Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 1, 1975, 297–307

THE HAUSDORFF DIMENSION OF THE BRANCH SET OF A QUASIREGULAR MAPPING

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1. Introduction

Let G be a domain in the n-dimensional euclidean space R^n, $n \geq 2$. Consider a non-constant quasiregular mapping $f: G \to R^n$. Let B_f denote the branch set of f.

By [6] $m(B_f) = m(fB_f) = 0$, where m is the n-dimensional Lebesgue measure in R^n. Then also $H^n(B_f) = H^n(fB_f) = 0$, where H^α, $\alpha > 0$, is the α-dimensional Hausdorff outer measure in R^n. On the other hand, in [3] it is shown by an example that $\dim_H B_f$ and $\dim_H fB_f$, the Hausdorff dimensions of B_f and fB_f, can be arbitrarily close to n.

In this paper we prove the following results. Let $i(x, f)$ denote the local topological index of f at x. If f is as above, then

$$\dim_H fB_f \leq c' < n, \tag{1.1}$$

where the constant c' depends only on n and the maximal dilatation $K(f)$ of f. If, in addition, $i(f) = \sup \{ i(x,f) \mid x \in B_f \} < \infty$, then

$$\dim_H B_f \leq c < n, \tag{1.2}$$

where the constant c depends only on n, $K(f)$ and $i(f)$. It remains an open question whether c actually depends on $i(f)$. If it does not, then always $\dim_H B_f < n$, too.

We shall prove (1.1) and (1.2) using a similar method to Rešetnjak's in [9] and Martio's and Rickman's in [5].

For more information on $\dim_H B_f$ and $\dim_H fB_f$ see, for example, [5].

2. Notation

We use the same notation and terminology as in [6]. If $A \subset R^n$, we write $\operatorname{cl} A$, $\operatorname{int} A$ and ∂A for the closure, the interior and the boundary

doi:10.5186/aasfm.1975.0121

of A. If $x \in R^n$ and $A \subset R^n$, $A \neq \emptyset$, we denote dy $d(x, A)$ the distance from x to A and by $d(A)$ the diameter of A. If $x \in R^n$ and $r > 0$ we write $B^n(x, r)$ for the open ball $\{ y \in R^n \mid |x-y| < r \}$ and abbreviate $B^n(r) = B^n(0, r)$, $B^n = B^n(1)$. We also write $S^{n-1}(x, r) = \partial B^n(x, r)$, $S^{n-1}(r) = S^{n-1}(0, r)$ and $S^{n-1} = S^{n-1}(1)$.

We let the notation $f: G \to R^n$ include the assumption that $G \subset R^n$ is a domain and f is continuous. If $x \in G$ and $r > 0$, put

$$l(x, f, r) = \inf_{|x-y|=r} |f(x) - f(y)|, \qquad L(x, f, r) = \sup_{|x-y|=r} |f(x) - f(y)|$$

whenever $B^n(x, r) \subset G$. We let $U(x, f, r)$ denote the x-component of $f^{-1}B^n(f(x), r)$ and write

$$l^*(x, f, r) = \inf\{ |x-y| \mid y \in \partial U(x, f, r) \} \text{ and}$$

$$L^*(x, f, r) = \sup\{ |x-y| \mid y \in \partial U(x, f, r) \}$$

whenever $\partial U(x, f, r) \neq \emptyset$. If $A \subset R^n$ and $y \in R^n$, put $N(y, f, A) = \operatorname{card}(A \cap f^{-1}(y))$ and $N(f, A) = \sup\{ N(y, f, A) \mid y \in R^n \}$. If $f: G \to R^n$ is quasiregular and $x \in G$, then there exists $r_x > 0$ such that if $r \in (0, r_x)$, then $U(x, f, r)$ is a normal neighborhood of x and $N(f, U(x, f, r)) = i(x, f)$; see [6; 2.9, 2.12].

Let $e_1, \ldots, e_n$ denote the coordinate unit vectors of R^n, Z the set of integers and N the set of positive integers.

3. On Hausdorff dimension in R^n

For $\alpha \in (0, \infty)$, the α-dimensional Hausdorff outer measure of a set $A \subset R^n$ is defined as

$$H^\alpha(A) = \lim_{r \to 0} (\inf \sum d(A_i)^\alpha),$$

where the infimum is taken over all countable coverings of A by sets A_i with $d(A_i) < r$. The Hausdorff dimension of $A \subset R^n$ is defined as

$$\dim_H A = \inf\{ \alpha > 0 \mid H^\alpha(A) = 0 \}.$$

Then $0 \leq \dim_H A \leq n$. Note that if A is the union of the sets A_i, $i = 1, 2, \ldots$, then $\dim_H A = \sup_i \dim_H A_i$.

To derive an upper bound for the Hausdorff dimension of a set in R^n we consider the following density quantity. For any $A \subset R^n$, $A \neq \emptyset$, and $x \in A$ define

$$
\begin{aligned}
\sigma(x, A) &= \liminf_{r\to 0}\Big(\sup_{0\le|x-y|<r}\frac{1}{r}\,d(y, A)\Big), \quad \text{and} \\
\sigma(A) &= \inf_{x\in A}\sigma(x, A).
\end{aligned} \tag{3.1}
$$

Then always $0 \le \sigma(A) \le 1$. The definition of $\sigma(A)$ is motivated by the following result.

3.2. Theorem. *If $A \subset R^n$ with $\sigma(A) > 0$, then $\dim_H A \le c < n$, where the constant c depends only on n and $\sigma(A)$.*

To prove this theorem we need a lemma essentially due to Gehring and Väisälä [3, Theorem 18]. We introduce a notation. If $Q \subset R^n$ is a closed cube of side $s > 0$ and $p \ge 2$ is an integer, we let $\mathscr{D}(Q, p)$ denote the collection of the cubes obtained by subdividing Q into p^n closed congruent cubes of side s/p.

3.3. Lemma. *Suppose that Q_0 is a closed cube in R^n, that A is a compact subset of Q_0 and p, q and i_0 are integers such that $p \ge 2$, $1 \le q \le p^n - 1$ and $i_0 \ge 0$. If for every integer $i > i_0$ and every $Q \in \mathscr{D}(Q_0, p^i)$ the set $Q \cap A$ can be covered with q cubes of $\mathscr{D}(Q, p)$, then*

$$\dim_H A \le \frac{\log q}{\log p} < n.$$

Proof. Let $\log q/\log p < \alpha < n$. We must prove $H^\alpha(A) = 0$. It is sufficient to show $H^\alpha(A \cap Q') = 0$, where Q' is any cube of $\mathscr{D}(Q_0, p^{i_0})$. By the assumptions of the lemma we can cover $A \cap Q'$ by q cubes of $\mathscr{D}(Q', p)$, say $Q_1, \dots, Q_q$. Similarly every set $A \cap Q_i$, $1 \le i \le q$, we can covered by q cubes of $\mathscr{D}(Q_i, p)$, and so we get a cover of $A \cap Q'$ by q^2 cubes of $\mathscr{D}(Q', p^2)$. Continuing in this way we get after j steps a cover $\mathscr{C}_j$ of $A \cap Q'$ by q^j cubes of $\mathscr{D}(Q', p^j)$. Then $d(Q) = d(Q')/p^j$ for every $Q \in \mathscr{C}_j$. Hence

$$\sum_{Q\in\mathscr{C}_j} d(Q)^\alpha = q^j\left(\frac{d(Q')}{p^j}\right)^\alpha = \left(\frac{q}{p^\alpha}\right)^j d(Q')^\alpha,$$

where $(q\,p^{-\alpha})^j \to 0$ as $j \to \infty$, since $q\,p^{-\alpha} < 1$ by the choice of α. This implies $H^\alpha(A \cap Q') = 0$ by the definition of H^α. The lemma is proved.

3.4. *Remark.* The upper bound in the above lemma is attained by a set A defined as follows. Let Q_0, p and q be as in the lemma. Put

$$\mathscr{A}_0 = \{Q_0\}, \qquad \mathscr{A}_i = \bigcup_{Q\in\mathscr{A}_{i-1}} \mathscr{B}(Q), \qquad i = 1, 2, \dots,$$

where $\mathscr{B}(Q)$ is a collection of q cubes of $\mathscr{D}(Q, p)$ for every $Q \in \mathscr{A}_{i-1}$. Define

$$A_i = \bigcup_{Q \in \mathscr{A}_i} Q\,, \qquad i = 0, 1, 2, \ldots\,, \qquad \text{and} \qquad A = \bigcap_{i=0}^{\infty} A_i\,.$$

Then A satisfies the assumptions of Lemma 3.3 with $i_0 = 0$. Hence $\dim_H A \leq \log q/\log p$. It is not difficult to see, for instance by [1, Corollary 2, p. 684], that $\dim_H A \geq \log q/\log p$.

Proof of Theorem 3.2. We may assume $A \subset Q_0$, where Q_0 is a closed cube of side 1. For every $j \in N$ let A_j be the set of all $x \in A$ with

$$\inf_{0 < r \leq 1/j} \left(\sup_{0 \leq |x-y| < r} \frac{1}{r} d(y\,, A) \right) < \frac{1}{2}\sigma(A) > 0\,.$$

Then A is the union of the sets A_j, $j = 1, 2, \ldots$. Let p be the smallest odd integer greater than 1 and not less than $13\, n^{1/2}/\sigma(A)$. Then p depends only on n and $\sigma(A)$.

Fix $j \in N$. Let $i_j \in N$ such that $p^{-i_j} < 1/j$. To apply Lemma 3.3 to $\operatorname{cl} A_j$ we show that the assumptions of the lemma are satisfied with p and $i_0 = i_j$ as above and $q = p^n - 1$. Choose any $i \geq i_j$ and $Q \in \mathscr{D}(Q_0\,, p^i)$. Then Q is a cube of side $t = p^{-i} < 1/j$. Let Q' be the cube in $\mathscr{D}(Q\,, p)$ which contains the center x_0 of Q. If $\operatorname{cl} A_j \cap \operatorname{int} Q' = \emptyset$, $\operatorname{cl} A_j \cap Q$ can be covered by $p^n - 1$ cubes of $\mathscr{D}(Q\,, p)$. Otherwise let $x \in A_j \cap Q'$. Then

$$d(x\,, \partial Q) > \frac{p-1}{2\,p}\, t \geq \frac{t}{3}\,.$$

So $B^n(x\,, t/6) \subset Q$, and because $t/6 < 1/j$, then by the definition of the set A_j there exists $y \in B^n(x\,, t/6)$ such that $B^n(y\,, r) \subset R^n \setminus \operatorname{cl} A \subset R^n \setminus \operatorname{cl} A_j$, where $r = \sigma(A)\, t/12$. Then $B^n(y\,, r) \subset Q$, and because $p > 12\, n^{1/2}/\sigma(A)$, we have $r > n^{1/2}\, t/p$. Therefore at least one of the cubes of $\mathscr{D}(Q\,, p)$ lies in $B^n(y\,, r) \subset R^n \setminus \operatorname{cl} A_j$. Hence $\operatorname{cl} A_j \cap Q$ can be covered by $p^n - 1$ cubes of $\mathscr{D}(Q\,, p)$ in this case, too. Lemma 3.3 implies

$$\dim_H A_j \leq \dim_H \operatorname{cl} A_j \leq \frac{\log\,(p^n - 1)}{\log p} = c < n\,, \tag{3.5}$$

where c depends only on n and $\sigma(A)$.

Since A is the union of the sets A_j, $j = 1, 2, \ldots$, (3.5) yields $\dim_H A \leq c$, and the proof is completed.

3.6. *Remark*. The converse of Theorem 3.2. is not true. In fact, if $I = \{\, t\, e_1 \mid -1 \leq t \leq 1 \,\}$ and for every $i \in N$

$$B_i = \left\{ \frac{1}{i} p_1\, e_1 + \frac{1}{i^2} \sum_{k=2}^{n} p_k\, e_k \mid p_k \in Z\,, \quad -i \leq p_k \leq i\,, \quad k = 1, 2, \ldots, n \right\},$$

then $A = I \cup \bigcup_{i=1}^{\infty} B_i$ is a compact set with $\sigma(A) = 0$ and $\dim_H A = 1$.

3.7 *Remark.* Theorem 3.2 fails to hold if in the definition (3.1) of $\sigma(x, A)$ we replace $\liminf$ by $\limsup$. To show this define for every $A \subset R^n$, $A \neq \emptyset$,

$$\eta(x, A) = \limsup_{r \to 0} \left(\sup_{0 \leq |x-y| < r} \frac{1}{r} d(y, A) \right) = \limsup_{y \to x} \frac{d(y, A)}{|x-y|}, \qquad x \in A,$$

and

$$\eta(A) = \inf_{x \in A} \eta(x, A).$$

Then $0 \leq \eta(A) \leq 1$. We construct a set $A \subset R^n$ with $\eta(A) = 1$ and $\dim_H A = n$. In fact, A will be locally so 'thin' that for every $x \in A$ and any $\varepsilon \in (0, 1)$ there exist arbitrarily small balls $B^n(x, r)$ such that

$$A \cap B^n(x, r) \subset B^n(x, \varepsilon r). \tag{3.8}$$

To define A we need the following notation. If Q is a closed cube in R^n and $i \geq 0$ is an integer, then $Q^{(i)}$ denotes the cube in $\mathscr{D}(Q, 3^i)$ which contains the center of Q. Now, let Q_0 be a closed cube in R^n. Put $\mathscr{A}_0 = \{ Q_0 \}$ and

$$\mathscr{A}_i = \bigcup_{Q' \in \mathscr{A}_{i-1}} \{ Q^{(i)} \mid Q \in \mathscr{D}(Q', 3^{i^2}) \}, \qquad i = 1, 2, \ldots .$$

Then define

$$A_i = \bigcup_{Q \in \mathscr{A}_i} Q, \qquad i = 1, 2, \ldots, \qquad \text{and} \qquad A = \bigcap_{i=1}^{\infty} A_i .$$

Clearly A has the property (3.8). On the other hand, every $\mathscr{A}_i$ consists of $N_i = 3^{nM_i}$ congruent cubes of diameter $\delta_i = 3^{-(M_i + m_i)} d(Q_0)$, where

$$M_i = \sum_{j=0}^{i} j^2 \quad \text{and} \quad m_i = \sum_{j=0}^{i} j, \qquad i = 1, 2, \ldots .$$

Then

$$\sum_{i=1}^{\infty} \left(\frac{\delta_{i-1}}{\delta_i} \right)^n (N_i \, \delta_i^{\alpha})^{-1} = d(Q_0)^{-\alpha} \sum_{i=1}^{\infty} 3^{-[(n-\alpha)M_i - \alpha m_i - n(i^2+i)]} < \infty$$

for every $\alpha \in (0, n)$, which implies $\dim_H A = n$ by [2, Theorem 5, p. 55].

4. The Hausdorff dimension of B_f

4.1. Lemma. [5, 3.2]. *Suppose that* $f_i : G \to R^n$, $i \in N$, *are open and discrete mappings,* $f : G \to R^n$ *is open and discrete or a constant mapping and* $f_i \to f$ *uniformly in compact subsets of* G. *If* $x_i \to x \in G$ *with* $x_i \in B_{f_i}$, *then* $x \in B_f$.

4.2. Lemma. *If* $f : G \to R^n$ *is* K*-quasiregular,* $B_f \neq \emptyset$ *and* $\sup \{ i(x, f) \mid x \in B_f \} \leq j$, *then* $\sigma(B_f) \geq s > 0$, *where the constant* s *depends only on* n, K *and* j.

Proof. Assume that the lemma is false for some $K \in [1, \infty)$ and $j \in [2, \infty)$. Then there exists a sequence of K-quasiregular mappings $f_i : G_i \to R^n$ with

$$x_i \in B_{f_i}, \; i(x_i, f_i) \leq j \text{ for every } i \in N, \text{ and} \tag{4.3}$$

$$\lim_{i \to \infty} \sigma(x_i, B_{f_i}) = 0 . \tag{4.4}$$

By [6, 4.5] there exists $C > 0$ such that the linear dilatation $H(x_i, f_i) < C$ for every $i \in N$. Set $\alpha_i = \sigma(x_i, B_{f_i}) + 1/3i$, $i \in N$. Then $\lim \alpha_i = 0$ and we may assume $0 < \alpha_i < 1/2$, $i \in N$. Furthermore, using similarity mappings we may also assume that for every $i \in N$

$$x_i = f(x_i) = 0 \text{ and } B^n \subset G_i , \tag{4.5}$$

$$N(f_i, B^n) = i(0, f_i) \leq j , \tag{4.6}$$

$$L(0, f_i, 1) = 1 \text{ and } \frac{L(0, f_i, 1)}{l(0, f_i, 1)} < C , \tag{4.7}$$

$$\sup_{0 \leq |y| < 1} d(y, B_{f_i}) < \alpha_i < 1/2 . \tag{4.8}$$

In particular, (4.8) implies

$$B^n(y, \alpha_i) \cap B_{f_i} \neq \emptyset \text{ for every } y \in B^n . \tag{4.9}$$

By [7, 3.17] the restrictions $f_i | B^n$, $i \in N$, form a normal family of K-quasiregular mappings and by [9, p. 664] we may assume that $\{ f_i \}$ converges uniformly in compact subsets of B^n to a K-quasiregular mapping $g : B^n \to R^n$.

We will show that $g : B^n \to R^n$ is not constant. It suffices to show that $\inf \{ d(f_i \operatorname{cl} B^n(1/2)) \mid i \in N \} > 0$. Fix $i \in N$. Put $l_i = l(0, f_i, 1)$ and $t_i = L(0, f_i, 1/2)$. Then $d(f_i \operatorname{cl} B^n(1/2)) \geq t_i$. We may assume $t_i < l_i$, since otherwise (4.7) implies $t_i \geq l_i \geq 1/C > 0$. Let Γ be the path family joining $S^{n-1}(1/2)$ to S^{n-1} in B^n, and let Γ_i' be the path family joining $S^{n-1}(t_i)$ to $S^{n-1}(l_i)$ in $B^n(l_i)$. Then $M(\Gamma'_i) \geq M(f_i\Gamma)$ and by the outer dilatation inequality [6, 3.2] and by (4.6)

$$\begin{aligned}\omega_{n-1}(\log 2)^{1-n} &= M(\Gamma) \leq K\,N(f_i\,, B^n)\,M(f_i\Gamma)\\ &\leq j\,K\,M(\Gamma'_i) = j\,K\,\omega_{n-1}(\log(l_i/t_i))^{1-n}\,.\end{aligned}$$

By (4.7) $l_i > 1/C$, and we get

$$d(f_i\,\mathrm{cl}\,B^n(1/2)) \geq t_i \geq (2^{(jK)^{1/(n-1)}}C)^{-1} > 0\,.$$

This holds for all $i \in N$, and thus $g : B^n \to R^n$ is not a constant mapping.

To complete the proof choose any $z \in B^n(1/2)$. By (4.9) there exists $z_i \in B^n(z\,, \alpha_i) \cap B_{f_i} \subset B^n$ for every $i \in N$. Then $\lim_{i\to\infty} z_i = z$ because $\lim_{i\to\infty} \alpha_i = 0$. Hence $z \in B_g$ by Lemma 4.1. But this implies $B^n(1/2) \subset B_g$, which is impossible, since $g : B^n \to R^n$ is a nonconstant quasiregular mapping. The lemma is proved.

Theorem 3.2 and Lemma 4.2 together imply:

4.10. Theorem. *If* $f : G \to R^n$ *is* K*-quasiregular and* $i(f) = \sup\{\, i(x\,, f) \mid x \in B_f \,\} < \infty$, *then* $\dim_H B_f \leq c < n$, *where the constant* c *depends only on* n, K *and* $i(f)$.

4.11. *Remark.* It is conjectured that if $f : G \to R^n$ is K-quasiregular, there exists a constant $k \geq 2$ depending only on n and K such that the set $\{\, x \in B_f \mid i(x\,, f) \geq k \,\}$ is discrete. If this conjecture holds, then Theorem 4.10 yields $\dim_H B_f \leq d < n$, where d depends only on n and K.

5. The Hausdorff dimension of fB_f

5.1. Lemma. [4, 6.8]. *Suppose that* $f : G \to R^n$ *is a non-constant quasiregular mapping and* F *is a compact set in* B_f *such that* $H^\alpha(fF) > 0$. *Then*

$$\alpha < n\left(\frac{K_I(f)}{\inf\limits_{x\in F} i(x\,, f)}\right)^{1/(n-1)}.$$

5.2. Lemma. *Let* $f : G \to R^n$ *be* K*-quasiregular,* $x \in G$ *and* $i(x\,, f) \leq m$. *Then there exist constants* $c\,, c^* \in (0\,, 1)$ *depending only on* n, K *and* m, *and* $r_x > 0$ *so that if* $r \in (0\,, r_x]$ *and* $l^*_x = l^*(x\,, f\,, r)$, *then*

(i) $U(x\,, f\,, r)$ *is a normal neighborhood of* x *and*

(ii) $U(x\,, f\,, c\,r) \subset B^n(x\,, c^*\,l^*_x)$.

Proof. By [6, 4.5] the linear dilatation $H(x\,, f) \in (0\,, \infty)$ has an upper bound $H < \infty$ in terms of n, K and m. Put $c = (1/2)\,(H+1)^{-1}$. By [6, 2.9] we can choose $r_x > 0$ such that if $0 < r \leq r_x$, then $U(x\,, f\,, r)$ is a normal neighborhood of x and

$$(5.3) \qquad l(x\,,f\,,l_x^*) \;\geq\; \frac{L(x\,,f\,,l_x^*)}{H+1} \;=\; 2\,c\,r\,,$$

where $l_x^* = l^*(x\,,f\,,r)$. Let $r \in (0\,,r_x]$ and l_x^* be as above. Let $t \in (0\,,1)$ such that $L^*(x\,,f\,,c\,r) = t\,l_x^*$. Let Γ be the path family joining $\mathrm{cl}\,U(x\,,f\,,c\,r)$ to $S^{n-1}(x\,,l_x^*)$ in $B^n(x\,,l_x^*)$. Then by the outer dilatation inequality [6, 3.2] and (5.3)

$$(5.4) \quad M(\Gamma) \;\leq\; N(f\,,B^n(x\,,l_x^*))\,K_0(f)\,M(f\Gamma) \;\leq\; m\,K\,\omega_{n-1}\,(\log 2)^{1-n}\,.$$

Because $\mathrm{cl}\,U(x\,,f\,,c\,r)$ is connected and $d(S^{n-1}(x\,,l_x^*)\,,\,\mathrm{cl}\,U(x\,,f\,,c\,r)) = (1-t)\,l_x^*$, then by [10, 11.9]

$$(5.5) \qquad M(\Gamma) \;\geq\; \varkappa_n\left(\frac{1-t}{t}\right) \;>\; 0\,,$$

where $\varkappa_n\colon (0\,,\infty) \to (0\,,\infty)$ is a decreasing function and $\varkappa_n(r) \to \infty$ as $r \to 0$. By (5.4) and (5.5) $t \leq c^* \in (0\,,1)$, where c^* depends only on n , K and m . The lemma is proved.

5.6. Lemma. *Suppose* $f\colon G \to R^n$ *is K-quasiregular,* $x \in B_f$ *and* $r > 0$ *such that* $U = U(x\,,f\,,r)$ *is a normal neighborhood of* x *. Then* $\sigma(f(U \cap B_f)) \geq s' > 0$ *, where* s' *is a constant depending only on* n *,* K *and* $i(x\,,f)$ *.*

Proof. Assume that the lemma is false. Then for some $K \geq 1$ and $m \geq 2$ there exists a sequence of K-quasiregular mappings $h_i\colon G_i \to R^n$ with $z_i \in G_i$ and $\delta_i > 0$ such that

(i) $i(z_i\,,h_i) = m$

(ii) $U_i = U(z_i\,,h_i\,,\delta_i)$ is a normal neighborhood of z_i ,

(iii) for every $i \in N$ there is $y_i \in h_i U_i$ so that $\lim_{i\to\infty} \sigma(y_i\,,h_i(U_i \cap B_{h_i})) = 0$.

Put $f_i = h_i|U_i$, $i \in N$. We may assume

$$(5.7) \qquad \sigma(y_i\,,f_iB_{f_i}) < \frac{1}{i} \quad \text{for every } i \in N\,.$$

Because $N(f_i\,,U_i) = i(z_i\,,f_i) = m$, then

$$m \;\geq\; p \;=\; \limsup_{i\to\infty}\,\mathrm{card}\,(f_i^{-1}(y_i)) \;\geq\; 1\,.$$

By passing to a subsequence, if necessary, we may assume $p = \mathrm{card}\,(f_i^{-1}(y_i))$ for every $i \in N$. Furthermore, $i(x\,,f_i) \leq m$ if $x \in f_i^{-1}(y_i)$ and $i \in N$.

Fix $i \in N$ and consider the mapping $f_i\colon U_i \to R^n$. Put $r_i' = \min\{\,r_x \mid x \in f_i^{-1}(y_i)\,\} > 0$, where $r_x > 0$ is as in Lemma 5.2. By (5.7) we can choose $r_i \in (0\,,r_i')$ such that

$$\sup \left\{ \frac{1}{r_i} d(y, f_i B_{f_i}) \mid 0 \le |y - y_i| < r_i \right\} < \frac{1}{i} . \tag{5.8}$$

Then $U(x, f_i, r_i)$ is a normal neighborhood for every $x \in f_i^{-1}(y_i)$ and

$$f_i^{-1} B^n(y_i, r_i) = \bigcup_{x \in f_i^{-1}(y_i)} U(x, f_i, r_i) . \tag{5.9}$$

Put $l_x^* = l^*(x, f_i, r_i)$, $x \in f_i^{-1}(y_i)$. By Lemma 5.2 and the choice of r_i

$$f_i^{-1} B^n(y_i, c\, r_i) \subset \bigcup_{x \in f_i^{-1}(y_i)} B^n(x, c^* l_x^*) , \tag{5.10}$$

where constants $c, c^* \in (0, 1)$ depend only on n, K and m. Let $T_i : B^n(y_i, r_i) \to B^n$ be the mapping $z \mapsto (1/r_i)(z - y_i)$. For every $x \in f_i^{-1}(y_i)$ define the mapping $g_i^x : B^n \to B^n$ by $g_i^x(z) = T_i \circ f_i(x + l_x^* z)$, $z \in B^n$. Say $f_i^{-1}(y_i) = \{x_1, \ldots, x_p\}$. Set

$$A_k = B^n(2\, k\, e_1, 1) , \quad k = 1, 2, \ldots, p , \quad \text{and} \quad A = \bigcup_{k=1}^{p} A_k ,$$

where e_1 is the first coordinate unit vector of R^n. Finally, define $g_i : A \to B^n$ by $g_i(z) = g_i^{x_k}(z - 2\, k\, e_1)$ if $z \in A_k$, $1 \le k \le p$. Then g_i is K-quasiregular in each A_k. Furthermore, by the definition of g_i and (5.10)

$$B^n(c) \cap g_i B_{g_i} = T_i(B^n(y_i, c\, r_i) \cap f_i B_{f_i}) ,$$

$$g_i^{-1} B^n(c) \subset \bigcup_{k=1}^{p} \operatorname{cl} B^n(2\, k\, e_1, c^*) \tag{5.11}$$

and, in particular, (5.8) implies for $y \in B^n$ and $i \in N$

$$B^n(y, 1/i) \cap g_i B_{g_i} \neq \emptyset \text{ whenever } B^n(y, 1/i) \subset B^n(c) . \tag{5.12}$$

Now, consider the sequence $g_i : A \to B^n$, $i \in N$. Since $\{ g_i | A_k \mid i \in N \}$ is a normal family for every $k = 1, \ldots, p$ by [7, 3.17], $\{ g_i \mid i \in N \}$ is also a normal family, and there is a subsequence, denoted again by $\{ g_i \}$, which converges uniformly in compact subsets of A to a mapping $g : A \to B^n$. By [9, p. 664] g is K-quasiregular in every A_k.

Consider any $w \in B^n(c)$. By (5.12) we can choose

$$w_i \in B^n(w, 1/i) \cap g_i B_{g_i} \subset B^n(c)$$

for every $i \in N$, $1/i < c - |w|$, and for each such w_i we choose $w_i^* \in B_{g_i} \cap g_i^{-1}(w_i)$. Then by (5.11) and by passing to a subsequence, if necessary, we may assume $w_i^* \to w^* \in A_k$. Because every $w_i^* \in B_{g_i}$ and $g_i \to g$ uniformly in compact subsets of A_k, $w^* \in B_g$ by Lemma 4.1. Thus $g(w^*) = \lim_{i \to \infty} g_i(w_i^*) = \lim_{i \to \infty} w_i = w$.

So $w \in gB_g$. It implies $B^n(c) \subset gB_g$. This is a contradiction, since $m(gB_g) = 0$ by [6, 2.27]. The lemma is proved.

5.13. Theorem. *If $f: G \to R^n$ is K-quasiregular, then*

$$\dim_H (fB_f) \leq c' < n ,$$

where the constant c' depends only on n and K.

Proof. We may suppose that $f: G \to R^n$ is non-constant. Let $m_K = K n^{n-1}$ and define $F = \{ x \in B_f \mid i(x, f) \geq m_K \}$. Then F is closed in G and by Lemma 5.1 it is easy to see that

$$\dim_H F \leq n \left(\frac{K}{m_K}\right)^{1/(n-1)} = 1 < n .$$

On the other hand, by Lemma 5.2 and Lemma 5.6 the set $B_f \setminus F = \{ x \in B_f \mid i(x, f) < m_K \}$ can be covered by countably many normal neighborhoods U such that $\dim_H (f(U \cap B_f)) \leq c'' < n$, where the constant c'' depends only on n, K and m_K. This proves the theorem.

References

[1] BEARDON, A. F.: On the Hausdorff dimension of general Cantor sets. - Proc. Cambridge Philos. Soc. 61, 1965, 679–694.

[2] EGGLESTON, H. G.: Sets of fractional dimensions which occur in some problems of number theory. - Proc. London Math. Soc. (2) 54, 1951, 42–93.

[3] GEHRING, F. W., and J. VÄISÄLÄ: Hausdorff dimension and quasiconformal mappings. - J. London Math. Soc. (2) 6, 1973, 504–512.

[4] MARTIO, O.: A capacity inequality for quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I 474, 1970, 1–17.

[5] MARTIO, O, and S. RICKMAN: Measure properties of the branch set and its image of quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I 541, 1973, 1–15.

[6] MARTIO, O., S. RICKMAN, and J. VÄISÄLÄ: Definitions for quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I 448, 1969, 1–40.

[7] —»— Distortion and singularities of quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I 465, 1970, 1–12.

[8] REŠETNJAK, JU. G.: Отображения с ограниченным искажением как экстремали интегралов типа дирихле. - Sibirsk. Mat. Ž. 9, 1968, 652–666. Translation: Mappings with bounded deformation as extremals of Dirichlet type integrals. - Siberian Math. J. 9, 1968, 487–498.

[9] —»— О множестве точек ветвления отображений с ограниченным искажением. - Sibirsk. Mat. Ž. 11, 1970, 1333–1339. Translation: On the set of of branch points of mappings with bounded distortion. - Siberian Math. J. 11, 1970, 982–986.

[10] VÄISÄLÄ, J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Mathematics 229, Springer-Verlag, Berlin–Heidelberg–New York, 1971.

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Received 23 May 1975