THE DOMAINS OF NORMALITY
OF AN ENTIRE FUNCTION

1. Introduction

If \( f \) is a rational or entire function of the complex variable \( z \) its natural iterates \( f_n \) are defined by \( f_1(z) = f(z), f_{n+1}(z) = f(f_n(z)), n = 1, 2, \ldots \). The theory developed by Fatou [7,8] and Julia [11] deals with the set \( \mathcal{C} = \mathcal{C}(f) \) of points of the complex plane in whose neighbourhood \( \{f_n(z)\} \) is a normal family. It is convenient to express many results in terms of the complement \( \mathfrak{F}(f) \) of \( \mathcal{C} \), i.e. the set of non-normality. We shall assume throughout that \( f \) is not a rational function of order 0 or 1. Then \( \mathfrak{F} = \mathfrak{F}(f) \) has the following properties (see [7] and [8]).

I. \( \mathfrak{F}(f) \) is a non-empty perfect set.

II. \( \mathfrak{F}(f) \) and \( \mathcal{C}(f) \) are completely invariant under the mapping \( z \rightarrow f(z) \).

In general a set \( S \) is called completely invariant under \( z \rightarrow f(z) \) if \( x \in S \) implies that \( f(x) \in S \) and that \( \beta \in S \) for every solution \( \beta \) of \( f(\beta) = x \).

The components \( G_i \) of \( \mathcal{C}(f) \) are maximal domains of normality for \( \{f_n\} \). The theory considers the various ways in which \( \mathfrak{F} \) may separate these components and the limit functions which arise from those subsequences of \( \{f_n\} \) which are locally uniformly convergent in \( G_i \).

It may happen for rational \( f \) that \( \mathfrak{F} \) is totally disconnected (a 'discontinuum') so that \( \mathcal{C} \) consists of a single domain. This occurs for \( f(z) = z^2 - p \), where \( p > 2 \) is a constant, in which case \( \mathfrak{F}(f) \) is a bounded, totally disconnected subset of the real axis (Myrberg [12]). At the end of [8] Fatou raises the question as to whether there are transcendental entire functions \( f \) for which \( \mathfrak{F}(f) \) is totally disconnected.

Concerning the set \( \mathcal{C}(f) \) H. Töpfer [15] has shown:

III. If \( f \) is transcendental and entire and if \( \mathcal{C}(f) \) has an unbounded component \( G \), then every other component of \( \mathcal{C}(f) \) is simply-connected. If in addition \( G \) is multiply connected, then \( G \) is completely invariant under the mapping \( z \rightarrow f(z) \).

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In this note we shall prove the

**Theorem 1.** If \( f \) is transcendental and entire, then \( \mathcal{C}(f) \) has no unbounded multiply-connected component.

Since the total discontinuity of \( \hat{\mathcal{X}}(f) \) implies that \( \mathcal{C}(f) \) is an unbounded connected domain Fatou's question is answered by the

**Corollary.** For transcendental entire \( f \) the set \( \hat{\mathcal{X}}(f) \) must contain non-degenerate continua.

Various authors e.g. Brolin [6], Garber [9], Oba and Pitcher [13] have investigated the metric properties of \( \hat{\mathcal{X}}(f) \), giving estimates of Hausdorff dimensions, capacities, and so on. The only significant lower estimate in the transcendental entire case was given in [9], where it was shown that the logarithmic capacity of \( \hat{\mathcal{X}} \) is strictly positive. Our corollary strengthens this result considerably. We remark also that the set \( \hat{\mathcal{X}} \) can even fill out the whole plane in some cases ([4]).

Returning to the components of \( \mathcal{C}(f) \) in the theorem: it is indeed possible that multiply-connected components exist for transcendental entire \( f \), as shown by an example in [1]. In this case the multiply-connected domains are of course bounded.

If \( f \) is a transcendental entire function, any completely invariant component of \( \mathcal{C}(f) \) is unbounded and hence, by our theorem, simply-connected. It was shown in [3] that there can be at most one such completely invariant component. P. Bhattacharyya [5] deduced from this that the number of components of \( \mathcal{C}(f) \) is either 1 or infinite. He also showed that for \( g(z) = e^{a+z} - e^z \), \( a < 0 \), \( \mathcal{C}(g) \) consists of a single (completely invariant) component. It is not clear whether the existence of a completely invariant component of \( \mathcal{C} \) precludes the existence of other components or not. We can prove

**Theorem 2.** If \( f \) is a transcendental entire function such that \( \mathcal{C}(f) \) has a completely invariant component \( G \), then in every other component of \( \mathcal{C}(f) \) \( f \) is univalent.

**Corollary.** A function \( f \) which satisfies the conditions of Theorem 2 can have at most one attractive fixpoint.

An attractive fixpoint \( z \) is a point for which \( f(z) = z \), \( |f(z)| < 1 \). Two different attractive fixpoints belong to different components of \( \mathcal{C}(f) \) and (c.f. [7,8]) \( f \) is not univalent in these components. The corollary follows. The example \( g(z) = e^{a+z} - e^z \), \( a < 0 \), shows that one attractive fixpoint is possible.
2. Lemmas needed in the Proofs

Additional results about $\mathfrak{N}(f)$ are proved in [7] for rational $f$ and in [8] for entire $f$, except where mentioned below.

IV. For any integer $n \geq 1$ we have $\mathfrak{N}(f_n) = \mathfrak{N}(f)$.

V. For every $z \in \mathfrak{N}(f)$ and for every complex $\beta$ (excluding at most two exceptional $\beta$-values) there exist a sequence of positive integers $\{n_k\}$ and a sequence of complex numbers $\{z_k\}$ such that

$$f_{n_k}(z_k) = \beta, \quad \lim z_k = z.$$  

A fixpoint $z$ of order $n$ of $f$ is a solution of $f_n(z) = z$; $z$ is said to have exact order $n$ if $f_k(z) \neq z$ for $1 \leq k < n$ and in this case the multiplier of $z$ is the number $f'_n(z)$. If $|f'_n(z)| > 1$ the fixpoint $z$ is called repulsive and belongs to $\mathfrak{N}(f)$. Moreover one has

VI. $\mathfrak{N}(f)$ is the derivative of the set of fixpoints of all orders of $f$. It is even true that the repulsive fixpoints are dense in $\mathfrak{N}$ (shown in [2] for entire $f$).

In addition we need

Lemma 1. (Pólya [14]). Let $e$, $g$ and $h$ be entire functions satisfying

(1) \hspace{1cm} e(z) = g(h(z)) \\
(2) \hspace{1cm} h(0) = 0.

There is a constant $c > 0$ (in fact $c = 1/8$) independent of $e$, $g$, $h$ such that

(3) \hspace{1cm} M(e, r) > M(g, cM(h, r/2)),

where $M(e, r)$ denotes $\max_{|z|=r} |e(z)|$.

Lemma 2. (Schottky's theorem, see e.g. [10]). There exists an absolute constant $C$ such that for every function $f(z)$ which is regular and satisfies $f(z) \neq 0, 1$ in $|z| < 1$ we have for

$$M(f, r) = \max_{|z|=r} |f(z)| < \exp \left[ \frac{1}{1-r} ((1 + r) \log \max (1, |f(0)|) + 2Cr) \right].$$

3. Proof of Theorem 1

Suppose that $f$ is a transcendental entire function and that $G$ is an unbounded, multiply connected component of $\mathfrak{N}(f)$. Property VI shows that there are in $\mathfrak{N}$ two repulsive fixpoints $z_1$, $z_2$ of order say $p$ and $q$ respectively, which may be taken to be different from the exceptional values in V. Both are repulsive fixpoints of $f_{pq}$ and IV shows we can replace $f_{pq}$ by $f$ and assume $z_1$, $z_2$ are repulsive fixpoints of $f$. Replacing $f(z)$
by \((f(a + bz) - a)/b, a, b\) constant, merely subjects \(\tilde{\gamma}\) and \(\mathbb{C}\) to a linear transformation, so we may without loss of generality assume that \(z_1 = 0\) and \(z_2 = 1\) are first order repulsive fixpoints of \(f\), i.e. \(0, 1 \in \tilde{\gamma}\), and that 0 is not an exceptional point in the sense of \(V\).

Now if any of the locally-convergent subsequences of \(\{f_n\}\) in \(G\) has a finite and hence regular limit it follows that the convergence remains uniform in the interior of any Jordan curve in \(G\), so that \(G\) is not multiply-connected. Thus \(f_n(z)\) must converge locally uniformly to \(\infty\) in \(G\).

The multiply-connected domain \(G\) must contain a Jordan curve \(\gamma\) in whose interior lie points of \(\tilde{\gamma}\), and so by \(V\)-points of the form \(f_n(0)\) for some arbitrarily large \(n\). Thus for sufficiently large \(n\) the set \(\gamma_1 = f_n(\gamma)\) is (by III) a curve in \(G\) which winds round 0 at least once and whose minimum distance \(r\) from 0 is as large as we please. We choose \(n\) so large that

\[
(1/8) M(f, t/4) > t \quad \text{for} \quad t \geq r.
\]

We next choose an \(m\) such that \(\gamma_2 = f_m(\gamma)\) is a curve in \(G\) which winds round 0 and which has a minimum distance \(s\) from 0 satisfying

\[
s > M(f_2, 2R),
\]

where \(R\) is the greatest distance of \(\gamma_1\) from 0. Join \(\gamma_1\) to \(\gamma_2\) by a path \(\gamma_3\) in \(G\) and denote by \(K\) the union of \(\gamma_1, \gamma_2\) and \(\gamma_3\).

Denote by \(\delta\) the distance of the compact set \(K\) from \(\tilde{\gamma}\). Then \(\delta > 0\). There is a finite collection \(C\) of say \(N\) discs of radius \(\delta\) whose centres lie on \(K\) and whose union covers \(K\). Since \(K\) is connected, there is for any pair \(t_1, t_2\) in \(K\) a chain of \(t \leq N\) points \(t_1 = w_1, w_2, ..., w_p = t_2\) in \(K\) such that \(w_i, w_{i+1}\) lie in a common disc of \(C\). Thus

\[
|w_{i+1} - w_i| < 2\delta.
\]

Suppose that in a \((3\delta)\)-neighbourhood \(L\) of \(K\) the function \(g\) is regular, satisfies \(|g(z)| > 1\) and omits the values 0 and 1. The disc \(|w - w_i| < 3\delta\) lies in \(L\) and contains \(w_{i+1}\). Applying Lemma 2 to the function \(g(w_i + 3\delta z)\) in the unit disc we see that there is an absolute constant \(A > 1\) such that

\[
|g(w_{i+1})| < A |g(w_i)|^5.
\]

Hence for \(t_1, t_2\) as above

\[
|g(t_2)| < B |g(t_1)|^C
\]

where the constants \(C = 5^N\), \(B = A^{1+5+...+5^N}\) are independent of \(g\) or of the choice of \(t_1, t_2\) in \(K\).

Since \(f_n \to \infty\) locally uniformly in \(G\), while \(f_n(G) \subseteq G\) so \(f_n(z) \neq 0, 1 \in \tilde{\gamma}\) for \(z \in G\), we see that for all sufficiently large \(n\) the
functions \( f_n \) satisfy \(|f_n(z)| > 1, f_n(z) \neq 0, 1\) in \( L \). Thus by (6) if \( t_1 \) is any point of \( \gamma_1 \) and if \( t_2 \) is the point of \( \gamma_2 \) at which \(|f_n|\) is a maximum, we have

\[
|f_n(t_2)| < B |f(t_1)|^C, \quad n \geq n_0.
\]

However by the choice of \( s \) in (5)

\[
|f_n(t_2)| \geq M(f_n, s) \\
\geq M(f_n, M(f_2, 2R)) \\
\geq M(f_n+2, 2R) \\
\geq M(f, (1/8) M(f_{n+1}, R))
\]

by Lemma 1. But on \( \gamma_1 \) we have \( f_n(z) \to \infty \) and so \( M(f_{n+1}, R) \to \infty \) as \( n \to \infty \). Thus the last expression above is, for all sufficiently large \( n \), greater than

\[
B (M(f_{n+1}, R))^C > B (M(f_n, (1/8) M(f, R/2)))^C \\
\geq B (M(f_n, R))^C \\
\geq B |f_n(t_1)|^C
\]

by (4). Thus we have a contradiction with (7). The theorem is proved.

**Proof of Theorem 2**

Suppose the transcendental entire function \( f \) has a completely invariant component \( G \) of \( \mathbb{C}(f) \). Then \( G \) is necessarily unbounded and simply connected. All other components of \( \mathbb{C} \) are simply connected. Suppose that there is a component \( H \neq G \) of \( \mathbb{C}(f) \) in which \( f \) is not univalent. Now by II \( f(H) \) lies in some component \( K \neq G \) of \( \mathbb{C}(f) \).

Take a value \( k = f(p) = f(q) \) where \( p \in H, \ q \in H, \ p \neq q, \ f'(p) \neq 0, \ f'(q) \neq 0 \). Thus there are branches \( z = P(w) \) and \( z = Q(w) \) of the inverse \( f^{-1} \) of \( w = f(z) \) which are regular at \( w = k \in K \) and satisfy \( p = P(k), \ q = P(k) \).

By Gross' star theorem we may continue \( P(w), \ Q(w) \) regularly to \( \infty \) along almost any ray starting at \( k \), in particular along some ray \( L \) which meets \( G \). Denote by \( \gamma \) the segment of \( L \) from \( k \) to a certain point \( g \in G \). Then \( P(\gamma), \ Q(\gamma) \) are disjoint curves joining \( p \in H \) to \( p' = P(g) \in G \) and \( q \in H \) to \( q' = Q(g) \in G \), respectively.

Join \( p \) to \( q \) by a simple arc \( \beta \) in \( H \), and \( p' \) to \( q' \) by a simple arc \( \beta' \in G \). Let \( \tilde{p} \) be the last intersection of \( \beta \) with \( P(\gamma) \), \( \tilde{q} \) the first intersection with \( Q(\gamma) \). Let \( \tilde{\beta} \) be the subarc of \( \beta \) which joins \( \tilde{p} \) to \( \tilde{q} \). Simi-
larly define \( \overline{p}' \) as the last intersection of \( \beta' \) with \( P(\gamma) \), \( \overline{q}' \) the first intersection with \( Q(\gamma) \) and \( \bar{\beta}' \) as the subarc \( \overline{p}' \overline{q}' \) of \( \beta' \). Denote by \( \pi \) the subarc \( \overline{pp}' \) of \( P(\gamma) \), by \( \bar{z} \) the subarc \( \overline{qq}' \) of \( Q(\gamma) \). Then \( \pi \bar{\beta}'(\bar{z})^{-1}(\beta')^{-1} \) is a Jordan curve \( C \) whose interior \( D \) maps under \( z \to f(z) \) into a bounded region \( f(D) \) whose boundary is contained in \( f(C) \subset f(\beta) \cup f(\beta') \cup \gamma \).

The \( f(\beta) \) and \( f(\beta') \) are closed bounded and disjoint curves. The unbounded component \( M \) of their complement contains \( \overline{\gamma}(f) \). Thus \( M \) meets \( \gamma \) since \( \overline{\gamma}(f) \) does. Now \( f(\pi) \) is a segment of \( \gamma \) which joins \( f(\beta) \) to \( f(\beta') \). If \( t \) is the last point of intersection of \( \gamma \) with \( f(\beta) \) and \( t' \) the first intersection with \( f(\beta') \), then the segment \( \overline{tt}' \) of \( \gamma \) is a crosscut of \( M \) whose ends belong to different components of the frontier. Thus \( \overline{tt}' \) does not disconnect \( M \). Since \( \overline{tt}' \) belongs to \( f(\pi) \) every point of \( \overline{tt}' \) is a boundary value of \( f(D) \). Thus \( f(D) \) must contain the whole of \( M - \overline{tt}' \), i.e. an unbounded set. This contradicts the boundedness of \( D \) and the result is proved.

References

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