AREA METHOD AND UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSIONS

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1. Introduction

For a function \( z \mapsto z + \sum b_n z^{-n} \), univalent for \(|z| > 1\), the area of the set of its omitted values can be easily expressed in terms of the coefficients \( b_n \). The obvious fact that this area is non-negative leads to the area theorem

\[
\sum_{n=1}^{\infty} n |b_n|^2 \leq 1.
\]

Let \( S \) be the class of functions \( f \), holomorphic and univalent in the unit disc \( D \), and so normalized that

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

Rolf Nevanlinna [4] was the first to observe that some basic properties of functions \( f \in S \) can be derived from the area theorem in a very simple manner: not only does the inequality \(|a_2| \leq 2\) follow immediately, but straightforward integration yields sharp bounds for \(|f(z)|\) and \(|f'(z)|\).

Our aim in this paper is to show that the area method and its modifications lead quite easily to general inequalities for univalent functions with quasiconformal extensions. In Section 2 we first give a (known) generalization of the area theorem for class \( S \). A sharp version of it is derived in Section 3, by means of Schwarz's lemma, for the subclasses of \( S \) whose functions admit quasiconformal extensions with uniformly bounded maximal dilatations. Some of its consequences are discussed in Sections 4 and 5.
2. Power inequality in $S$

For a function $f \in S$ we write

$$[f(z)]^{-\nu} = \sum_{n=-\nu}^{\infty} b_{\nu n} z^n, \quad \nu = 1, 2, \ldots.$$  \hspace{1cm} (2.1)

Owing to normalization, the first coefficients $b_{-\nu}$ are equal to 1. Choose complex parameters $x_1, x_2, \ldots, x_N$ and denote

$$F(z) = \sum_{\nu=1}^{N} x_{\nu} [f(z)]^{-\nu}.$$  \hspace{1cm} (2.2)

Then

$$F(z) = \sum_{n=-N}^{\infty} y_n z^n$$

with

$$y_n = \sum_{\nu=1}^{N} b_{\nu n} x_{\nu},$$  \hspace{1cm} (2.3)

if we set $b_{\nu n} = 0$ for $\nu < -N$.

For a positive $\rho < 1$, direct computation gives

$$\int_{|z|=\rho} \overline{F(z)} \, dF(z) = 2\pi i \sum_{n=-N}^{\infty} n |y_n|^2 \rho^{2n}. $$  \hspace{1cm} (2.4)

Application of Green's formula shows that

$$i \int_{|z|=\rho} \overline{F(z)} \, dF(z) \geq 0.$$  \hspace{1cm} (2.5)

Hence, letting $\rho \to 1$ we obtain from (2.4) the "Power inequality"  

$$\sum_{n=1}^{\infty} n |y_n|^2 \leq \sum_{n=1}^{N} n |y_n|^2.$$  \hspace{1cm} (2.6)

(Schiffer – Tammi [6], Ahlfors [1]).

This formula contains some well-known inequalities as special cases. For $N = 1$, (2.5) reduces to the classical area theorem

$$\sum_{n=1}^{\infty} n |b_{n1}|^2 \leq 1.$$  \hspace{1cm} (2.6)

The Grunsky inequalities also follow from (2.5) with a suitable choice of the parameters $x_n$.  

3. Power inequality in $S_k$

Let $S_k$, $0 < k < 1$, be the class of quasiconformal homeomorphisms $f$ of the plane for which $f \mid D \in S$ and whose complex dilatation $\mu$ satisfies the condition $||\mu||_\infty \leq k$. The subclass of $S_k$ whose functions map the point $\zeta$ to infinity is denoted by $S_k(\zeta)$. The class $S_0(\zeta)$ contains the single element

$$z \mapsto z(1-z/\zeta)^{-1}.$$  

Suppose $\mu$ is a measurable function which satisfies $||\mu||_\infty < k$ and vanishes in $D$. If $\zeta$ and $w$ are given complex numbers, $|\zeta| \geq 1$, $|w| < 1$, then there is a unique mapping $f(\ ,w)$ of class $S_{|w|}(\zeta)$ with complex dilatation $w\mu/k$. We denote by $a_n(w)$, $b_n(w)$ and $y_n(w)$ the coefficients of $f(\ ,w) \mid D$, defined by (1.1), (2.1) and (2.2).

**Theorem 1.** If $f \in S_k$, then

$$\sum_{n=1}^{\infty} n |y_n|^2 \leq k^2 \sum_{n=1}^{N} n |y_n|.$$  

**Proof.** Let $\mu$ be the complex dilatation of $f$, and $f(\ ,w)$ the mapping with complex dilatation $w\mu/k$ which belongs to class $S_{|w|}(f^{-1}(\infty))$. Then $w \mapsto f(z,w)$ is defined in the unit disc, and $f = f(\ ,k)$.

The $N \times N$ determinant whose rows and columns consist of the coefficients $b_{nv}$ of $f$, $n, v = 1, 2, \ldots, N$, is equal to 1. In view of (2.3), we can thus associate with $f(\ ,w)$ the parameters $x_v$ so that $y_{-n}(w) = y_{-n}$, $n = 1, 2, \ldots, N$, for each $w \in D$.

The functions $w \mapsto a_n(w)$ are known to be holomorphic in $D$ ([3]). Thus every $w \mapsto b_n(w)$, being a polynomial of the coefficients $a_n(w)$, is holomorphic. By formula (2.3), the same is true of $w \mapsto y_n(w)$.

From (3.1) we see that $b_n(0) = 0$ for all positive values of $n$. Hence, by (2.3),

$$y_n(0) = 0, \quad n = 1, 2, \ldots.$$  

Set $\lambda_n = |y_n|^2/y_n^2$, $n = 1, 2, \ldots$, if $y_n \neq 0$; otherwise $\lambda_n = 1$. Having fixed a natural number $M$, we consider the function $\psi$ defined by

$$\psi(w) = \sum_{n=1}^{M} n \lambda_n(y_n(w))^2.$$  

It is holomorphic in the unit disc, and by (3.3) has a double zero at the origin. From (2.5) it follows that

$$|\psi(w)| \leq \sum_{n=1}^{N} n |y_n|^2.$$
Hence, applying Schwarz’s lemma to the function $w \mapsto \psi(w) / w$ we obtain

$$|\psi(w)| \leq |w|^2 \sum_{n=1}^{N} |y_n|^2.$$  

For $w = k$ and $M \to \infty$ this yields (3.2).

Setting $u_n = n y_n$ and applying Schwarz’s inequality to (3.2) we conclude that

$$\sum_{n=1}^{N} u_n y_n \leq k \sum_{n=1}^{N} |u_n|^2 / n.$$  

(3.4)

4. Functional $a_2^2 - a_3$

An immediate consequence of Theorem 1 is the (known) area theorem for $S_h$:

$$\sum_{n=1}^{\infty} n |b_{n1}|^2 \leq k^2.$$  

Since $b_{11} = a_2^2 - a_3$, it follows that in $S_h$

$$|a_2^2 - a_3| \leq k.$$  

Equality can hold only if $b_{n1} = 0$ for $n > 1$. Then

$$f(z) = z (1 - a_2 z + k e^{i \theta} z^2)^{-1}, \quad z \in D.$$  

This function is univalent. It is holomorphic if and only if $k e^{i \theta} z^2 - a_2 z + 1 \neq 0$ in $D$. This is equivalent to the condition

$$a_2 \in E_{\theta},$$

where $E_{\theta}$ is the closed ellipse onto whose exterior $z \mapsto 1 / z + k e^{i \theta} z$ maps $D$. Condition (4.3) implies

$$|a_2| \leq 1 + k.$$  

(4.4)

If this inequality holds, there is at least one $\theta$ for which (4.3) is true.

Suppose (4.3) is fulfilled. Then (4.2), together with

$$f(z) = z \bar{z} (z - a_2 z \bar{z} + k e^{i \theta} z)^{-1}, \quad |z| \geq 1,$$

(4.5)

defines an element of $S_h$. By a result of Strebel [7], (4.5) is the only extension of (4.2) with this property. Hence, equality holds in (4.1) if and only if $f$ is defined by (4.2) and (4.5), and condition (4.3) is satisfied.

The restriction (4.3) (or (4.4)) is not void if $0 < k < 1$; Schiffer and Schober [5] have proved that

$$\max_{S_h} |a_2| = 2 - 4 \zeta^2, \quad \zeta = (\arccos k) / \pi.$$
Since this is greater than $1 + k$ for $0 < k < 1$, we have $\max |a_2^2 - a_3| = k$ for $0 \leq |a_3| \leq 1 + k$, while $|a_2^2 - a_3| < k$ for $1 + k < |a_3| \leq 2 - 4 z^2$. Because the functions maximizing $|a_3|$ are unique up to trivial rotations, direct computation gives $|a_2^2 - a_3| = 1 - 16 z^2 / 3 + 16 z^4 / 3$ for $|a_2| = 2 - 4 z^2$.

In the subclass $S_k(\zeta)$, the equations (4.2) and (4.5) define an extremal function, provided that $|a_2| \leq (1 + k) / |\zeta|$. In particular, for $\zeta = \infty$ we have $\max |a_2^2 - a_3| = k$ if and only if $a_2 = 0$. The maximum value of $|a_3|$ is $2 k$, the corresponding functions $f$ being defined by $f(z) = z(1 + k e^{i\theta} z)^{-1}$ in $D$. Consequently, $|a_2^2 - a_3| = k^2$ for $|a_2| = 2 k$.

5. Coefficient $a_4$

Let $f$ belong to $S_k(\infty)$ and have the power series coefficients $a_n$. The function $\varphi$, defined by $\varphi(z) = (f(z^2))^{1/2}$, is then also in $S_k(\infty)$. The standard way to estimate $a_4$ is to apply (3.4) to $\varphi$, with the choice $N = 3$, $u_1 = u$, $u_2 = 0$, $u_3 = 1$. It follows that

$$|a_4| / 2 - a_2 a_3 + 13 a_2^3 / 24 + a_2 u^2 / 2 + (a_3 - 3 a_2^2 / 4) u| \leq k(1 / 3 + |u|^2).$$

In estimating $|a_4|$ we can suppose, without loss of generality, that $a_4$ is positive. Choose $u = a_2$. Since $|a_3| \leq 2 k$, the inequality (5.1) then yields (Kühnau [2])

$$|a_4| \leq 2 k / 3 + 38 k^3 / 3.$$ 

This estimate is asymptotically correct as $k \to 0$, but becomes quite inaccurate as $k \to 1$. Kühnau [2] has proved that

$$|a_4| \leq 2 k / 3 + 10 k^3 / 3$$

if $k \geq (7 / 15)^{1/2}$. For $k = 1$, this gives the sharp estimate $|a_4| \leq 4$.

Using (5.1) we shall show that if the coefficients $a_2, a_3, a_4$ are real, then (5.2) remains valid for $k \geq 0.41$. Again we can assume that $a_4 > 0$. If (5.1) is written in the form $a u^2 + 2 b u + c \leq 0$, one sees that $u = -b / a$ is an optimal choice. This gives the inequality

$$a_4 \leq 2 k / 3 + d,$$

with

$$d = 2 a_2 a_3 - 13 a_2^3 / 12 - (a_3 - 3 a_2^2 / 4)^2 (2 k - a_2)^{-1}.$$ 

Wanting to establish (5.2) we can exclude the case $a_2 = 2 k$ (then $a_4 = 4 k^3$). Hence $2 k - a_2 > 0$. 

Rearranging the terms we obtain
\[ d = 2k a_2^2 - 7a_2^3 / 12 - (a_3 + a_2^2 / 4 - 2k a_2^2) (2k - a_2)^{-1}. \]

With \( a_2 \) fixed and \( a_3 \) variable, \( d \) attains its maximum \( M(a_2) = 2k a_2^2 - 7a_2^3 / 12 \) for
\[ a_3 = -a_2^2 / 4 + 2k a_2. \]

We observe that
\[ M(2k) = 10k^3 / 3. \]

Condition \( |a_2^3 - a_3| \leq k \), coupled with (5.4), yields
\[ a_2 \geq h = 4k / 5 - 2(4k^2 + 5k)^{1/2} / 5. \]

For a fixed \( a_2 \) satisfying
\[ -2k \leq a_2 \leq h, \]
we have \( a_3 + a_2^2 / 4 - 2k a_2 \geq 0 \) if \( a_3 \geq -k + a_2^2 \). Hence, on the interval (5.6) the choice \( a_3 = -k + a_2^2 \) gives an upper bound for \( d \), i.e.
\[ d \leq 2k a_2^2 - 7a_2^3 / 12 - (5a_2^2 / 4 - 2k a_2 - k) (2k - a_2)^{-1}. \]

Thus far, the computations have been easily carried out. It remains to determine the maximum value of the majorant in (5.7) for \(-2k \leq a_2 \leq h\). We were glad to leave it to the computer to show that the maximum does not exceed \( 10k^3 / 3 \) if \( k \geq 0.41 \). In view of (5.3) and (5.5), we thus obtain the desired estimate (5.2).

References


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