_1(z)\,(u_0 u_k' - u_0' u_k) + A_1$$
$$= \sum_{l=0}^{1} a_l \sum_{j=0}^{1-l} (u_j u'_{k+1-l-j} - u'_j u_{k+1-l-j}) + A_1\,,$$

where $a_0 = c_0 z^{b+\nu}$, $a_1 = a_1(z)$ is a polynomial in z, and A_1 is a homogeneous polynomial of degree two with respect to $u(k-1)$, $u'(k-1)$.

Next, let us assume that for some $i < s$ we have

$$P = \sum_{l=0}^{i} a_l \sum_{j=0}^{i-l} (u_j u'_{k+1-l-j} - u'_j u_{k+1-l-j}) + A_i\,,$$

where A_i is a homogeneous polynomial of degree two with respect to $u(k-i)$, $u'(k-i)$, and $a_l = a_l(z)$ are polynomials in z. Let us again denote

$$A_i = A_{i+1} + P_{10} u_{k-i} + P_{01} u'_{k-i} + P_{20} u^2_{k-i} + P_{11} u_{k-i} u'_{k-i} + P_{02} u'^2_{k-i}\,.$$

The substitution $\bar{u}(k-i-1) = u(k-i-1)$, $\bar{u}_{k-i} = u_{k-i} + \sigma\, u_0$, $\bar{u}_{k+1-i} = u_{k+1-i} + \sigma\, u_1, \ldots, \bar{u}_{k+1} = u_{k+1} + \sigma\, u_{i+1}$ in (14) now gives

$$\sum_{l=0}^{i} a_l \sum_{j=0}^{i-l} (u_j u'_{k+1-l-j} - u'_j u_{k+1-l-j} + \sigma\,(u_j u'_{i+1-l-j} - u'_j u_{i+1-l-j}))$$

$$\begin{aligned}(17)\quad &+ A_{i+1} + P_{10}(u_{k-i} + \sigma\, u_0) + P_{01}(u'_{k-i} + \sigma\, u'_0) + P_{20}(u_{k-i} + \sigma\, u_0)^2 + \\ &+ P_{11}(u_{k-i} + \sigma\, u_0)\,(u'_{k-i} + \sigma\, u'_0) + P_{02}(u'_{k-i} + \sigma\, u'_0)^2 = c(\sigma)\, z^b e^z\,.\end{aligned}$$

By differentiating this equation with respect to σ and putting $\sigma = 0$, we obtain

$$P = \sum_{l=0}^{i+1} a_l \sum_{j=0}^{i+1-l} (u_j u'_{k+1-l-j} - u'_j u_{k+1-l-j}) + A_{i+1}\,.$$

It follows that

$$P = \sum_{l=0}^{s} a_l \sum_{j=0}^{s-l} (u_j u'_{k+1-l-j} - u'_j u_{k+1-l-j}) + A_s\,, \tag{18}$$

where again A_s is a homogeneous polynomial of degree two with respect to $u(k-s)$, $u'(k-s)$. We now have two possibilities, either $2s = k$ or $2s + 1 = k$.

1°. $2s = k$. Denote

$$P_l = \sum_{j=0}^{s-l} (u_j u'_{2s+1-l-j} - u'_j u_{2s+1-l-j})\,.$$

Since the polynomial $P = \sum_{l=0}^{s} a_l P_l + A_s$ satisfies the differential equation (14) with $a = 1$, we obtain

$$(19) \qquad \sum_{l=0}^{s} (z\,a_l' - (b+\nu)\,a_l)\,P_l + \sum_{l=0}^{s} a_l u_{s-l} u_s + z\,A_s' = (z+b)\,A_s\,.$$

We have $a_0 = c_0 z^{b+\nu}$, and so $z\,a_0' - (b+\nu)\,a_0 = 0$. Thus (19) with the induction hypothesis gives $a_l = c_l z^{b+\nu}$, $l = 1, \ldots, s$, where c_l are constants. We now immediately obtain the equation

$$(20) \qquad \sum_{l=0}^{s} c_l z^{b+\nu} u_{s-l} u_s + z\,A_s' = (z+b)\,A_s\,.$$

Denote

$$A_s = A_{s+1} + P_{10} u_s + P_{01} u_s' + P_{20} u_s^2 + P_{11} u_s u_s' + P_{02} u_s'^2\,,$$

where A_{s+1}; P_{10}, P_{01} and P_{20}, P_{11} P_{02} are homogeneous polynomials of degree two; one and zero, respectively, in $u(s-1)$, $u'(s-1)$. So P_{20}, P_{11}, P_{02} are polynomials in z satisfying, by (20) and the induction hypothesis,

$$(21) \qquad \begin{aligned} z\,P_{20}' - (z+b)\,P_{20} + \mu\,P_{11} + c_0 z^{b+\nu} &= 0\,,\\ z\,P_{11}' - (b+\nu)\,P_{11} + 2\,z\,P_{20} + 2\,\mu\,P_{02} &= 0\,,\\ z\,P_{02}' + (z - b - 2\,\nu)\,P_{02} + z\,P_{11} &= 0\,. \end{aligned}$$

Let d_1, d_2 and d_3 be the degrees of the polynomials P_{20}, P_{11} and P_{02} respectively. Further, let c be the coefficient of z^{d_1} in P_{20}. If $d_1 > b + \nu - 1$, then it follows from (21), that $d_2 = d_3 = d_1 + 1$. The second equation of (21) now gives

$$d_2 c/\mu - (b+\nu)\,c/\mu + 2\,c - 2\,c = (d_1 - b - \nu + 1)\,c/\mu = 0\,,$$

which is impossible. If $d_1 < b + \nu - 1$, then $d_2 = d_3 = b + \nu$, and from the second equation of (21) it follows that

$$-\,d_2 c_0/\mu + (b+\nu)\,c_0/\mu + 2\,c_0 = 2\,c_0 = 0\,,$$

but this, too, is impossible, since $c_0 \neq 0$. So we are left with the case $d_1 = b + \nu - 1$. Then we must have $d_2 = d_3 = d_1 + 1$. Thus (21) again gives the following contradiction,

$$d_2(c - c_0)/\mu - (b+\nu)\,(c - c_0)/\mu + 2\,c + 2\,(c_0 - c) = 2\,c_0 = 0\,.$$

2°. $2\,s + 1 = k$. By substituting $\bar{u}(s) = u(s)$, $\bar{u}_{s+1} = u_{s+1} + \sigma\,u_0$, $\bar{u}_{s+2} = u_{s+2} + \sigma\,u_1, \ldots, \bar{u}_{2s+2} = u_{2s+2} + \sigma\,u_{s+1}$ in (14), we obtain

$$\begin{aligned} &\sum_{l=0}^{s} a_l \sum_{j=0}^{s-l} (u_j u_{2s+2-l-j}' - u_j' u_{2s+2-l-j} + \sigma(u_j u_{s+1-l-j}' - u_j' u_{s+1-l-j}))\\ &\quad + A_{s+1} + P_{10}(u_{s+1} + \sigma\,u_0) + P_{01}(u_{s+1}' + \sigma\,u_0') + P_{20}(u_{s+1} + \sigma\,u_0)^2 +\\ &\quad + P_{11}(u_{s+1} + \sigma\,u_0)\,(u_{s+1}' + \sigma\,u_0') + P_{02}(u_{s+1}' + \sigma\,u_0')^2 = c(\sigma) z^b e^z\,, \end{aligned}$$

where we have presented A_s in the form

$$A_s = A_{s+1} + P_{10}u_{s+1} + P_{01}u'_{s+1} + P_{20}u^2_{s+1} + P_{11}u_{s+1}u'_{s+1} + P_{02}u'^2_{s+1},$$

and here A_{s+1}; P_{10}, P_{01} and P_{20}, P_{11}, P_{02} are homogeneous polynomials of degree two; one and zero, respectively, in $u(s)$, $u'(s)$.

We differentiate this equation with respect to σ and put $\sigma = 0$. In this way we obtain the equation

$$\sum_{l=0}^{s} a_l(u_0u'_{s+1-l} - u'_0u_{s+1-l}) + P_{10}u_0 + P_{01}u'_0 + 2\,u_0u_{s+1}P_{20} + \\ + P_{11}(u_0u'_{s+1} + u'_0u_{s+1}) + 2\,u'_0u'_{s+1}P_{02} = c'(0)\,z^be^z.$$

We see from the induction hypothesis that this equation is impossible. This contradiction completes the proof of step A.

5. *Step* B. In order to prove the algebraic independence of the functions $u(k+1)$, $u'(k+1)$ over $\mathbf{C}(z)$, let us imagine, contrary to this proposition, that these functions are algebraically dependent over $\mathbf{C}(z)$. Then the functions $u(k)$, $u'(k)$, t and $v = u_{k+1}/u_0$ are also algebraically dependent over $\mathbf{C}(z)$. From step A it follows that v is algebraic over the field $\mathbf{F} = \mathbf{C}(z, u(k), u'(k), t)$. Now

$$v' = (u_0u'_{k+1} - u'_0u_{k+1})/u_0^2 = t/u_0^2$$

is an element of $\mathbf{F}$. Thus Lemma 1 of [2] implies that v itself is also an element of $\mathbf{F}$. Thus there exist two polynomials Q and P in z, $u(k)$, $u'(k)$, t, having no common factors, such that

$$Q\,v - P = 0. \tag{22}$$

When we differentiate this equation and multiply the result by $z\,u_0^2$, we obtain

$$z\,u_0^2Q'\,v + z\,Q\,t - z\,u_0^2P' = 0. \tag{23}$$

If $Q' = 0$, then $Q\,t = u_0^2P'$, and this is impossible by step A. So it follows that $Q' \neq 0$.

The left-hand side of this equation (23) is a polynomial in the same variables as $Q\,v - P$. That polynomial must be divisible by $Q\,v - P$, since otherwise we would obtain a contradiction with step A by eliminating v from the equations (22) and (23). This means that there exists a polynomial R in z, $u(k)$, $u'(k)$, t such that we have, identically in z, $u(k)$, $u'(k)$, t,

$$\begin{aligned} z\,u_0^2Q' &= R\,Q, \\ z\,Q\,t - z\,u_0^2P' &= -R\,P. \end{aligned} \tag{24}$$

If R is not divisible by u_0, then (24) yields the result that both polynomials Q and P are divisible by u_0^2. This contradiction gives the result $R = u_0S$, where S is a polynomial in z, $u(k)$, $u'(k)$, t.

Denote

$$Q = \sum_{i=0}^{m} A_i t^i , \qquad A_m \neq 0 ,$$

where A_i are polynomials in z , $u(k)$, $u'(k)$. The first equation (24) gives

$$u_0(z A'_m + m\,(z - \nu)\,A_m) = S\,A_m .$$

From this and step A it follows that S cannot contain t , and is at most of degree one with respect to z or with respect to $u(k)$, $u'(k)$.

First let us assume that S is not divisible by u_0 . Then (24) implies that Q is divisible by u_0 , say $Q = u_0 Q_0$. By (24),

$$z\,u_0 Q'_0 = (S - z\,u'_0)\,Q_0 ,$$
$$z\,Q_0 t - z\,u_0 P' = -S\,P .$$

From these equations it follows that $S - z\,u'_0$ must be divisible by u_0 , since otherwise Q_0 , and also P , would be divisible by u_0 , which is impossible. Thus we obtain

$$\begin{aligned} z\,Q'_0 &= (a\,z + b)\,Q_0 , \\ z\,Q_0 t &= z\,u_0 P' - \{(a\,z + b)\,u_0 + z\,u'_0\}\,P \end{aligned} \tag{25}$$

with constants a and b .

Let us denote

$$Q_0 = \sum_{i=0}^{m} B_i t^i , \qquad P = \sum_{i=0}^{n} P_i t^i , \qquad B_m P_n \neq 0 ,$$

where B_i and P_i are polynomials in z , $u(k)$, $u'(k)$. Now the first equation of (25) gives

$$z\,B'_m + m\,(z - \nu)\,B_m = (a\,z + b)\,B_m ,$$

by which

$$B_m = c_m z^{b+m\nu} e^{(a-m)z} .$$

The induction hypothesis implies that $a = m$. Further, $b + m\,\nu$ must be a non-negative integer, and

$$B_m = c_m z^{b+m\nu} .$$

If $n > m + 1$, then the second equation (25) gives, by step A , the following equation

$$u_0(z\,P'_n + n\,(z - \nu)\,P_n) - \{(m\,z + b)\,u_0 + z\,u'_0\}\,P_n = 0 .$$

This yields

$$P_n = c_n z^{b+n\nu} u_0 e^{(m-n)z} ,$$

but this contradicts the induction hypothesis. Thus $n = m + 1$. By comparing the coefficients of t^n in (25), we obtain

$$c_m z^{b+m\nu+1} = u_0(z P_n' + (z - b - n\nu) P_n) - z u_0' P_n .$$

Since $c_m \neq 0$ and the right-hand side of this equation is of degree $\geqq 1$ with respect to $u(k)$, $u'(k)$, we have obtained a contradiction.

Next let $S = (a z + b) u_0$ with constants a and b. Then (24) is of the form

$$\begin{aligned} z Q' &= (a z + b) Q , \\ z Q t - z u_0^2 P' &= -(a z + b) u_0^2 P . \end{aligned}$$

Thus Q must be divisible by u_0^2. On the other hand, if we denote as before

$$Q = \sum_{i=0}^{m} A_i t^i , \qquad A_m \not\equiv 0 ,$$

and use step A, we obtain

$$z A_m' + m (z - \nu) A_m = (a z + b) A_m .$$

Hence,

$$A_m = c_m z^{b+m\nu} e^{(a-m)z} , \qquad c_m \neq 0 .$$

The induction hypothesis yields $a = m$. Thus A_m is a polynomial in z alone, and Q cannot be divisible by u_0^2. This is the desired contradiction, and completes step B.

6. *Step* C. Here we prove that the functions $u(k+1)$, $u'(k+1)$, $e^{\gamma z}$, $\gamma \in \mathbf{Q}$ and $\gamma \neq 0$, are algebraically independent over $\mathbf{C}(z)$. Let us assume that these functions are algebraically dependent over $\mathbf{C}(z)$.

From Lemma 4 and step B it follows that our assumption implies an equation

$$P = P_m e^{m\gamma z} + P_0 = 0 ,$$

where P_m and P_0 are polynomials in z, $u(k+1)$, $u'(k+1)$, and one of them is a polynomial in z alone. Further, we have

$$z P' = (a z + b) P , \qquad a,\ b \in \mathbf{Q}$$

identically in z, $u(k+1)$, $u'(k+1)$, $e^{\gamma z}$. So $P_m = c_m z^b$ or $P_0 = c_0 z^b$, where c_m and c_0 are non-zero constants and b must be a non-negative integer.

We first assume that $\gamma = 1$. If $P_m = c_m z^b$, then $a = m$ and P_0 satisfies the differential equation

$$z P_0' = (m z + b) P_0 .$$

In the same way as on p. 7 of [7] we can now deduce that

(26) $$P_0 = P_{00} + \sum_{i=1}^{l} Q_i(u_0' u_{k+1} - u_0 u_{k+1}')^i, \qquad Q_l \neq 0\,,$$

where P_{00} and Q_i are polynomials in z, $u(k)$, $u'(k)$. Further, as on p. 8 of [7], we obtain

$$c_1'(0)\, z^b e^{mz} - \sum_{i=1}^{l} i\, Q_i(u_0' y - u_0 y')\,(u_0' u_{k+1} - u_0 u_{k+1}')^{i-1} = 0\,,$$

where y is a solution of (4), linearly independent of u_0. Since $u_0' y - u_0 y' = c\, e^z/z^\nu$, $c \neq 0$, we have

(27) $$c_1'(0)\, z^{b+\nu} e^{(m-1)z} - c \sum_{i=1}^{l} i\, Q_i(u_0' u_{k+1} - u_0 u_{k+1}')^{i-1} = 0\,.$$

If $c_1'(0) = 0$, then we have a contradiction with step B. If $c_1'(0) \neq 0$, then $m = 1$ and also, by step B, $l = 1$. Thus $Q_1 = d\, z^{b+\nu}$, $d = c_1'(0)/c$. So it follows that the polynomial

$$P_0 = P_{00} + d\, z^{b+\nu}(u_0' u_{k+1} - u_0 u_{k+1}') = P_{00} - d\, z^{b+\nu} t$$

satisfies the differential equation

$$z P_0' = (z + b)\, P_0\,.$$

This is impossible as was proved in step A (see pp. 98—101).

If $P_0 = c_0 z^b$, then $a = 0$ and P_m satisfies the differential equation

$$z P_m' = (-m z + b)\, P_m\,.$$

We can now deduce that P_m has the same form (26) as P_0. The equation analogous to (27) is now of the form

(28) $$c_1'(0)\, z^{b+\nu} - c\, e^{(m+1)z} \sum_{i=1}^{l} i\, Q_i(u_0' u_{k+1} - u_0 u_{k+1}')^{i-1} = 0\,.$$

The left-hand side of this equation must be divisible by the polynomial $P = P_m e^{mz} + c_0 z^b$. This is impossible, and thus the functions $u(k+1)$, $u'(k+1)$, e^z are algebraically independent over $\mathbf{C}(z)$.

Let $\gamma \neq 0$ be an arbitrary rational number. If $P_m = c_m z^b$, then we can again deduce that P_0 has the form (26), and the equation analogous to (27) is of the form

$$c_1'(0)\, z^{b+\nu} e^{m\gamma z} - c\, e^z \sum_{i=1}^{l} i\, Q_i(u_0' u_{k+1} - u_0 u_{k+1}')^{i-1} = 0\,.$$

From this it follows that

$$c_1'(0)\, z^\nu P_0/c_m + c\, e^z \sum_{i=1}^{l} i\, Q_i(u_0' u_{k+1} - u_0 u_{k+1}')^{i-1} = 0\,.$$

This contradicts with the case $\gamma = 1$, which we just proved. If $P_0 = c_0 z^b$, then we obtain a similar contradiction as before, and thus step C holds. Our Lemma 1 is now proved.

7. We now denote $u_{0,j}(z, \nu, \mu) = u_{0,j}$, $u_{j,0}(z, \nu, \mu) = u_{j,0}$, $j = 0, 1, \ldots$. In the following let the notations $u(m, n)$ and $u'(m, n)$ denote the functions $u_{i,0}$, $u_{0,j}$ and $u'_{i,0}$, $u'_{0,j}$, $i = 0, 1, \ldots, m$, $j = 1, 2, \ldots, n$, respectively.

From (5) and (1) it follows that the functions $u(p, n)$ satisfy the following system of differential equations

$$\begin{aligned} & u''_{0,j} + (\nu/z - 1)\, u'_{0,j} - (\mu/z)\, u_{0,j} - (1/z)\, u_{0,j-1} = 0, \quad u_{0,-1} \equiv 0, \\ (29)\quad & u''_{i,0} + (\nu/z - 1)\, u'_{i,0} - (\mu/z)\, u_{i,0} + (1/z)\, u'_{i-1,0} = 0, \quad u_{-1,0} \equiv 0, \\ & i = 0, 1, \ldots, p, \qquad j = 1, 2, \ldots, n. \end{aligned}$$

Lemmas 2 and 3 will now be proved simultaneously. The proofs will be performed by induction. If $m = 0$, then Lemmas 2 and 3 follow from Lemma 1, which we just proved.

Now, let us assume that the functions

$$u(m, n), \quad u'(m, n), \quad e^{\gamma z} \tag{30}$$

are algebraically independent over $\mathbf{C}(z)$. Using this assumption, we prove that the functions

$$u(m+1, n), \quad u'(m+1, n), \quad e^{\gamma z}, \tag{31}$$

too, are algebraically independent over $\mathbf{C}(z)$. Let us assume the contrary case that the functions (31) are algebraically dependent over $\mathbf{C}(z)$. We shall prove that this leads to a contradiction.

The proof is divided into three steps, called here A 1, B 1 and C 1.

8. Step A 1. By analogy with step A we denote

$$t = u_{00} u'_{m+1,0} - u'_{00} u_{m+1,0},$$

and prove that the functions

$$u(m, n), \qquad u'(m, n) \tag{32}$$

and t are algebraically independent over $\mathbf{C}(z)$. Otherwise we would have an equation

$$P = \sum_{i=0}^{l} P_i t^i = 0,$$

where P is an irreducible polynomial in z, t and (32); and P_i, $i = 0, 1, \ldots, l$, are polynomials in z and (32) such that $P_l \not\equiv 0$. By the induction hypothesis, $l \geqq 1$.

In a completely analogous way to step A (pp. 97—98) we can now deduce that P is of the form

$$(33) \qquad P = c_0 z^{b+\nu} t + P_0 = 0$$

and satisfies

$$(34) \qquad z P' = (z + b) P , \quad b \in \mathbf{Q} ,$$

identically in z, t and (32). Further, P_0 is a homogeneous polynomial of degree two with respect to (32).

Since (34) is an identity, it follows that (34) holds if we replace the solution $u(m+1, n)$ by any other solution $\bar{u}(m+1, n)$ of (29) with $p = m+1$.

We now write $s = [m/2]$. By analogy with reasoning employed on pp. 98—99 we can deduce that P must be of the form

$$(35) \qquad P = \sum_{l=0}^{s} a_l \sum_{j=0}^{s-l} (u_{j0} u'_{m+1-l-j,0} - u'_{j0} u_{m+1-l-j,0}) + A ;$$
$$a_0 = c_0 z^{b+\nu} \neq 0 ,$$

where A is a homogeneous polynomial of degree two with respect to $u(0, n)$, $u'(0, n)$, u_{j0}, u'_{j0}, $j = 1,2,\ldots, m-s$. From (34) it now follows that

$$(36) \qquad \sum_{l=1}^{s} (z a'_l - (b+\nu) a_l - a_{l-1}) P_l - \sum_{l=0}^{s} a_l u_{s-l,0} u'_{m-s,0} + z A' = (z+b) A ,$$

where we have denoted

$$P_l = \sum_{j=0}^{s-l} (u_{j,0} u'_{m+1-l-j,0} - u'_{j,0} u_{m+1-l-j,0}) .$$

If $s > 0$, which means $m > 1$, then the induction hypothesis yields

$$z a'_1 - (b+\nu) a_1 - c_0 z^{b+\nu} = 0 .$$

If $b + \nu = 0$, then $z a'_1 = c_0$. But this is impossible. If $b + \nu > 0$, then $a_1 = d z^i + e z^{i+1} + \ldots$, $d \neq 0$, $1 \leqq i \leqq b + \nu$. Thus we obtain

$$i d - (b+\nu) d - \delta_{i,b+\nu} c_0 = 0 ,$$

and this leads to a contradiction.

If $m = 1$, then P is of the form

$P = c_0 z^{b+\nu} t + A_1 + P_{10} u_{10} + P_{01} u'_{10} + P_{20} u_{10}^2 + P_{11} u_{10} u'_{10} + P_{02} u'^2_{10}$, where A_1; P_{10}, P_{01} and P_{20}, P_{11}, P_{02} are homogeneous polynomials of degree two; one and zero with respect to $u(0, n)$, $u'(0, n)$. When in (34) we replace $u(2, n)$ by the solution $\bar{u}(0, n) = u(0, n)$, $\bar{u}_{i0} = u_{i0} + \sigma u_{i-1,0}$,

$i = 1,2$, integrate (34), differentiate the result with respect to σ and put $\sigma = 0$, we obtain

$$c_0 z^{b+\nu}(u_{00}u'_{10} - u'_{00}u_{10}) + P_{10}u_{00} + P_{01}u'_{00} + 2\,P_{20}u_{10}u_{00} + P_{11}(u_{10}u'_{00} + u'_{10}u_{00}) + 2\,P_{02}u'_{10}u'_{00} = c'(0)\, z^b e^z .$$

The induction hypothesis tells us that this is impossible.

Next, let $m = 0$, and let $r = [n/2]$. If $r > 0$, then we obtain, using the induction hypothesis and a similar technique as on pp. 98—99,

$$P = c_0 z^{b+\nu} t + \sum_{l=0}^{r-1} a_l \sum_{j=0}^{r-1-l} (u_{0j}u'_{0,n-l-j} - u'_{0j}u'_{0,n-l-j}) + A_r , \tag{37}$$

where $a_l = a_l(z)$ are polynomials in z and A_r is a homogeneous polynomial of degree two with respect to $u(0, n-r)$, $u'(0, n-r)$. Now either $2\,r = n$ or $2\,r = n-1$.

1°. $2\,r = n$. The substitution $\bar{u}(0, r-1) = u(0, r-1)$, $\bar{u}_{0j} = u_{0j} + \sigma\, u_{0, j-r}$, $j = r, \ldots, n$, in (34) now gives, after similar steps as before,

$$\sum_{l=0}^{r-1} a_l\,(u_{00}u'_{0,r-l} - u'_{00}u_{0,r-l}) + P_{10}u_{00} + P_{01}u'_{00} + 2\,P_{20}u_{0r}u_{00} + {} + P_{11}(u_{0r}u'_{00} + u'_{0r}u_{00}) + 2\,P_{02}u'_{0r}u'_{00} = c'(0)\, z^b e^z ,$$

where we have presented A_r in the typical form

$$A_r = A_{r+1} + P_{10}u_{0r} + P_{01}u'_{0r} + P_{20}u^2_{0r} + P_{11}u_{0r}u'_{0r} + P_{02}u'^2_{0r} .$$

From the induction hypothesis it follows that $a_0(z) \equiv 0$.

2°. $2\,r = n-1$. By the substitution $\bar{u}(0, r) = u(0, r)$, $\bar{u}_{0j} = u_{0j} + \sigma\, u_{0, j-r-1}$, $j = r+1, \ldots, n$, in (34) we get

$$P = c_0 z^{b+\nu} t + \sum_{l=0}^{r} a_l \sum_{j=0}^{r-l} (u_{0j}u'_{0,2r+1-l-j} - u'_{0j}u_{0,2r+1-l-j}) + A \tag{38}$$

where A is a homogeneous polynomial of degree two with respect to $u(0, r)$, $u'(0, r)$. By (34) we have

$$-c_0 z^{b+\nu} u_{00}u'_{00} + \sum_{l=0}^{r} (z\, a'_l - (b+\nu)\, a_l)\, P_l + \sum_{l=0}^{r} a_l u_{0,r-l} u_{0r} + z\, A' = (z+b)\, A , \tag{39}$$

where

$$P_l = \sum_{j=0}^{r-l} (u_{0j}u'_{0,2r+1-l-j} - u'_{0j}u_{0,2r+1-l-j}) .$$

By the induction hypothesis we get $a_l = d_l z^{b+\nu}$, where d_l are constants. As on pp. 100—101 we obtain a contradiction if $d_0 \neq 0$. Thus $a_0(z) \equiv 0$. Thus we see that the functions $u_{0,n}$, $u'_{0,n}$ do not occur in P.

By continuing in the same way we can deduce that none of the functions $u_{0,i}$, $u'_{0,i}$, $[i/2] > 0$, can occur in P. Thus we finally see that P must be of the form

$$(40)\quad P = c_0 z^{b+\nu} t + c\, z^{b+\nu}(u_{00}u'_{01} - u'_{00}u_{01}) + P_{20}u_{00}^2 + P_{11}u_{00}u'_{00} + P_{02}u'^2_{00}\,,$$

where P_{20}, P_{11}, P_{02} are polynomials in z. These polynomials must satisfy the equations (see p. 100)

$$(41)\quad \begin{aligned} &z P'_{20} - (z+b) P_{20} + \mu P_{11} + c\, z^{b+\nu} = 0\,,\\ &z P'_{11} - (b+\nu) P_{11} + 2 z P_{20} + 2 \mu P_{02} - c_0 z^{b+\nu} = 0\,,\\ &z P'_{02} + (z - b - 2\nu) P_{02} + z P_{11} = 0\,. \end{aligned}$$

Let us assume, at first, that $c = 0$. If d_1, d_2 and d_3 are the degrees of P_{20}, P_{11} and P_{02} respectively, then (41) gives $d_2 = d_3 = d_1 + 1 = b + \nu$. If d is the coefficient of z^{d_3} in P_{02}, then it follows from (41) that

$$-d_2 d + (b+\nu)\, d - 2\mu\, d + 2\mu\, d - c_0 = -c_0 = 0\,,$$

which is impossible, since $c_0 \neq 0$. This gives step A 1 of Lemma 2 (we have not yet used the assumption $\nu \neq 1,2,\ldots$ of Lemma 3).

Next, let $c \neq 0$. The degrees $d_2 = d_3$ and $d_1 + 1$ cannot exceed $b + \nu$. Further, since $\nu \neq 1,2,\ldots$ in Lemma 3, it follows from (41) that P_{11} and P_{02} are not $\equiv 0$. Now let $P_{02} = k\, z^i + l\, z^{i+1} + \ldots$, $k \neq 0$. By putting $P_{11} = k_1 z^{i-1} + l_1 z^i + \ldots$ and by using the last equation (41) we obtain

$$i\, k - (b + 2\nu)\, k + k_1 = 0\,.$$

This yields $k_1 \neq 0$, since $i - b - 2\nu = i - n_0 - \nu \neq 0$, $n_0 = b + \nu$. From the second equation (41) it follows that $P_{20} = k_2 z^{i-2} + l_2 z^{i-1} + \ldots$, where $k_2 \neq 0$. Then we use the first equation (41), which gives

$$(i - 2)\, k_2 - b\, k_2 = 0\,.$$

Thus b is an integer, and so $\nu = n_0 - b$ must also be an integer. This contradiction completes the proof of step A 1.

9. Step B 1. Here we can use a word to word repetition of step B (pp. 101—103), now defining $v = u_{m+1,0}/u_{00}$. We thus obtain the algebraic independence of the functions (32) and $u_{m+1,0}$, $u'_{m+1,0}$ over $\mathbf{C}(z)$.

10. Step C 1. The functions (30) and $u_{m+1,0}$, $u'_{m+1,0}$ can now be proved to be algebraically independent over $\mathbf{C}(z)$ in a completely analogous way to the proof of step C. (It should be noted that Lemma 4 remains valid if we replace the functions $u(n)$, $u'(n)$ by the functions $u(m+1\,, n)$, $u'(m+1\,, n)$.) Thus Lemmas 2 and 3 are true.

11. The functions $u(m, n)$ and $u'(m, n)$ are E-functions (for the definition of E-functions see p. 33 of [6]). Further, by (29), the functions $u(0, n)$, $u'(0, n)$, e^z; $u(m, 0)$, $u'(m, 0)$, e^z and $u(m, n)$, $u'(m, n)$, e^z satisfy

$$(u_{0j})' = u'_{0j}, \quad (u'_{0j})' = (1 - \nu/z)\, u'_{0j} + (\mu/z)\, u_{0j} + (1/z)\, u_{0,\, j-1}, \quad y' = y,$$
$$u_{0,\,-1} \equiv 0, \quad j = 0,1,\ldots, n;$$
$$(u_{i0})' = u'_{i0}, \quad (u'_{i0})' = (1 - \nu/z)\, u'_{i0} + (\mu/z)\, u_{i0} - (1/z)\, u'_{i-1,\, 0}, \quad y' = y,$$
$$u_{-1,\, 0} \equiv 0, \quad i = 0,1,\ldots, m,$$

and

$$(u_{0j})' = u'_{0j}, \quad (u'_{0j})' = (1 - \nu/z)\, u'_{0j} + (\mu/z)\, u_{0j} + (1/z)\, u_{0,\, j-1}, \quad u_{0,\,-1} \equiv 0,$$
$$(u_{i0})' = u'_{i0}, \quad (u'_{i0})' = (1 - \nu/z)\, u'_{i0} + (\mu/z)\, u_{i0} - (1/z)\, u'_{i-1,\, 0}, \quad u_{-1,\, 0} \equiv 0,$$
$$y' = y; \quad i = 0,1,\ldots, m, \quad j = 1,2,\ldots, n.$$

Thus we can end our paper by establishing that the truth of our theorems follows from our lemmas and Šidlovskiĭ's theorem [5].

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