SOME COEFFICIENT PROBLEMS FOR STARLIKE FUNCTIONS

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0. Introduction

Consider the class $S^*$ of normalized starlike functions $f$ in the unit disc $D = \{ |z| < 1 \}$, whose basic results were established by Rolf Nevanlinna in his paper [6], and write

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

The mapping

$$A_n : f \mapsto (a_2, \ldots, a_n)$$

associates to each $f$ a point in $C^{n-1}$ and takes $S^*$ onto some compact set $S_n^*$ which is called the $n$-th coefficient body for the class $S^*$. This paper deals with some basic properties of $S_n^*$. It will be proved that $S_n^*$ is homeomorphic to a ball in $C^{n-1}$, that for each boundary point $a$ of $S_n^*$ there is only one function $f$ in $S^*$ such that $A_n(f) = a$ and that $A_n(f)$ is on the boundary of $S_n^*$ if and only if $f$ takes the unit disc onto a domain which is bounded by at most $n-1$ rays

$$R_j = \{ z = te^{izj} \mid t \leq r_j \}, \quad \alpha_1 < \alpha_2 < \ldots < \alpha_m < \alpha_1 + 2\pi,$$

$1 \leq m < n$. Contrary to the analogous problem for the class $S$ (cf. [9] for example) the situation here is very explicit and elementary. The basic idea is to consider an analogous coefficient problem for the Carathéodory class $C$ of functions $g$ holomorphic in the unit disc which have positive real part and are normalized by the condition $g(0) = 1$. For each $n$ the expansion

$$g(z) = 1 + 2 c_1 z + \ldots + 2 c_n z^n + \ldots$$

defines a mapping

$$\gamma_n : g \mapsto (c_1, \ldots, c_n)$$

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of $C$ onto some compact set $C_n$ in $C^n$, which is called the $n$-th coefficient body for the class $C$. The basic properties of $C_n$ (Theorem A) are due to C. Carathéodory and O. Toeplitz; for completeness a full proof will be given in the third part of the paper.

$C_n$ is a convex body in $C^n$, hence $C_n$ admits for each boundary point a supporting hyperplane. This fact gives a set of inequalities for $a_2, \ldots, a_{n+1}$ relative to a boundary point $(a_2^0, \ldots, a_{n+1}^0)$ of $S_{n+1}^*$. Among them there are coefficient inequalities for the class $S^*_n$ (Theorem 2) which are quite similar to those the extended general coefficient theorem of J. A. Jenkins gives for the class $S$. They imply that some sections of $S^*_n$ are convex, i.e. if $a^0 = (a_2^0, \ldots, a_{n}^0)$ is a point of $S^*_n$ and if $W_n(a_2^0, \ldots, a_{n}^0)$ is the set of points $(a_{n+1}^0, \ldots, a_{n})$ in $C^{n-2}$ such that $(a_2^0, \ldots, a_{n}^0, a_{n+1}^0, \ldots, a_{n})$ is in $S^*_n$, then $W_n(a_2^0, \ldots, a_{n}^0)$ is strictly convex for each point $a^0$ in $S^*_n$ provided that $n \leq 2q$, and this bound for $n$ is sharp.

1. The $n$-th coefficient body

1.1. In this section we will prove the following

Theorem 1. The $n$-th coefficient body $S^*_n$ is homeomorphic to a ball in $C^{n-1}$. For each point $a$ on the boundary of $S^*_n$ there is only one function in $S^*$ which is taken onto $a$ by the mapping $A_n$ while $A_n^{-1}(a)$ is an infinite set in $S^*$ if $a$ is in the interior of $S^*_n$. $A_n(f)$ is on the boundary of $S^*_n$ if and only if there are distinct points $z_1, \ldots, z_m$ on the unit circumference $\{ |z| = 1 \}$ and positive numbers $\mu_1, \ldots, \mu_m$ where $\sum_{j=1}^{m} \mu_j = 1$ and $1 \leq m < n$ such that

$$f(z) = z \prod_{j=1}^{m} (1 - z \xi_j) - 2 \mu_j.$$

Given the boundary point $a$, the numbers $m$, $\xi_j$ and $\mu_j$ are unique.

Remark. $f$ takes the unit disc onto a domain which is bounded by $m$ rays $(0, 3)$, where $\xi_{j+1} - \xi_j = 2 \pi \mu_j$, $j = 1, \ldots, m$. Conversely, any $m$ such rays, $1 \leq m < n$, up to a suitable homothety $z \rightarrow rz$, $r > 0$, determine via the mapping function a point on the boundary of $S^*_n$.

Since $f$ belongs to $S^*$ if and only if $zf'(z)/f(z)$ is in $C$, the differential equation

$$zf'(z) = g(z)f(z)$$

establishes a homeomorphism between $C$ and $S^*$ if $C$ and $S^*$ are provided with the topology of uniform convergence on compact subsets of $D$. Equation (1.2) implies the following relations between the coefficients $a_j$ and $c_j$ in (0.1) and (0.4) respectively:
Some coefficient problems for starlike functions

\[ a_2 = 2 c_1, \]
\[ 2 a_3 = 2 (c_2 + a_2 c_1), \]
\[ (n - 1) a_n = 2 (c_{n-1} + a_2 c_{n-2} + \ldots + a_{n-1} c_1). \]

For each \( n, \ n = 2, 3, \ldots \), they define a homeomorphism of \( S_n^* \) onto \( C_{n-1} \). Hence, for some basic properties of \( S_n^* \), it suffices to study \( C_{n-1} \).

1.2. The following result is due to C. Carathéodory [1] and O. Toeplitz [10] (cf. also [2]).

Theorem A. \( C_n \) is a convex body in \( C^n \) containing the origin. To each point \( \zeta = (c_1, \ldots, c_n) \) in the interior of \( C_n \) there correspond infinitely many functions in \( C \), i.e. \( \gamma_n^{-1}(\zeta) \) is infinite; but for each point \( \zeta \) on the boundary of \( C_n \), there is only one \( g \) in \( C \) which is taken onto \( \zeta \) by the mapping \( \gamma_n \cdot \gamma_n(g) \) is on the boundary of \( C_n \) if and only if there are distinct points \( x_1, \ldots, x_m \) on the unit circumference \( \{ |z| = 1 \} \) and positive numbers \( \mu_1, \ldots, \mu_m \) such that

\[ g(z) = \sum_{j=1}^m \frac{1 + x_j \overline{z}}{1 - x_j \overline{z}} \mu_j, \]

where \( 1 \leq m \leq n \) and \( \sum_{j=1}^m \mu_j = 1 \). The numbers \( m, x_j, \mu_j \) are determined uniquely by the boundary point \( \zeta \).

Proof of Theorem 1. Theorem A, together with (1.2) and (1.3), immediately implies Theorem 1. Since (1.3) establishes a homeomorphism between the boundaries of \( C_{n-1} \) and \( S_n^* \), by integration of (1.2), the functions (1.4) give exactly those functions in \( S_n^* \) which are taken onto the boundary of \( S_n^* \) under the mapping \( A_n \).

1.3. The implication of Theorem 1 for extremal problems within the class \( S_n^* \) is immediate. Let \( F(a_2, \ldots, a_n) \) be a real valued function of the complex variables \( a_2, \ldots, a_n \), which is defined and continuously differentiable with respect to the real variables \( x_j = \text{Re} a_j, \ y_j = \text{Im} a_j, \ j = 2, \ldots, n, \) in some neighborhood \( N \) of \( S_n^* \), such that \( |\text{grad} F| \) is positive there. Then the function \( F \) attains its maximum on \( S_n^* \) only on the boundary. Hence each function \( f \) which maximizes \( F \) (considered as a functional on \( S_n^* \) on \( S_n^* \) is necessarily of type (1.1). This result was proved by J. A. Hummel by variational methods of Schiffer's type within the class \( S_n^* \) (cf. [4]).

Furthermore, let \( F \) attain its maximum at a point \( (a_2^0, \ldots, a_n^0) \) of the boundary \( \partial S_n^* \). Then, from
\[
F(a_2, \ldots, a_n) = F(a^0_2, \ldots, a^0_n) + 2 \Re \sum_{j=2}^{n} \frac{\partial F}{\partial a_j} \Delta a_j \\
+ o(\max_j \{|\Delta a_j|\}) \quad \Delta a_j = a_j - a^0_j,
\]

it follows that
\[
\Re \left\{ \sum_{j=2}^{n} \frac{\partial F}{\partial a_j} (a^0_2, \ldots, a^0_n) \Delta a_j \right\} + o(\max_j \{|\Delta a_j|\}) \leq 0
\]

and this shows that \((\partial F/\partial a_2, \ldots, \partial F/\partial a_n)\) is an outer normal vector to \(S^*_n\) at \((a^0_2, \ldots, a^0_n)\). (Cf. also Paragraph 2.1.)

1.4. If \(\zeta = (c_1, \ldots, c_n)\) is an interior point of \(C_n\), then \(\gamma_n^{-1}(\zeta)\) is an infinite set in \(C\). It is possible, however, to define in a natural way a subset of \(C\) which is homeomorphic to \(C_n\) under the mapping \(\gamma_n\). Assume first that \(\zeta \neq (0, \ldots, 0)\). Since \(C_n\) is a convex body containing the origin, there is a unique number \(t > 1\) such that \(t \zeta\) is on the boundary of \(C_n\). By Theorem A there is a unique set of numbers \(\mu_j\) and \(\kappa_j\), \(j = 1, \ldots, m\), \(1 \leq m \leq n\), such that
\[
g(z) = \sum_{j=1}^{m} \frac{1 + \kappa_j z}{1 - \kappa_j z} \mu_j
\]
corresponds to \(t \zeta\), i.e. \(\gamma_n(g) = t \zeta\). The function \(g^* = g / t + 1 - 1 / t\), which can be written in the form
\[
g^*(z) = 1 + 2 \sum_{j=1}^{m} \frac{\kappa_j z}{1 - \kappa_j z} \mu_j^* \quad \sum_{j=1}^{m} \mu_j^* = \frac{1}{t},
\]
is in \(C\) and \(\gamma_n(g^*) = \zeta\). If \(\zeta = (0, \ldots, 0)\) we choose \(1 / t = 0\), i.e. \(\mu\) vanishes and \(g^*\) is just the constant 1, which case may be characterized also by setting \(m = 0\). Thus we proved

**Theorem A'.** To each point \(\zeta = (c_1, \ldots, c_n)\) of \(C_n\) there corresponds a unique set of distinct points \(\kappa_1, \ldots, \kappa_m\) on the unit circumference and a set of positive numbers \(\mu_1, \ldots, \mu_m\), where \(\sum_{j=1}^{m} \leq 1\) and \(0 \leq m \leq n\), such that
\[
g(z ; \zeta) = 1 + 2 \sum_{j=1}^{m} \frac{\kappa_j z}{1 - \kappa_j z} \mu_j
\]
is in \(C\) and \(\gamma_n(g( ; \zeta)) = \zeta\). The set of the numbers \(\kappa_j\) and \(\mu_j\) might be empty in which case we set \(m = 0\). \(\zeta\) is on the boundary of \(C_n\) if and only if \(\sum_{j=1}^{m} \mu_j = 1\). The correspondence \(\zeta \mapsto g( ; \zeta)\) defines a homeomorphic mapping of \(C_n\) onto the subset \(\{ g( ; \zeta) \}\) of \(C\).
As before, we obtain from this Theorem $A'$, together with (1.3) and (1.2),

**Theorem 1'.** Let $M$ denote the set of measures defined by $m$ distinct points $x_1, ..., x_m$ on the unit circumference with assigned positive numbers $\mu_1, ..., \mu_m$ such that $\sum_j^m \mu_j \leq 1$ and $0 \leq m < n$. Then by (1.1) there corresponds to $M$ a subset of functions in $S^*$ which is homeomorphic to $S_n^*$.

2. Coefficient inequalities

2.1. Here we derive coefficient inequalities relative to a given boundary point $a^0 = (a_2^0, ..., a_n^0)$ of $S_n^*$. By (1.3) there corresponds to $a^0$ a point $\zeta^0 = (c_1^0, ..., c_{n-1}^0)$ on the boundary of $C_{n-1}$. Choose a supporting hyper-plane at $\zeta^0$, with normal direction $\alpha = (\alpha_1, ..., \alpha_{n-1})$. Then, for any point $\zeta = (c_1, ..., c_{n-1})$ of $C_{n-1}$ we have the inequality

$$\text{Re} <\zeta - \zeta^0, \alpha> \leq 0.$$  

(2.1)

Since by (1.3) $c_j - c_j^0$ depends on the coefficients $a_2, a_2^0, ..., a_{j+1}, a_{j+1}^0$ only, (2.1) represents an inequality involving the boundary point $a^0$ and an arbitrary point $a$ of $S_n^*$. We transform now (2.1) considering that $\zeta - \zeta^0$ could be small. Let $g$ and $g_0$ be functions in $C$ such that $\gamma_{n-1}(g) = \zeta$ and $\gamma_{n-1}(g_0) = \zeta^0$,

$$g(z) = \sum_0^\infty 2 c_j z^j, \quad g_0(z) = \sum_0^\infty 2 c_j^0 z^j, \quad 2 c_0 = 1.$$  

Define

$$\Phi(z) = \int_0^z (g(t) - g_0(t)) \frac{dt}{t} = \varphi_1 z + ... + \varphi_n z^n + ...$$  

(2.2)

where $2 (c_j - c_j^0) = j \varphi_j, \ j = 1, 2, ...$. Let the functions $f$ and $f_0$ in $S^*$ correspond to $g$ and $g_0$. Equation (1.2) implies then

$$f(z) = z \exp \int_0^z (g(t) - 1) \frac{dt}{t} = f_0(z) \exp \Phi(z).$$  

(2.3)

Write now

$$x(z) = \frac{\alpha_{n-1}}{z_{n-1}} + ... + \frac{\alpha_1}{z},$$  

(2.4)

$$k(z) = - \frac{x'(z)}{f_0(z)} = \frac{k_1}{z^{n+1}} + ... + \frac{k_2}{z^3} + ...$$  

(2.5)
and correspondingly $\overline{\alpha}(z) = \overline{\alpha}(\zeta)$, $\overline{k}(z) = \overline{k(\zeta)}$. It follows then

\[
(2.6) \quad 2 < \zeta - \zeta' , \alpha > = \frac{1}{2\pi i} \oint \overline{\alpha}(z) (g(z) - g_0(z)) \frac{dz}{z} = - \frac{1}{2\pi i} \oint \overline{\alpha}'(z) \Phi(z) \, dz = \frac{1}{2\pi i} \oint \overline{k}(z) f_0(z) \Phi(z) \, dz
\]

and by (2.1)

\[
(2.7) \quad \text{Re} \frac{1}{2\pi i} \oint \overline{k}(z) f_0(z) \Phi(z) \, dz \leq 0 ,
\]

where the integration is taken along a positively oriented circuit around the origin.

**2.2.** Choose a variation of $f_0$ in $S^*$, i.e. a mapping $\varepsilon \mapsto f_\varepsilon$ of some interval $(0, \varepsilon_0)$ into $S^*$ such that

\[
(2.8) \quad f_\varepsilon = f_0 + \varepsilon f_1 + o(\varepsilon) ,
\]

where $f_1$ is holomorphic in $D$ with $f_1(z) = O(z^2)$, and $o(\varepsilon) / \varepsilon$ converges to zero uniformly on compact subsets of $D$. If

\[
f_\varepsilon(z) = \sum_{j=1}^{\infty} a_j(\varepsilon) z^j \quad \text{and} \quad f_1(z) = \sum_{j=2}^{\infty} a'_j z^j ,
\]

then $a' = (a'_2, ..., a'_n)$ is a tangent vector to the curve $\varepsilon \mapsto a(\varepsilon) = (a_2(\varepsilon), ..., a_n(\varepsilon))$, $0 \leq \varepsilon \leq \varepsilon_0$ at $a(0) : a(\varepsilon) = a(0) + \varepsilon a' + o(\varepsilon)$ for $\varepsilon \to 0$. Corresponding to (2.8) we have

\[
(2.9) \quad g_\varepsilon = g_0 + \varepsilon g_1 + o(\varepsilon) \quad \text{and} \quad \Phi_\varepsilon = \varepsilon \Phi_1 + o(\varepsilon) ,
\]

where $\Phi_1(z) = \int_0^1 g_1(\zeta) \, d\zeta / \zeta$. From (2.2) and (2.3) it then follows that

\[
(2.10) \quad f_1 = f_0 \Phi_1 .
\]

With $g_\varepsilon(z) = \sum_{j=0}^{\infty} c_j(\varepsilon) z^j$ and $g_1(z) = \sum_{j=1}^{\infty} c'_j z^j$ we get $\zeta(\varepsilon) = \zeta(0) + \varepsilon \zeta' + o(\varepsilon)$ for $\varepsilon \to 0$, where $\zeta(\varepsilon) = (c_1(\varepsilon), ..., c_{n-1}(\varepsilon))$ and $\zeta' = (c'_1, ..., c'_{n-1})$. If $\alpha = (\alpha_1, ..., \alpha_{n-1})$ is an outer normal vector to $C_{n-1}$ at the boundary point $\zeta(0)$, it follows from (2.1), (2.6), (2.9) and (2.2) that

\[
\text{Re} \frac{1}{2\pi i} \oint \overline{k}(z) f_1(z) \, dz = \text{Re} \{ \overline{k_2 a'_2 + ... + k_n a'_n} \} \leq 0
\]

for all tangent vectors $a'$, i.e. $(k_2, ..., k_n)$ is an outer normal vector to $S^*_n$ at $a^0$. 
Conversely, let \( k \) be an outer normal to \( S_{n}^{k} \) at \( a(0) \). For any \( g \in C \) define \( g_{\varepsilon} = g_{0} + \varepsilon (g - g_{0}) \), \( 0 \leq \varepsilon \leq 1 \). This is a variation of \( g_{0} \) in \( C \).

It follows from (2.5) and (2.4). From (2.6) and (2.10) it follows that

\[
\sum_{j=1}^{n-1} \bar{z}_{j} (c_{j} - c_{j}^{0}) \leq 0
\]

for all points \( \zeta = (c_{1}, \ldots, c_{n-1}) \) of \( C_{n-1} \) and this shows that \( \alpha = (x_{1}, \ldots, x_{n-1}) \) is an outer normal vector to \( C_{n-1} \). Thus, by (2.5), we proved

**Proposition 2.2.** There is a one to one correspondence between the outer normal vectors to \( C_{n-1} \) and \( S_{n}^{k} \) at associated boundary points, which is given by the equations

\[
\begin{align*}
\alpha_{1} &= k_{n} a_{n-1} + k_{n-1} a_{n-2} + \ldots + k_{2} a_{2} + k_{2} \\
2 \alpha_{2} &= k_{n} a_{n-2} + k_{n-1} a_{n-3} + \ldots + k_{3} \\
& \vdots \\
(n-2) \alpha_{n-2} &= k_{n} a_{2} + k_{n-1} \\
(n-1) \alpha_{n-1} &= k_{n}.
\end{align*}
\]

**2.3.** The preceding considerations suggest to develop (2.3) into powers of \( \Phi \) and to write (2.7) in the form

\[
\text{Re} \left\{ \frac{1}{2\pi i} \oint \frac{-k(z)}{z} \left( f(z) - f_{0}(z) - \frac{1}{21} f_{0}(z) (\Phi^{2}(z) - \ldots) \right) dz \right\} \leq 0.
\]

Only the powers \( \Phi^{2}, \ldots, \Phi^{n-1} \) are relevant for the evaluation of the integral, since \( \Phi \) has a zero at the origin. However, the higher is the order of this zero the less powers of \( \Phi \) are needed. Observe that by (2.2) we have

\[
\Phi(z) = \phi_{q} z^{q} + \phi_{q+1} z^{q+1} + \ldots, \quad 1 \leq q < n,
\]

or \( c_{j} = c_{j}^{0} \) for \( j = 1, \ldots, q - 1 \) if and only if \( a_{j} = a_{j}^{0}, \ j = 2, \ldots, q \), and that in case of (2.13) we have

\[
\bar{k}(z) f_{0}(z) \Phi^{2}(z) = \left( \frac{k_{n}}{z^{n+1}} + \ldots \right) (z + a_{0} z^{2} + \ldots) (\phi_{q}^{0} z^{2q} + \ldots)
\]

\[
= \bar{k}_{n} \phi_{q}^{0} z^{2q-n} + \ldots.
\]

We consider two cases.

1. \( q \geq n \). In this case inequality (2.12) reduces to

\[
\text{Re} \left\{ \sum_{j=q+1}^{n} \bar{k}_{j} (a_{j} - a_{j}^{0}) \right\} \leq 0.
\]
Equality occurs if and only if it holds in (2.7) also. \( k = (k_2, \ldots, k_n) \) is an outer normal vector to \( S^*_n \) at the point \( \alpha^0 = (a^0_2, \ldots, a^0_n) \). To \( \alpha^0 \) and \( k \) there corresponds the outer normal vector \( \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \) to \( C_{n-1} \) at \( \zeta^0 = (c^0_1, \ldots, c^0_{n-1}) \). From (2.6) it follows that equality holds in (2.7) if and only if
\[
\text{Re} \left< \zeta^0, \alpha \right> = \text{Re} \sum_{j=1}^{n-1} (c_j - c^0_j) \alpha_j = 0,
\]
i.e. the point \( \zeta = (c_1, \ldots, c_{n-1}) \) lies on the supporting hyperplane through \( \zeta^0 \) with normal direction \( \alpha \). Since, by assumption, \( c_j = c^0_j \) for \( j = 1, \ldots, q - 1 \) and \( 2q \geq n \) it follows from Lemma 3.5 (in Paragraph 3.5, with \( q = 1 \) and \( n-1 \) instead of \( q \) and \( n \) respectively) that this occurs only if \( \zeta = \zeta^0 \), hence only if \( a = a^0 \) or \( f = f_0 \).

If \( a_{q+1} = a^0_{q+1} \) equality occurs only if \( a = a^0 \), because then we are in the preceding case, i.e. \( n < 2(q + 1) \). Thus we proved

**Theorem 2.** Let \( a^0 = (a^0_2, \ldots, a^0_n) \) be a boundary point of the coefficient body \( S^*_n \), let \( k = (k_2, \ldots, k_n) \) be an outer normal vector to \( S^*_n \) at \( a^0 \) and let the integer \( q \) satisfy the condition \( q < n \leq 2q + 1 \). If \( \epsilon = 0 \) for \( n \leq 2q \) and \( \epsilon = 1 \) for \( n = 2q + 1 \), then the inequality
\[
(2.16) \quad \text{Re} \left\{ \sum_{j=q+1}^{n} \overline{k_j} (a_j - a^0_j) - \epsilon \frac{k_n}{2} (a_{q+1} - a^0_{q+1})^2 \right\} \leq 0
\]
holds for all points \( a \) of \( S^*_n \) such that \( a_j = a^0_j \), \( j = 2, \ldots, q \). In the case that \( n \leq 2q \) or that \( n = 2q + 1 \) and \( a_{q+1} = a^0_{q+1} \) equality occurs in (2.16) if and only if \( f = f_0 \).

**Remark.** Theorem 2 gives a coefficient inequality which is quite similar to the one J. A. Jenkins has given in his general coefficient Theorem [5] for the particular case of the normalized schlicht functions in the unit disc.

**2.4.** In the case \( n \leq 2q \) Theorem 2 has an interesting corollary. Choose in \( S^*_n \) a fixed point \( (a^0_2, \ldots, a^0_n) \), \( 1 < q < n \). Denote by \( W_n(a^0_2, \ldots, a^0_n) \) the set of all points \( (a^0_{q+1}, \ldots, a^0_n) \) in \( C^{n-q} \) such that \( (a^0_2, \ldots, a^0_q, a^0_{q+1}, \ldots, a^0_n) \) is in \( S^*_n \). Let \( (a^0_{q+1}, \ldots, a^0_n) = A^0 \) be on the boundary of \( W_n \). Then \( (a^0_2, \ldots, a^0_q, a^0_{q+1}, \ldots, a^0_n) = a^0 \) is a boundary point of \( S^*_n \). Choose there an outer normal vector \( k = (k_2, \ldots, k_n) \). By Theorem 2 we have (2.15) for all points \( (a^0_{q+1}, \ldots, a^0_n) \) in \( W_n(a^0_2, \ldots, a^0_n) \).
This shows that at each boundary point $W_n$ has a supporting hyperplane, hence $W_n$ is convex. It is even strictly convex, i.e. each supporting hyperplane to $W_n$ contains only one point of $W_n$ because equality occurs in (2.15) only if $a_j = a_j^0$, $j = q + 1, \ldots, n$.

Now we show that $W_n(a_2, \ldots, a_q)$ is no more convex for arbitrary points $(a_2, \ldots, a_q)$ in $S^*_\varrho$, if $n > 2 \varrho$. More precisely, we show that $W_{2\varrho + 1}(0, \ldots, 0)$ is not convex. For this purpose we consider the two functions

$$f_1(z) = (k(z^\varrho))^{1/\varrho} = z + \frac{2}{\varrho} z^{\varrho + 1} + \frac{\varrho + 2}{\varrho^2} z^{2\varrho + 1} + \ldots \text{ and}$$

$$f_2(z) = \varrho^{-1} f_1(\varrho z) = z - \frac{2}{\varrho} z^{\varrho + 1} + \frac{\varrho + 2}{\varrho^2} z^{2\varrho + 1} + \ldots$$

in $S^*$, where $\varepsilon^\varrho = -1$ and $k$ is the Koebe function $k(z) = z / (1 - z)^2$. They show that $W_{2\varrho + 1}(0, \ldots, 0)$ contains the points

$$(2 / \varrho, 0, \ldots, 0, (\varrho + 2) / \varrho^2) \text{ and } (-2 / \varrho, 0, \ldots, 0, (\varrho + 2) / \varrho^2).$$

But the midpoint $(0, \ldots, 0, (\varrho + 2) / \varrho^2)$ of them does not belong to $W_{2\varrho + 1}(0, \ldots, 0)$, because for any schlicht function, hence for any starlike function $f(z) = z + a_{2\varrho + 1} z^{2\varrho + 1} + \ldots$ we have $|a_{2\varrho + 1}| \leq 1 / \varrho$ by a result of Prawitz ([8]) and because $1 / \varrho < (\varrho + 2) / \varrho^2$. Thus we proved

Theorem 3. Associate to a point $(a_2^0, \ldots, a_q^0)$ in $S^*_\varrho$ and an integer $n > \varrho$ the set $W_n(a_2^0, \ldots, a_q^0)$ of those points $(a_{q + 1}, \ldots, a_n)$ in $C^{n-\varrho}$ for which $(a_2^0, \ldots, a_q^0, a_{q + 1}, \ldots, a_n)$ is in $S^*_n$. Then $W_n(a_2^0, \ldots, a_q^0)$ is a strictly convex body if $n \leq 2 \varrho$. However, $W_{2\varrho + 1}(0, \ldots, 0)$ is no longer convex.

A similar theorem holds for the coefficient bodies of the class $S$ (cf. [7]).

Let consider the particular case that $\varrho = n - 1$. If $(a_2^0, \ldots, a_{n-1}^0)$ is on the boundary of $S^*_n$, then obviously $W_n(a_2^0, \ldots, a_{n-1}^0)$, the range of $a_n$, is a point. Thus we may assume that $(a_2^0, \ldots, a_{n-1}^0)$ is in the interior of $S^*_n$. The corresponding point $(c_1^0, \ldots, c_{n-2}^0)$ is in the interior of $C_{n-2}$. As was remarked by Carathéodory ([1]), the range of $C_n$ is a disc. Hence, by (1.3), it follows: For a given point $(a_2^0, \ldots, a_{n-1}^0)$ in $S^*_n$ the range of $a_n$ is either a disc or a point. Based on a different method this result was given by J. A. Hummel in [3].

3. Proof of Theorem A

3.1. $C_n$ is a compact and convex set in $C^n$, since $C$ is convex and compact (in the topology of uniform convergence on compact subsets of $D$), and $\gamma_n$ is continuous and linear.
3.2. Let $\zeta$ be an interior point of $C_n$. There is a $\lambda$, $\lambda > 1$, such that $\lambda \zeta = (\lambda c_1, \ldots, \lambda c_n)$ is still in $C_n$. Choose in $C$ a function $g$ such that $\gamma_n(g) = \lambda \zeta$. Then $g_1 = \lambda^{-1} g + (1 - 1/\lambda)$ is in $C$ and satisfies $\Re g_1(z) > 1 - 1/\lambda$. Hence, $g_1 + h$ is in $C$ and $\gamma_n(g_0 + h)$ equals to $\zeta$ for each function $h(z) = b_{n+1} z^{n+1} + \ldots$ which is holomorphic in $D$, such that $\sup_{z \in D} |h(z)| \leq 1 - 1/\lambda$. This proves that $\gamma_n^{-1}(\zeta)$ is an infinite set in $C$.

3.3. Let $P$ denote the set of probability measures supported by the unit circumference $\{ |z| = 1 \}$. According to a result of Herglotz $g$ belongs to the class $C$ if and only if

$$g(z) = \int_0^{2\pi} \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z} \, d\mu_\theta, \quad \mu \in P,$$

or equivalently if and only if the coefficients $c_n$ of $g$ (in $(0, \pi)$) are the trigonometric moments of a probability measure, i.e.

$$c_n = \int_0^{2\pi} e^{in\theta} \, d\mu_\theta, \quad \mu \in P, \quad n = 0, 1, 2, \ldots.$$  

In the sequel we represent points in $\mathbb{R}^{2n}$ in the form $\zeta = (\zeta_1, \ldots, \zeta_n)$, $\zeta_j \in C$, as points in $C^n$, and consequently, we write the standard scalar-product in $\mathbb{R}^{2n}$ as $\Re \langle \zeta, \zeta^* \rangle = \Re \sum_{j=1}^n \bar{\zeta}_j \zeta_j^*$. Hence, the norm of $\zeta$ is given by $|\zeta| = (\langle \zeta, \zeta \rangle)^{1/2}$.

Defining

$$\zeta(\mu) = \int_0^{2\pi} (e^{i\theta}, \ldots, e^{in\theta}) \, d\mu_\theta$$

for any real measure (supported by the unit circumference) we have

$$C_n = \{ \zeta(\mu) \mid \mu \in P \},$$

i.e. $C_n$ is the convex hull of the curve

$$\Gamma: \theta \mapsto (e^{i\theta}, \ldots, e^{in\theta}), \quad 0 \leq \theta \leq 2\pi.$$

Let $\alpha$ be a unit vector:

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| = 1.$$

Define

$$h(\alpha) = \max_\theta \Re \{ \bar{\alpha}_1 e^{i\theta} + \ldots + \bar{\alpha}_n e^{in\theta} \}$$

and
Some coefficient problems for starlike functions

(3.4) \[ T(e^{i\theta}, \alpha) = h(\alpha) - \text{Re}\left(\sum_{j=1}^{n} \bar{z}_j e^{i\theta}\right). \]

Obviously \( T(e^{i\theta}, \alpha) \geq 0 \) for all \( \theta \) and \( \alpha \). Hence, by a lemma of Fejér and Riesz, there is a polynomial \( p(z) = \xi_0 + \xi_1 z + \ldots + \xi_n z^n \) such that

(3.5) \[ T(e^{i\theta}, \alpha) = |p(e^{i\theta})|^2. \]

Let now \( \zeta_0 = \zeta(\mu_0), \mu_0 \in P \), be a point on the boundary of \( C_n \). Since \( C_n \) is convex there is a supporting hyperplane

\[ \text{Re} \langle \zeta - \zeta_0, \alpha \rangle = 0, \quad |\alpha| = 1, \text{ i.e. } \max_{\zeta \in C_n} \text{Re} \langle \zeta, \alpha \rangle = \text{Re} \langle \zeta_0, \alpha \rangle. \]

Since \( C_n \) is the convex hull of \( \Gamma \) we have also

\[ \max_{\theta} \text{Re} \sum_{j=1}^{n} \bar{z}_j e^{i\theta} = \text{Re} \langle \zeta_0, \alpha \rangle \]

or by (3.3) \( h(\alpha) - \text{Re} \langle \zeta_0, \alpha \rangle = 0 \). With the notations (3.4) and (3.5) and with (3.1) it follows

\[ \int_{0}^{2\pi} T(e^{i\theta}, \alpha) d\mu_0 = \int_{0}^{2\pi} |p(e^{i\theta})|^2 d\mu_0 = 0 \]

and this shows that a measure \( \mu_0, \mu_0 \in P \), such that \( \zeta(\mu_0) \) is on the boundary of \( C_n \), is a measure supported by at most \( n \) points on the unit circumference.

Conversely, for an integer \( m, 1 \leq m \leq n \), choose \( m \) distinct points \( \xi_j \) on the unit circumference and positive numbers \( \mu_j, j = 1, \ldots, m \) such that \( \sum \mu_j = 1 \). Let the pairs \( \{ (\xi_j, \mu_j) \} \) define the measure \( \mu_0 \). Then \( \zeta(\mu_0) \) is on the boundary of \( C_n \). In fact, setting \( p(z) = \Pi(z - \xi_j) \) we have

(3.6) \[ \int_{0}^{2\pi} |p(e^{i\theta})|^2 d\mu_0 = \sum_{1}^{m} p(\xi_j) \mu_j = 0. \]

But

\[ |p(e^{i\theta})|^2 = \alpha_0 - \text{Re} \sum_{j=1}^{n} \bar{z}_j e^{i\theta} \]

for suitably chosen numbers \( \bar{z}_j \). As a positive factor is not relevant we can assume that \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a unit vector and this implies \( \alpha_0 = h(\alpha) \).

Equivalently to (3.6) we have then
\[ \int_0^{2\pi} T(e^{i\theta}, x) \, d\mu_0 = 0, \quad \text{and} \quad \int_0^{2\pi} T(e^{i\theta}, x) \, d\mu \geq 0 \quad \text{for all} \quad \mu \in P, \]
i.e.

\[
\max_{\zeta \in C_n} \Re <\zeta, x> = \Re <\zeta(\mu_0), x>.
\]

This shows that \( \zeta(\mu_0) \) is on the boundary of \( C_n \) ending the proof that \( \zeta = \gamma_n(g) \) belongs to the boundary of \( C_n \) if and only if \( g \) is given by (1.4).

3.4. It will be shown now that for a point \( \zeta^0 = (c_1, \ldots, c_n) \) on the boundary of \( C_n \) there is only one measure \( \mu \) in \( P \) such that \( \zeta(\mu) = \zeta^0 \), and then this implies that there is a unique \( g \) in \( C \) satisfying \( \gamma_n(g) = \zeta_0 \).

Let \( \mu \) be a measure in \( P \) such that \( \zeta(\mu) = \zeta^0 \). \( \mu \) is carried by some points \( x_1, \ldots, x_m \), \( 1 \leq m \leq n \); hence,

\[
\zeta^0 = \mu_1 \zeta^1 + \ldots + \mu_m \zeta^m, \quad \mu_j > 0, \quad \sum_{j=1}^m \mu_j = 1,
\]

where

\[
(3.7) \quad \zeta^j = (x_j, x_j^2, \ldots, x_j^n), \quad j = 1, \ldots, m.
\]

These vectors \( \zeta^j \) are linearly independent (because the points \( x_j \) are distinct). Their convex hull \( \text{coh } \{ \zeta^1, \ldots, \zeta^m \} \) lies on the \( m-1 \) dimensional hyperplane

\[
(3.8) \quad \zeta = \sum_{j=1}^m \lambda_j \zeta^j, \quad \sum_{j=1}^m \lambda_j = 1,
\]

and contains \( \zeta^0 \) in its interior, since all the weights \( \mu_j \) are positive.

Let now \( \mu' \) be another measure in \( P \) such that \( \zeta(\mu') = \zeta(\mu) = \zeta^0 \). \( \mu' \) is carried by some points \( x_{1j}, j = 1, \ldots, m_1 \), \( 1 \leq m_1 \leq n \) such that

\[
\zeta^0 = \sum_{j=1}^{m_1} \zeta_{1j} \mu_j', \quad \mu_j > 0, \quad \sum_{j=1}^{m_1} \mu_j = 1,
\]

where

\[
(3.8) \quad \zeta_{1j} = (x_{1j}, x_{1j}^2, \ldots, x_{1j}^n), \quad j = 1, \ldots, m_1.
\]

Since \( \zeta^0 \) is in the interior of either convex hull, say of \( \text{coh } \{ \zeta', \ldots, \zeta^m \} \) and of \( \text{coh } \{ \zeta^1, \ldots, \zeta^m \} \), these convex hulls lie on the same hyperplane of dimension \( m-1 = m_1-1 \). Choose a \( x_{1q}, q = 1, \ldots, m_1 \), and write \( x_{1q} = \zeta \). From \( \zeta^0 = \sum_{j=1}^{m_1} \lambda_j \zeta^j \), \( \sum_{j=1}^{m_1} \lambda_j = 1 \), according to (3.8) it follows that the vectors \( \zeta_{1j} - \zeta^1, \zeta^2 - \zeta^1, \ldots, \zeta^m - \zeta^1 \) are linearly dependent, hence
Consider this determinant as a polynomial in \( x \). It is of degree \( m \) and does not vanish identically; its roots are \( \kappa_1, \ldots, \kappa_m \); hence, \( \kappa_1 = \infty \) equals to one of these roots and this implies that the two sets \( \{ \kappa_1, \ldots, \kappa_m \} \) and \( \{ \kappa_1, \ldots, \kappa_m \} \) are identical, i.e. the two measures \( \mu \) and \( \mu' \) have the same support \( \{ \kappa_1, \ldots, \kappa_m \} \). Their values \( \mu_j \) carried by the points \( \kappa_j \) have to satisfy the linear system

\[
\sum_{j=1}^{m} \kappa_j^k \mu_j = c_k, \quad \lambda = 1, \ldots, n.
\]

Since the matrix \( (\kappa_j^k)_{j=1, \ldots, m} \) has rank \( m \), the point \( \zeta^0 = (c_1, \ldots, c_n) \) uniquely determines the weights \( \mu_j \) and this implies \( \mu^1 = \mu \). We conclude that for a boundary point \( \zeta^0 \) of \( C_n \) there is a unique function \( g \) in \( C \) such that \( \gamma_n(g) = \zeta^0 \), and this completes the proof of Theorem A.

**3.5.** The lemma we used in Paragraph 2.3 easily follows by a similar argument as used just ahead. Let a supporting hyperplane to \( C_n \), with normal direction \( \pi \), be given. If the polynomial \( h(\pi) - \text{Re} \sum_{j=1}^{n} \pi_j \zeta^j \) has the zeros \( \kappa_1, \ldots, \kappa_m \) on the unit circumference, then the intersection of \( C_n \) with the given supporting hyperplane is the convex hull of the points \( \zeta^j \), \( j = 1, \ldots, m \), where the \( \zeta^j \) are given by (3.7). Furthermore, let the coefficients \( c_1, \ldots, c_q \) be given. With \( c_{-k} = c_k \), \( k = 1, \ldots, q \) and \( c_0 = 1 \) they have to satisfy the equations

\[
\sum_{j=1}^{m} \kappa_j^k \mu_j = c_k, \quad k = 0, \pm 1, \ldots, \pm q,
\]

because the \( \mu_j \) are real. The matrix \( (\kappa_j^k)_{j=1, \ldots, m} \) has rank \( m \) if \( 2q + 1 \geq m \). This shows that the coefficients \( c_1, \ldots, c_q \) uniquely determine the point \( (c_1, \ldots, c_q) \) on the supporting hyperplane with the given normal direction \( \pi \). Thus we proved

**Lemma 3.5.** Let a given supporting hyperplane to \( C_n \) touch the curve

\[ \Gamma: \theta \mapsto (e^{i0}, e^{2i0}, \ldots, e^{ni0}), \quad 0 \leq \theta \leq 2\pi, \]

in \( m \) distinct points, \( 1 \leq m \leq n \). A point \( (c_1, \ldots, c_q) \) of \( C_n \) on this hyperplane is then uniquely determined by its coordinates \( c_1, \ldots, c_q \), if \( 2q + 1 \geq m \).
References


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