THE HAUSDORFF DIMENSION OF THE BRANCH
SET OF A QUASIREGULAR MAPPING

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1. Introduction

Let \( G \) be a domain in the \( n \)-dimensional euclidean space \( \mathbb{R}^n \), \( n \geq 2 \). Consider a non-constant quasiregular mapping \( f : G \to \mathbb{R}^n \). Let \( B_f \) denote the branch set of \( f \).

By [6] \( m(B_f) = m(fB_f) = 0 \), where \( m \) is the \( n \)-dimensional Lebesgue measure in \( \mathbb{R}^n \). Then also \( H^x(B_f) = H^x(fB_f) = 0 \), where \( H^x, x > 0 \), is the \( x \)-dimensional Hausdorff outer measure in \( \mathbb{R}^n \). On the other hand, in [3] it is shown by an example that \( \dim_H B_f \) and \( \dim_H fB_f \), the Hausdorff dimensions of \( B_f \) and \( fB_f \), can be arbitrarily close to \( n \).

In this paper we prove the following results. Let \( i(x, f) \) denote the local topological index of \( f \) at \( x \). If \( f \) is as above, then

\[
\dim_H fB_f \leq c' < n,
\]

where the constant \( c' \) depends only on \( n \) and the maximal dilatation \( K(f) \) of \( f \). If, in addition, \( i(f) = \sup \{ i(x, f) | x \in B_f \} < \infty \), then

\[
\dim_H B_f \leq c < n,
\]

where the constant \( c \) depends only on \( n \), \( K(f) \) and \( i(f) \). It remains an open question whether \( c \) actually depends on \( i(f) \). If it does not, then always \( \dim_H B_f < n \), too.

We shall prove (1.1) and (1.2) using a similar method to Rešetnjak’s in [9] and Martio’s and Rickman’s in [5].

For more information on \( \dim_H B_f \) and \( \dim_H fB_f \) see, for example, [5].

2. Notation

We use the same notation and terminology as in [6]. If \( A \subseteq \mathbb{R}^n \), we write \( \text{cl} A \), \( \text{int} A \) and \( \partial A \) for the closure, the interior and the boundary.
of $A$. If $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, $A \neq \emptyset$, we denote by $d(x, A)$ the distance from $x$ to $A$ and by $d(A)$ the diameter of $A$. If $x \in \mathbb{R}^n$ and $r > 0$ we write $B^n(x, r)$ for the open ball $\{ y \in \mathbb{R}^n \mid |x - y| < r \}$ and abbreviate $B^n_r = B^n(0, r)$, $B^n = B^n(1)$. We also write $S^{n-1}(x, r) = \partial B^n(x, r)$, $S^{n-1}(0, r) = S^n = S^{n-1}(1)$.

We let the notation $f : G \to \mathbb{R}^n$ include the assumption that $G \subset \mathbb{R}^n$ is a domain and $f$ is continuous. If $r \in G$ and $r > 0$, put

$$l(x, f, r) = \inf_{|x - y| = r} |f(x) - f(y)|,$$
$$L(x, f, r) = \sup_{|x - y| = r} |f(x) - f(y)|$$

whenever $B^n(x, r) \subset G$. We let $U(x, f, r)$ denote the $x$-component of $f^{-1}B^n(f(x), r)$ and write

$$l^s(x, f, r) = \inf \{|x - y| \mid y \in \partial U(x, f, r)\} \quad \text{and} \quad L^s(x, f, r) = \sup \{|x - y| \mid y \in \partial U(x, f, r)\}$$

whenever $\partial U(x, f, r) \neq \emptyset$. If $A \subset \mathbb{R}^n$ and $y \in \mathbb{R}^n$, put $N(y, f, A) = \text{card}(A \cap f^{-1}(y))$ and $N(f, A) = \sup \{ N(y, f, A) \mid y \in \mathbb{R}^n \}$. If $f : G \to \mathbb{R}^n$ is quasiregular and $x \in G$, then there exists $r_x > 0$ such that if $r \in (0, r_x)$, then $U(x, f, r)$ is a normal neighborhood of $x$ and $N(f, U(x, f, r)) = i(x, f)$; see [6; 2.9, 2.12].

Let $e_1, \ldots, e_n$ denote the coordinate unit vectors of $\mathbb{R}^n$, $\mathbb{Z}$ the set of integers and $\mathcal{N}$ the set of positive integers.

### 3. On Hausdorff dimension in $\mathbb{R}^n$

For $x \in (0, \infty)$, the $x$-dimensional Hausdorff outer measure of a set $A \subset \mathbb{R}^n$ is defined as

$$H^x(A) = \lim_{r \to 0} (\inf \sum d(A_i)^x),$$

where the infimum is taken over all countable coverings of $A$ by sets $A_i$ with $d(A_i) < r$. The Hausdorff dimension of $A \subset \mathbb{R}^n$ is defined as

$$\dim_H A = \inf \{ x > 0 \mid H^x(A) = 0 \}.$$

Then $0 \leq \dim_H A \leq n$. Note that if $A$ is the union of the sets $A_i$, $i = 1, 2, \ldots$, then $\dim_H A = \sup_i \dim_H A_i$.

To derive an upper bound for the Hausdorff dimension of a set in $\mathbb{R}^n$ we consider the following density quantity. For any $A \subset \mathbb{R}^n$, $A \neq \emptyset$, and $x \in A$ define
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\[
\sigma(x, A) = \lim \inf_{r \to 0} \left( \sup_{0 \leq |x-y| < r} \frac{1}{r} d(y, A) \right), \quad \text{and} \\
\sigma(A) = \inf_{x \in A} \sigma(x, A).
\]

Then always \( 0 \leq \sigma(A) \leq 1 \). The definition of \( \sigma(A) \) is motivated by the following result.

3.2. Theorem. If \( A \subset \mathbb{R}^n \) with \( \sigma(A) > 0 \), then \( \dim_H A \leq c < n \), where the constant \( c \) depends only on \( n \) and \( \sigma(A) \).

To prove this theorem we need a lemma essentially due to Gehring and Väisälä [3, Theorem 18]. We introduce a notation. If \( Q \subset \mathbb{R}^n \) is a closed cube of side \( s > 0 \) and \( p \geq 2 \) is an integer, we let \( \mathcal{D}(Q, p) \) denote the collection of the cubes obtained by subdividing \( Q \) into \( p^n \) closed congruent cubes of side \( s/p \).

3.3. Lemma. Suppose that \( Q_0 \) is a closed cube in \( \mathbb{R}^n \), that \( A \) is a compact subset of \( Q_0 \) and \( p, q \) and \( i_0 \) are integers such that \( p \geq 2 \), \( 1 \leq q \leq p^n-1 \) and \( i_0 \geq 0 \). If for every integer \( i > i_0 \) and every \( Q \in \mathcal{D}(Q_0, p^i) \) the set \( Q \cap A \) can be covered with \( q \) cubes of \( \mathcal{D}(Q, p) \), then

\[
\dim_H A \leq \frac{\log q}{\log p} < n.
\]

Proof. Let \( \log q/\log p < \alpha < n \). We must prove \( H^\alpha(A) = 0 \). It is sufficient to show \( H^\alpha(A \cap Q') = 0 \), where \( Q' \) is any cube of \( \mathcal{D}(Q_0, p^i) \). By the assumptions of the lemma we can cover \( A \cap Q' \) by \( q \) cubes of \( \mathcal{D}(Q', p) \), say \( Q_1, \ldots, Q_q \). Similarly every set \( A \cap Q_{i_0}, 1 \leq i \leq q \), we can covered by \( q \) cubes of \( \mathcal{D}(Q_{i_0}, p) \), and so we get a cover of \( A \cap Q' \) by \( q^2 \) cubes of \( \mathcal{D}(Q', p^2) \). Continuing in this way we get after \( j \) steps a cover \( \mathcal{C}_j \) of \( A \cap Q' \) by \( q^j \) cubes of \( \mathcal{D}(Q', p^j) \). Then \( d(Q) = d(Q')/p^j \) for every \( Q \in \mathcal{C}_j \). Hence

\[
\sum_{Q \in \mathcal{C}_j} d(Q)^\alpha = q^j \left( \frac{d(Q')}{p^j} \right)^\alpha = \left( \frac{q}{p^\alpha} \right)^j d(Q')^\alpha,
\]

where \( (q/p^\alpha)^j \to 0 \) as \( j \to \infty \), since \( q/p^\alpha < 1 \) by the choice of \( \alpha \). This implies \( H^\alpha(A \cap Q') = 0 \) by the definition of \( H^\alpha \). The lemma is proved.

3.4. Remark. The upper bound in the above lemma is attained by a set \( A \) defined as follows. Let \( Q_0, p \) and \( q \) be as in the lemma. Put

\[
\mathcal{A}_0 = \{ Q_0 \}, \quad \mathcal{A}_i = \bigcup_{Q \in \mathcal{A}_{i-1}} \mathcal{D}(Q), \quad i = 1, 2, \ldots,
\]

where \( \mathcal{D}(Q) \) is a collection of \( q \) cubes of \( \mathcal{D}(Q, p) \) for every \( Q \in \mathcal{A}_{i-1} \). Define
\[ A_i = \bigcup_{Q \in \mathcal{A}_i} Q, \quad i = 0, 1, 2, \ldots, \quad \text{and} \quad A = \bigcap_{i=0}^{\infty} A_i. \]

Then \( A \) satisfies the assumptions of Lemma 3.3 with \( i_0 = 0 \). Hence \( \dim_H A \leq \log q / \log p \). It is not difficult to see, for instance by [1, Corollary 2, p. 684], that \( \dim_H A \geq \log q / \log p \).

**Proof of Theorem 3.2.** We may assume \( A \subseteq Q_0 \), where \( Q_0 \) is a closed cube of side 1. For every \( j \in \mathbb{N} \) let \( A_j \) be the set of all \( x \in A \) with
\[
\inf_{0 < r \leq 1/j} \left( \sup_{0 \leq \|x-y\| < r} \frac{1}{r} d(y, A) \right) < \frac{1}{2} \sigma(A) > 0.
\]
Then \( A \) is the union of the sets \( A_j, j = 1, 2, \ldots \). Let \( p \) be the smallest odd integer greater than 1 and not less than \( 13 n^{1/2} / \sigma(A) \). Then \( p \) depends only on \( n \) and \( \sigma(A) \).

Fix \( j \in \mathbb{N} \). Let \( i_j \in \mathbb{N} \) such that \( p^{-i_j} < 1/j \). To apply Lemma 3.3 to \( \text{cl} A_{i_j} \) we show that the assumptions of the lemma are satisfied with \( p \) and \( i_0 = i_j \) as above and \( q = p^{n-1} \). Choose any \( i \geq i_j \) and \( Q \in \mathcal{D}(Q_0, p^n) \). Then \( Q \) is a cube of side \( t = p^{-i} < 1/j \). Let \( Q' \) be the cube in \( \mathcal{D}(Q, p) \) which contains the center \( x_0 \) of \( Q \). If \( \text{cl} A_j \cap \text{int} Q' = \emptyset \), \( \text{cl} A_j \cap Q \) can be covered by \( p^n - 1 \) cubes of \( \mathcal{D}(Q, p) \). Otherwise let \( x \in A_j \cap Q' \). Then
\[
d(x, \partial Q) > \frac{p-1}{2p} t \geq \frac{t}{3}.
\]
So \( B^n(x, t/6) \subset Q \), and because \( t/6 < 1/j \), then by the definition of the set \( A_j \) there exists \( y \in B^n(x, t/6) \) such that \( B^n(y, r) \subset R^n \setminus \text{cl} A \subset R^n \setminus \text{cl} A_{i_j} \), where \( r = \sigma(A) t/12 \). Then \( B^n(y, r) \subset Q \), and because \( p > 12 n^{1/2} / \sigma(A) \), we have \( r > n^{1/2} t / p \). Therefore at least one of the cubes of \( \mathcal{D}(Q, p) \) lies in \( B^n(y, r) \subset R^n \setminus \text{cl} A_{i_j} \). Hence \( \text{cl} A_j \cap Q \) can be covered by \( p^n - 1 \) cubes of \( \mathcal{D}(Q, p) \) in this case, too. Lemma 3.3 implies
\[
\dim_H A_j \leq \dim_H \text{cl} A_j \leq \frac{\log (p^n - 1)}{\log p} = c < n,
\]
where \( c \) depends only on \( n \) and \( \sigma(A) \).

Since \( A \) is the union of the sets \( A_j, j = 1, 2, \ldots \), (3.5) yields \( \dim_H A \leq c \), and the proof is completed.

3.6. **Remark.** The converse of Theorem 3.2. is not true. In fact, if \( I = \{ t e_i \mid -1 \leq t \leq 1 \} \) and for every \( i \in \mathbb{N} \)
\[
B_i = \left\{ \frac{1}{i} p_1 e_1 + \frac{1}{i^2} \sum_{k=2}^{n} p_k e_k \mid p_k \in Z, \quad -i \leq p_k \leq i, \quad k = 1, 2, \ldots, n \right\},
\]
then \( A = I \cup \bigcup_{i=1}^{\infty} B_i \) is a compact set with \( \sigma(A) = 0 \) and \( \dim_H A = 1 \).
3.7 Remark. Theorem 3.2 fails to hold if in the definition (3.1) of $\sigma(x, A)$ we replace $\lim \inf$ by $\lim \sup$. To show this define for every $A \subset \mathbb{R}^n$, $A \neq \emptyset$,

$$
\eta(x, A) = \lim_{r \to 0} \sup \left( \frac{1}{r} \sup_{0 \leq |x-y| < r} d(y, A) \right)
$$

$$
= \lim_{y \to x} \sup \frac{d(y, A)}{|x-y|}, \quad x \in A,
$$

and

$$
\eta(A) = \inf_{x \in A} \eta(x, A).
$$

Then $0 \leq \eta(A) \leq 1$. We construct a set $A \subset \mathbb{R}^n$ with $\eta(A) = 1$ and $\dim_H A = n$. In fact, $A$ will be locally so 'thin' that for every $x \in A$ and any $\varepsilon \in (0, 1)$ there exist arbitrarily small balls $B^n(x, r)$ such that

$$
(3.8) \quad A \cap B^n(x, r) \subset B^n(x, \varepsilon r).
$$

To define $A$ we need the following notation. If $Q$ is a closed cube in $\mathbb{R}^n$ and $i \geq 0$ is an integer, then $Q^{(i)}$ denotes the cube in $\mathcal{D}(Q, 3^i)$ which contains the center of $Q$. Now, let $Q_0$ be a closed cube in $\mathbb{R}^n$. Put $\mathcal{A}_0 = \{Q_0\}$ and

$$
\mathcal{A}_i = \bigcup_{Q \in \mathcal{A}_{i-1}} \{Q^{(i)} \mid Q \in \mathcal{D}(Q', 3^i)\}, \quad i = 1, 2, \ldots.
$$

Then define

$$
A_i = \bigcup_{Q \in \mathcal{A}_i} Q, \quad i = 1, 2, \ldots, \quad \text{and} \quad A = \bigcap_{i=1}^{\infty} A_i.
$$

Clearly $A$ has the property (3.8). On the other hand, every $\mathcal{A}_i$ consists of $N_i = 3^{nM_i}$ congruent cubes of diameter $\delta_i = 3^{-(M_i + m_i)} d(Q_0)$, where

$$
M_i = \sum_{j=0}^{i} j^2 \quad \text{and} \quad m_i = \sum_{j=0}^{i} j, \quad i = 1, 2, \ldots.
$$

Then

$$
\sum_{i=1}^{\infty} \left( \frac{\delta_i}{\delta_{i-1}} \right) \left( N_i \delta_i^2 \right)^{-1} = d(Q_0)^{-\alpha} \sum_{i=1}^{\infty} 3^{-[(n-\alpha)M_i - zm_i - in(i+1)]} < \infty
$$

for every $\alpha \in (0, n)$, which implies $\dim_H A = n$ by [2, Theorem 5, p. 55].
4. The Hausdorff dimension of $B_f$

4.1. Lemma. [5, 3.2]. Suppose that $f_i : G \to R^n$, $i \in N$, are open and discrete mappings, $f : G \to R^n$ is open and discrete or a constant mapping and $f_i \to f$ uniformly in compact subsets of $G$. If $x_i \to x \in G$ with $x_i \in B_{f_i}$, then $x \in B_f$.

4.2. Lemma. If $f : G \to R^n$ is $K$-quasiregular, $B_f \neq \emptyset$ and \(\sup \{ i(x, f) \mid x \in B_f \} \leq j \), then $\sigma(B_f) \geq s > 0$, where the constant $s$ depends only on $n$, $K$, and $j$.

Proof. Assume that the lemma is false for some $K \in [1, \infty)$ and $j \in (2, \infty)$. Then there exists a sequence of $K$-quasiregular mappings $f_i : G_i \to R^n$ with

\[
(4.3) \quad x_i \in B_{f_i}, \quad i(x_i, f_i) \leq j \quad \text{for every} \quad i \in N, \quad \text{and}
\]

\[
(4.4) \quad \lim_{i \to \infty} \sigma(x_i, B_{f_i}) = 0.
\]

By [6, 4.5] there exists $C > 0$ such that the linear dilatation $H(x_i, f_i) < C$ for every $i \in N$. Set $x_i = \sigma(x_i, B_{f_i}) + 1/3i$, $i \in N$. Then $\lim x_i = 0$ and we may assume $0 < x_i < 1/2$, $i \in N$. Furthermore, using similarity mappings we may also assume that for every $i \in N$

\[
(4.5) \quad x_i = f(x_i) = 0 \quad \text{and} \quad B^n \subset G_i,
\]

\[
(4.6) \quad N(f_i, B^n) = i(0, f_i) \leq j,
\]

\[
(4.7) \quad L(0, f_i, 1) = 1 \quad \text{and} \quad \frac{L(0, f_i, 1)}{L(0, f_i, 1)} < C,
\]

\[
(4.8) \quad \sup_{0 \leq |z| < 1} d(y, B_{f_i}) < x_i < 1/2.
\]

In particular, (4.8) implies

\[
(4.9) \quad B^n(y, x_i) \cap B_{f_i} \neq \emptyset \text{ for every } y \in B^n.
\]

By [7, 3.17] the restrictions $f_i|B^n$, $i \in N$, form a normal family of $K$-quasiregular mappings and by [9, p. 664] we may assume that $\{ f_i \}$ converges uniformly in compact subsets of $B^n$ to a $K$-quasiregular mapping $g : B^n \to R^n$.

We will show that $g : B^n \to R^n$ is not constant. It suffices to show that $\inf \{ d(f_i \circ B^n(1/2)) \mid i \in N \} > 0$. Fix $i \in N$. Put $l_i = L(0, f_i, 1)$ and $t_i = L(0, f_i, 1/2)$. Then $d(f_i \circ B^n(1/2)) \geq t_i$. We may assume $t_i < l_i$, since otherwise (4.7) implies $t_i \geq l_i \geq 1/C > 0$. Let $I'$ be the path family joining $S^{n-1}(1/2)$ to $S^{n-1}$ in $B^n$, and let $I'_i$ be the path
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family joining $S^{n-1}(l_i) \to S^{n-1}(l_i)$ in $B^n(l_i)$. Then $M(I'_i) \geq M(f, \Gamma)$ and by the outer dilatation inequality [6, 3.2] and by (4.6)

$$\omega_{n-1} \left( \log 2 \right)^{1-n} = M(\Gamma) \leq K N(f, B^n) M(f, \Gamma) \leq j K M(I'_i) = j K \omega_{n-1} \left( \log (l_i/l_i) \right)^{1-n}.$$ 

By (4.7) $l_i > 1/C$, and we get

$$d(f, cl B^n(1/2)) \geq t_i \geq (2^{(jk)}(n-1) C)^{-1} > 0.$$ 

This holds for all $i \in N$, and thus $g : B^n \to R^n$ is not a constant mapping.

To complete the proof choose any $z \in B^n(1/2)$. By (4.9) there exists $z_i \in B^n(z, z_i) \cap B_i \subset B^n$ for every $i \in N$. Then $\lim_{i \to \infty} z_i = z$ because $\lim_{i \to \infty} z_i = 0$. Hence $z \in B^n$ by Lemma 4.1. But this implies $B^n(1/2) \subset B^n$, which is impossible, since $g : B^n \to R^n$ is a nonconstant quasiregular mapping. The lemma is proved.

Theorem 3.2 and Lemma 4.2 together imply:

4.10. Theorem. If $f : G \to R^n$ is $K$-quasiregular and $i(f) = \sup \{ i(x, f) \mid x \in B_j \} < \infty$, then $\dim H B_j \leq c < n$, where the constant $c$ depends only on $n$, $K$ and $i(f)$.

4.11. Remark. It is conjectured that if $f : G \to R^n$ is $K$-quasiregular, there exists a constant $k \geq 2$ depending only on $n$ and $K$ such that the set $\{ x \in B_j \mid i(x, f) \geq k \}$ is discrete. If this conjecture holds, then Theorem 4.10 yields $\dim H B_j \leq d < n$, where $d$ depends only on $n$ and $K$.

5. The Hausdorff dimension of $fB_j$

5.1. Lemma. [4, 6.8]. Suppose that $f : G \to R^n$ is a non-constant quasiregular mapping and $F$ is a compact set in $B_j$ such that $H^s(fF) > 0$. Then

$$\alpha < n \left( K_j(f) \right) \left( \inf_{x \in F} i(x, f) \right)^{1/(n-1)}.$$ 

5.2. Lemma. Let $f : G \to R^n$ be $K$-quasiregular, $x \in G$ and $i(x, f) \leq m$. Then there exists constants $c, c^* \in (0, 1)$ depending only on $n$, $K$ and $m$, and $r_x > 0$ so that if $r \in (0, r_x]$ and $l_x^w = l^w(x, f, r)$, then

(i) $U(x, f, r)$ is a normal neighborhood of $x$ and

(ii) $U(x, f, c r) \subset B^n(x, c^* l_x^w)$.

Proof. By [6, 4.5] the linear dilatation $H(x, f) \in (0, \infty)$ has an upper bound $H < \infty$ in terms of $n$, $K$ and $m$. Put $c = (1/2) (H + 1)^{-1}$. By [6, 2.9] we can choose $r_x > 0$ such that if $0 < r \leq r_x$, then $U(x, f, r)$ is a normal neighborhood of $x$ and
\[ l(x, f, l^*_x) \geq \frac{L(x, f, l^*_x)}{H + 1} = 2c r, \]

where \( l^*_x = l^*(x, f, r) \). Let \( r \in (0, r_1) \) and \( l^*_x \) be as above. Let \( t \in (0, 1) \) such that \( l^*(x, f, c r) = t \). Let \( \Gamma \) be the path family joining \( cl U(x, f, c r) \) to \( S^{n-1}(x, l^*_x) \) in \( B^n(x, l^*_x) \). Then by the outer dilatation inequality [6, 3.2] and (5.3)

\[ M(\Gamma) \leq N(f, B^n(x, l^*_x)) K_0(f) M(f \Gamma) \leq m K \omega_{n-1} (\log 2)^{1-n}. \]

Because \( cl U(x, f, c r) \) is connected and \( d(S^{n-1}(x, l^*_x), cl U(x, f, c r)) = (1-t) l^*_x \), then by [10, 11.9]

\[ M(\Gamma) \geq \kappa_n \left( \frac{1-t}{t} \right) > 0, \]

where \( \kappa_n : (0, \infty) \to (0, \infty) \) is a decreasing function and \( \kappa_n(r) \to \infty \) as \( r \to 0 \). By (5.4) and (5.5) \( t \leq c^* \in (0, 1) \), where \( c^* \) depends only on \( n \), \( K \) and \( m \). The lemma is proved.

5.6. Lemma. Suppose \( f : G \to R^n \) is \( K \)-quasiregular, \( x \in B_I \) and \( r > 0 \) such that \( U = U(x, f, r) \) is a normal neighborhood of \( x \). Then \( \sigma(f(U \cap B_I)) \geq s' > 0 \), where \( s' \) is a constant depending only on \( n \), \( K \) and \( i(x, f) \).

Proof. Assume that the lemma is false. Then for some \( K \geq 1 \) and \( m \geq 2 \) there exists a sequence of \( K \)-quasiregular mappings \( h_i : G_i \to R^n \) with \( z_i \in G_i \) and \( \delta_i > 0 \) such that

(i) \( i(z_i, h_i) = m \)

(ii) \( U_i = U(z_i, h_i, \delta_i) \) is a normal neighborhood of \( z_i \),

(iii) \( \text{for every } i \in N \text{ there is } y_i \in h_i U_i \text{ so that } \lim_{t \to \infty} \sigma(y_i, h_i(U_i \cap B h_i)) = 0 \).

Put \( f_i = h_i | U_i, i \in N \). We may assume

\[ \sigma(y_i, f_i B_{f_i}) < \frac{1}{i} \text{ for every } i \in N. \]

Because \( N(f_i, U_i) = i(z_i, f_i) = m \), then

\[ m \geq p = \limsup_{i \to \infty} \text{card } (f_i^{-1}(y_i)) \geq 1. \]

By passing to a subsequence, if necessary, we may assume \( p = \text{card } (f_i^{-1}(y_i)) \) for every \( i \in N \). Furthermore, \( i(x, f_i) \leq m \) if \( x \in f_i^{-1}(y_i) \) and \( i \in N \).

Fix \( i \in N \) and consider the mapping \( f_i : U_i \to R^n \). Put \( r'_i = \min \{ r_x | x \in f_i^{-1}(y_i) \} > 0 \), where \( r_x > 0 \) is as in Lemma 5.2. By (5.7) we can choose \( r_i \in (0, r'_i) \) such that
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(5.8) \[ \sup \left\{ \frac{1}{r_i} d(y, f_i B_{r_i}) \mid 0 \leq |y - y_i| < r_i \right\} < \frac{1}{i}. \]

Then \( U(x, f_i, r_i) \) is a normal neighborhood for every \( x \in f_i^{-1}(y_i) \) and

(5.9) \[ f_i^{-1} B^n(y_i, r_i) = \bigcup_{x \in f_i^{-1}(y_i)} U(x, f_i, r_i). \]

Put \( l_x^* = l^*(x, f_i, r_i) \), \( x \in f_i^{-1}(y_i) \). By Lemma 5.2 and the choice of \( r_i \)

(5.10) \[ f_i^{-1} B^n(y_i, c r_i) \subset \bigcup_{x \in f_i^{-1}(y_i)} B^n(x, c^* l_x^*), \]

where constants \( c, c^* \in (0, 1) \) depend only on \( n, K \) and \( m \). Let

\[ T_i : B^n(y_i, r_i) \to B^n \]

be the mapping \( z \mapsto (1/r_i)(z - y_i) \). For every \( x \in f_i^{-1}(y_i) \) define the mapping \( g_i^* : B^n \to B^n \) by \( g_i^*(z) = T_i \circ f_i(x + l_x^* z) \), \( z \in B^n \). Say \( f_i^{-1}(y_i) = \{x_i, \ldots, x_p\} \). Set

\[ A_k = B^n(2k e_1, 1), \quad k = 1, 2, \ldots, p, \quad \text{and} \quad A = \bigcup_{k=1}^p A_k, \]

where \( e_1 \) is the first coordinate unit vector of \( R^n \). Finally, define \( g_i : A \to B^n \) by \( g_i(z) = g_i^*(z - 2k e_1) \) if \( z \in A_k, 1 \leq k \leq p \). Then \( g_i \) is \( K \)-quasiregular in each \( A_k \). Furthermore, by the definition of \( g_i \) and (5.10)

\[ B^n(c) \cap g_i B_{e_1} = T_i(B^n(y_i, c r_i) \cap f_i B_{r_i}), \]

(5.11) \[ g_i^{-1} B^n(c) \subset \bigcup_{k=1}^p \text{cl } B^n(2k e_1, c^*) \]

and, in particular, (5.8) implies for \( y \in B^n \) and \( i \in N \)

(5.12) \[ B^n(y, 1/i) \cap g_i B_{e_1} \neq \emptyset \quad \text{whenever} \quad B^n(y, 1/i) \subset B^n(c). \]

Now, consider the sequence \( g_i : A \to B^n, i \in N \). Since \( \{ g_i A_k \mid i \in N \} \)

is a normal family for every \( k = 1, \ldots, p \) by [7, 3.17], \( \{ g_i \mid i \in N \} \) is also a normal family, and there is a subsequence, denoted again by \( \{ g_i \} \),

which converges uniformly in compact subsets of \( A \) to a mapping \( g : A \to B^n \). By [9, p. 664] \( g \) is \( K \)-quasiregular in every \( A_k \).

Consider any \( w \in B^n(c) \). By (5.12) we can choose

\[ w_i \in B^n(w, 1/i) \cap g_i B_{e_1} \subset B^n(c) \]

for every \( i \in N, 1/i < c - |w| \), and for each such \( w_i \) we choose \( w_i^* \in B_{e_1} \cap g_i^{-1}(w_i) \). Then by (5.11) and by passing to a subsequence, if necessary, we may assume \( w_i^* \to w^* \in A_k \). Because every \( w_i^* \in B_{e_1} \) and \( g_i \to g \) uniformly in compact subsets of \( A_k \), \( w^* \in B_g \) by Lemma 4.1. Thus \( g(w^*) = \lim_{i \to \infty} g_i(w_i^*) = \lim_{i \to \infty} w_i = w \).
So \( w \in gB_\varepsilon \). It implies \( B^n(c) \subset gB_\varepsilon \). This is a contradiction, since \( m(gB_\varepsilon) = 0 \) by [6, 2.27]. The lemma is proved.

5.13. Theorem. If \( f: G \to \mathbb{R}^n \) is \( K \)-quasiregular, then
\[
\dim_H (fB_1) \leq c' < n,
\]
where the constant \( c' \) depends only on \( n \) and \( K \).

Proof. We may suppose that \( f: G \to \mathbb{R}^n \) is non-constant. Let \( m_K = Kn^{n-1} \) and define \( F = \{ x \in B_1 : \hat{i}(x,f) \geq m_K \} \). Then \( F \) is closed in \( G \) and by Lemma 5.1 it is easy to see that
\[
\dim_H F \leq n \left( \frac{K}{m_K} \right)^{1/(n-1)} = 1 < n.
\]
On the other hand, by Lemma 5.2 and Lemma 5.6 the set \( B_1 \setminus F = \{ x \in B_1 : \hat{i}(x,f) < m_K \} \) can be covered by countably many normal neighborhoods \( U \) such that \( \dim_H (f(U \cap B_1)) \leq c'' < n \), where the constant \( c'' \) depends only on \( n \), \( K \) and \( m_K \). This proves the theorem.

References

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