ASYMPTOTIC PATHS FOR SUBHARMONIC FUNCTIONS IN $\mathbb{R}^n$

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1. The purpose of this note is to prove the following theorem.

Theorem. Let $u(x)$ be subharmonic in $\mathbb{R}^n$ and assume $\sup u(x) = +\infty$. Then there is polygonal path $\gamma$ to $\infty$ so that

$$\lim_{\gamma} u(x) = \infty.$$

Theorem is a generalization of Iversen's theorem. It was inspired by a recent manuscript of B. Fuglede where, among other results, the theorem was proved for a continuous path. Fuglede used finely harmonic functions and probability and the above result was obtained in an effort to find a classical proof. See also the work of M. N. M. Talpur (W.K. Hayman: Einige Verallgemeinerungen des Iversenschen Satzes auf subharmonische Funktionen. – Jber. Deutsch. Math.-Verein. 71, 1969, 115–122).

2. We first assume $u(x)$ continuous. Let $O_n$ be the open set where $u(x) > n$. There are two cases.

   a) $O_n$ has only one component for every $n$. We then choose $x_n$ with $u(x_n) = n$ and connect $x_n$ to $x_{n+1}$ inside $O_{n-1}$ with a polygon. This gives the desired path.

   b) Some $O_n$ has two components (or more). Let $A$ and $B$ be two components. By the maximum principle both are unbounded. We say that $A$ has the Phragmén-Lindelöf property if every harmonic function in $A$ which is bounded and $\leq 0$ on $\partial A$ is $\leq 0$. The following criterion is easy to prove.

   Lemma. $A$ has the Phragmén-Lindelöf property iff the complement of $A^{-1} = \{ x \mid x \, |x|^2 \in A \}$ is thin at $x = 0$.

   Corollary. At least one of $A$ and $B$ has the Phragmén-Lindelöf property.

   This follows e.g. from the Wiener criterion since

$$ (R^n \setminus A^{-1}) \cup (R^n \setminus B^{-1}) = R^n \setminus \{0\}.$$

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To complete the proof choose \( A = A_n \) as above. Then \( u(x) \) has to be unbounded in \( A \). Choose \( x_{n+1} \in A_n \) and let \( A_{n+1} \) be the corresponding component of \( O_{n+1} \cdot A_{n+1} \) also has the Phragmén–Lindelöf property and we can choose \( x_{n+2} \in A_{n+1} \) etc. — The proof in this case is complete.

3. In the general case we have to find a method of constructing \( \gamma \) inside the set where the potential representing \( u(x) \) converges uniformly. We do this by approximating \( u(x) \) by smooth subharmonic functions which are negative on the set where \( u(x) \) misbehaves. The construction is quite explicit and depends on a dyadic subdivision which we are next going to describe.

We may assume that \( u(x) \geq 0 \). Let \( K_r \) be the symmetric cube of side \( 2^{r+1} \) and centre at \( x = 0 \) and set \( R_r = K_{r+1} \setminus K_r \). We write \( (n \geq 3) \)

\[
(3.1) \quad u(x) = H_r(x) - \int_{K_{r+1}} \frac{d\mu(y)}{|x-y|^{n-2}}, \quad x \in K_{r+1},
\]

where \( H_r(x) \) is harmonic in \( K_{r+1} \). Set

\[
(3.2) \quad M_r = \max_{K_r} H_r(x) + \mu(K_{r+1}).
\]

C will denote constants only depending on the dimension \( n \).

We are now going to describe a subdivision of \( \mathbb{R}^n \) into a grid \( G \) of dyadic cubes of sizes tending to zero at \( \infty \). The construction depends on a given sequence of numbers \( \delta_r \searrow 0 \) and the sides \( s(Q) \) of a cube \( Q \subset K_r \) will be \( < \delta_r \).

We may assume that \( \delta_r = 2^{-N_r} \), \( N_r \) integers. For \( Q \subset K_r \) choose \( G \) so that \( s(Q) = 2^{-N_r} \). Assume that \( G \) is constructed in \( K_r \). We choose

\[
K_r \subset K_r^{(1)} \subset K_r^{(2)} \subset ... \subset K_r^{(N_r+1-N_r)} \subset K_{r+1}
\]

so that the cube \( K_r^{(i)} \) has side \( 2^{r+1} (1 + 1/2 + 1/4 + ... + 1/2^i) \). In \( K_r^{(i+1)} \setminus K_r^{(i)} \) we construct \( Q \in G \) with sides \( 2^{-N_r-i} \) where we set \( K_r^{(N_r+1-N_r+1)} = K_{r+1} \). This defines \( G \) completely. It is important that \( s(Q) \) changes slowly in the following sense. If \( Q \in G \), \( Q \subset R_r \), then \( s(Q') \leq 2 s(Q) \) for all \( Q' \in G \) with distance \( < 2^{N_r} s(Q) \) from \( Q \).

In the formula (3.1) we now replace the measure \( \mu \) by the following continuous measure \( \mu' \):

\[
d\mu' = \frac{\mu(Q)}{m(Q)} \, dx, \quad x \in Q \in G.
\]

More precisely, we define
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\[ u'(x) = u(x) + \int_{\mathbb{R}^n} \frac{d\mu(y) - d\mu'(y)}{|x-y|^{n-2}} ; \]

$u'(x)$ is continuous and subharmonic if the integral converges in a suitable sense. We have for $x \in K_v$

\[ u'(x) = H_v(x) - \int_{K_{v+1}^+} \frac{d\mu'(y)}{|x-y|^{n-2}} + \int_{\mathbb{R}^n \setminus K_{v+1}^-} \frac{d(\mu - \mu')}{|x-y|^{n-2}}. \]

The last term can be estimated (in $K_v$) by

\[ (3.3) \quad \left| \int_{\mathbb{R}^n \setminus K_{v+1}^-} \frac{d\mu - d\mu'}{|x-y|^{n-2}} \right| \leq C \sum_{i \geq v} M_i \delta_i. \]

We can also show that $u(x) - u'(x)$ is small in general. Let $x \in K_v$ and let $Q^*$ be the union of all $Q$'s in $G$ with distance $< M_v \delta_v$ from $x$. We find

\[ (3.4) \quad |u(x) - u'(x)| \]

\[ \leq \int_{Q^*} \frac{d\mu(y) + d\mu'(y)}{|x-y|^{n-2}} + C \int_{K_{v+1}^+ \setminus Q^*} \frac{\delta_v}{|x-y|^{n-1}} d\mu(y) + \sum O(M_j \delta_j) \]

\[ \leq \int_{Q^*} \frac{d\mu + d\mu'}{|x-y|^{n-2}} + \frac{C}{M_v} \int_{K_{v+1}^+} \frac{d\mu(y)}{|x-y|^{n-2}} + \sum O(M_j \delta_j) \]

\[ \leq \int_{Q^*} \frac{d\mu(y) + d\mu'(y)}{|x-y|^{n-2}} + O(1) \]

if we assume $\sum M_j \delta_j < \infty$ and observe the definition of $M_v$ and $u(x) \geq 0$.

If now $u(x_v) \to \infty$ it follows that if we choose $\delta_v$ small enough $u'(x)$ is a subharmonic continuous function so that $u'(x)$ is unbounded. Hence $\gamma$ exists for $u'(x)$ and if we could make the estimate (3.4) uniformly by controlling $\int_{Q^*}$ we would have solved our problem. This however is not possible, so an additional construction is needed to make $\gamma$ avoid these bad cubes.

4. We fix some grid $G$ and consider the set of cubes $Q \in G$ in $R_v$. We increase each such $Q$ in the scale $M_v$ and denote the resulting cubes $Q^*$. They cover $R_v M_v^*$ times ($\delta_v < M_v^{-1}$). Denote by $A^*_v$ the set of such cubes such that
\( \Delta^* : \mu(Q^*) \geq s(Q)^{n-2} \)

and set

\[ E^*_\nu = \bigcup_{Q^* \in d^*_\nu} Q^*. \]

By Egorov's theorem

\[ \int_{|x-y| < M^*_\nu} \frac{d\mu(y)}{|x-y|^{n-2}} \leq M^{-2n}_\nu \]

except for \( x \in E^*_\nu \) in \( R^*_\nu \) such that \( \mu(E^*_\nu) \leq M^{-n}_\nu \) provided \( \delta^*_\nu \) is small enough. Clearly \( E^*_\nu \subset E^*_\nu'. \)

We can now finally fix our grid \( G \) so that all conditions above are satisfied. We replace \( u'(x) \) considered above by

\[ U(x) = u'(x) - \sum_{\nu=1}^{\infty} M^{n-1}_\nu \int_{E^*_\nu} \frac{d\mu(y)}{|x-y|^{n-2}} = u' - p'. \]

Since \( p'(x) \) is the potential of a bounded measure (\( \sum 1/M_\nu \) is supposed finite) and \( u'(x) \) is an unbounded subharmonic function, it follows easily that \( U(x) \) is also unbounded. Hence there is a continuous path \( \gamma \) so that \( U(x) \to \infty \) along \( \gamma \). The important improvement is that if \( \mu(Q^*) > s(Q)^{n-2} \) then \( \gamma \cap Q = \emptyset \). This is clear since \( u'(x) \leq M^*_\nu \) and for \( x \in Q \) \( p'(x) > M^*_\nu \) so \( U(x) \leq 0 \).

5. It is clear that \( u'(x) \to \infty \) along \( \gamma \). However \( u(x) \) may not but we do have the estimate (3.4).

Let \( Q_1 \in G \) be the "first" cube intersected by \( \gamma \) and let \( x_2 \) be the last point on \( \gamma \) in \( Q_1 \). Then \( x_2 \in Q_2 \) also. Let \( x_3 \) be the last point in \( Q_2 \) etc. \( x_2 \) belongs to the face \( F'_1 \) of \( Q_1 \) and \( F'_2 \) of \( Q_2 \). \( x_3 \in F'_2 \) in \( Q_2 \) and \( F'_3 \) in \( Q_3 \) etc. Observe that \( F''_i = F'_i \cap F'_{i+1} = F'_i \) or \( F'_{i+1} \) and that each face includes at least \( 100 \cdot 2^{-n} \% \) of the other.

We now join \( F''_i \) to \( F''_{i+1} \) by a line-segment \( l'_i \) in \( Q_i \). Since \( \mu(Q^*_i) \leq s(Q_i)^{n-2} \) it is a well-known property of Newtonian potentials that except for a small fraction of endpoints in \( F''_i \) and \( F''_{i+1} \) (with respect to normalized \((n-1)\)-dimensional measure)

\[ \int_{Q^*_i} \frac{d\mu(y)}{|x-y|^{n-2}} \leq C \]

along \( l'_i \).

We now modify \( x_i \) in the following way. Consider faces \( F''_i \) with even index \( i \). By Fubini, for every \( \xi_i \in F''_i \) except a set of small relative \((n-1)\)-dimensional measure there is an \( l_i \) going forward to every \( \xi_{i+1} \)
except a small exceptional set and one \( l_{i-1} \) going backward to a corresponding set of \( \xi_{i-1} \)'s. We choose these \( \xi_2 \)'s in this manner. Then clearly they can be joined via \( \xi_{2i} \) to each other. This now gives the desired polygon.

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