ON THE GREEN'S FUNCTION OF FUCHSIAN GROUPS

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1. Introduction

Let \( \Gamma \) be a Fuchsian group, that is a discontinuous group of Möbius transformations

\[
\gamma(z) = e^{i\alpha} \frac{a - z}{1 - \overline{a}z} \quad (0 \leq \alpha < 2\pi, \quad |a| < 1)
\]

of the unit disk \( D = \{ |z| < 1 \} \) onto itself. For simplicity we assume throughout the paper that \( 0 \) is not an elliptic fixed point. Let \( \iota \) denote the identity \( \iota(z) = z \) and let

\[
F = \{ z \in D : |\gamma'(z)| < 1 \text{ for all } \gamma \in \Gamma, \quad \gamma \neq \iota \}
\]

denote the normal fundamental domain with respect to \( 0 \).

A character of \( \Gamma \) is a complex-valued function \( v(\gamma) \) satisfying

\[
v(\varphi \circ \gamma) = v(\varphi) v(\gamma), \quad |v(\gamma)| = 1 \quad (\varphi, \gamma \in \Gamma).
\]

An analytic function \( f(z) \ (z \in D) \) is called character-automorphic if

\[
f(\gamma(z)) = v(\gamma) f(z) \quad (\gamma \in \Gamma)
\]

for some character \( v \) of \( \Gamma \). This is true if and only if \( |f(\gamma(z))| = |f(z)| \) for all \( \gamma \in \Gamma \).

We assume now that \( \Gamma \) is of convergence type, that is

\[
\sum_{\gamma \in \Gamma} (1 - |\gamma(z)|^2) = (1 - |z|^2) \sum_{\gamma \in \Gamma} |\gamma'(z)| < \infty \quad (z \in D).
\]

Then the Green's function of \( \Gamma \) with respect to \( 0 \) is defined as the Blaschke product (compare (1.1))

\[
g(z) = \prod_{\gamma \in \Gamma} [e^{-i\theta(\gamma)} \gamma(z)] \quad (\theta(\gamma) = \arg \gamma(0), \quad \theta(\iota) = 0);
\]

see Poincaré [15], Myrberg [13] and Nevanlinna [14, p. 214]. We have

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The Green's function is character-automorphic and satisfies $g(0) = 0$, $|g(z)| < 1 \ (z \in D)$, and if $f(z)$ is any function with these properties then $|f(z)| \leq |g(z)|$. Projecting $-\log |g(z)|$ to the Riemann surface $D / \Gamma$, we obtain the Green's function of $D / \Gamma$, the smallest positive harmonic function with a logarithmic pole at a certain point.

We say that an analytic function $f(z) \ (z \in D)$ has the angular limit $f(\zeta)$ at $\zeta \in \partial D$ if $f(z) \to f(\zeta)$ as $z \to \zeta$ in every Stolz angle at $\zeta$. The angular limit of the derivative is called the angular derivative and is denoted by $f'(\zeta)$ if it exists. The function is called of bounded characteristic ("beschränktartig") if

$$
\frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \leq K \quad (0 < r < 1).
$$

This is true if and only if $f(z)$ is the quotient of two bounded analytic functions [14, p. 189].

The Green's function $g(z)$ of $\Gamma$ has angular limits with $|g(\zeta)| = 1$ for almost all $\zeta \in \partial D$. Considering the angular derivative we define:

(a) $\Gamma$ is of accessible type if $g'(\zeta)$ exists on a set of positive measure on $\partial D$;
(b) $\Gamma$ is of fully accessible type if $g'(\zeta)$ exists almost everywhere on $\partial D$;
(c) $\Gamma$ is of Widom type if the function $g'(z)$ is of bounded characteristic in $D$.

Since every function of bounded characteristic has finite angular limits almost everywhere [14, p. 208], it is clear that (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

We shall give a number of characterizations of these concepts. In Theorem 1 we show, for instance, that

$$\Gamma \text{ is of accessible type } \iff \text{mes } (\partial F \cap \partial D) > 0.$$ 

In Theorems 2 and 3 we characterize groups of accessible type in terms of their Riemann surface $D / \Gamma$, using results of J. E. McMillan [11] on the angular derivative of univalent functions. We construct a new example of a group of convergence type that is not of accessible type (compare [19, p. 515]).

Let $H^\infty(\Gamma, v)$ denote the Banach space of bounded analytic functions satisfying (1.3) for the character $v$ of $\Gamma$. If $\Gamma$ is of Widom type and $g^*(z)$ is the inner factor in the canonical representation of $g'(z)$ [6, p. 25], we show (Theorem 5) that
On the Green's function of Fuchsian groups

\[ f(z) = \frac{g(z)}{g'(z)} \sum_{\gamma \in \Gamma} \frac{v(\gamma) h(\gamma(z)) g'(\gamma(z))}{\gamma(z)} \]
defines a bounded linear operator \( h \in H^\omega \mapsto f \in H^\omega(I', v) \); this is a modification of a construction of Earle and Marden [7, p. 206]. We give an explicit formula for \( g'(z) \) in Theorem 8. It follows from Theorem 7 and from the remarkable results of Widom [21] (see also [20]) that

\[ I' \text{ is of Widom type } \iff H^\omega(I', v) \neq \{ \text{const} \} \text{ for every } v \]

if \( I' \) has no elliptic elements. Hardy classes of regular Riemann surfaces of Widom type were also considered by Hasumi [10].

Our definition (c) was suggested by a paper of Ahern and Clark [1] on the angular derivative of Blaschke products. In Theorem 6 we show that

\[ \sum_{\gamma \in \Gamma} l(\gamma) = 2\pi, \quad \sum_{\gamma \in \Gamma} l(\gamma) \log \frac{2\pi}{l(\gamma)} < \infty \Rightarrow I' \text{ is of Widom type} \]

where \( l(\gamma) = \text{mes} \gamma(\partial F \cap \partial D) \), and in Theorem 7 that

\[ I' \text{ is of Widom type } \iff \sum_{\gamma \in \Gamma} |\gamma'(0)| \log \frac{1}{|\gamma'(0)|} < \infty. \]

2. Groups of accessible type

An oricycle at \( \zeta \in \partial D \) is a disk in \( D \) touching \( \partial D \) at \( \zeta \). We call \( \zeta \in \partial D \) an oricyclic point (with respect to \( I' \)) if every oricycle at \( \zeta \) contains only finitely many points \( \gamma(0) \) (\( \gamma \in \Gamma' \)); it is easy to deduce that, for each \( z \in D \), every oricycle contains only finitely many points \( \gamma(z) \) (\( \gamma \in \Gamma \)). This concept is motivated by the following lemma.

L e m m a 1. For every oricyclic point \( \zeta \), with at most countably many exceptions, there exists \( \gamma \in \Gamma' \) such that the normal fundamental domain \( \gamma(F) \) with respect to \( \gamma(0) \) is tangential to \( \partial D \) at \( \zeta \).

The domain \( H \subset D \) is called tangential to \( \partial D \) at \( \zeta \) if \( H \) contains every Stolz angle

\[ S = \left\{ z \in D : |\arg(1 - \overline{\zeta}z)| < \frac{\pi}{2} - \delta, \ |1 - \overline{\zeta}z| < \varrho \right\} \]

for \( \delta > 0 \) and some \( \varrho = \varrho(\delta) > 0 \).

T h e o r e m 1. Let \( I' \) be a Fuchsian group and let \( 0 < \beta \leq 2\pi \).

Then the following four conditions are equivalent:

(i) The normal fundamental domains \( \gamma(F) \) satisfy

\[ \sum_{\gamma \in \Gamma} \text{mes} [\partial D \cap \partial \gamma(F)] \geq \beta; \]
(ii) there exists a measurable set \( B \subseteq \partial D \) containing no two \( \Gamma \)-equivalent points such that
\[
\sum_{\gamma \in \Gamma} \meas \gamma(B) \geq \beta ;
\]
(iii) \( \Gamma \) is of convergence type and
\[
\meas \{ \zeta \in \partial D : g'(\zeta) \text{ exists} \} \geq \beta ;
\]
(iv) the set of oricyclic points has measure \( \geq \beta \).

The Fuchsian group \( \Gamma \) is called of accessible type if it satisfies the above (equivalent) conditions for some \( \beta > 0 \). We can replace (i) and (ii) by the more concise conditions

(i') \( \meas (\partial F \cap \partial D) > 0 \);

(ii') there exists a set of positive measure on \( \partial D \) that contains no two \( \Gamma \)-equivalent points.

The group \( \Gamma \) is called of fully accessible type if it satisfies the above conditions with \( \beta = 2\pi \). Every group of the second kind is of accessible kind as (i') shows, but need not be of fully accessible type as Example 2 will show.

Remark. One might attempt to "prove" (ii) for all groups as follows: We choose a representative in each \( \Gamma \)-equivalence class. Their union \( B \) contains no two \( \Gamma \)-equivalent points and satisfies
\[
\bigcup_{\gamma \in \Gamma} \gamma(B) = \partial D ,
\]
and this would seem to imply (ii) with \( \beta = 2\pi \). Unfortunately, the set \( B \) need not be measurable as the existence of groups not of accessible type shows.

We need the following result of Frostman [9] on Blaschke products; see also Ahern and Clark [1].

Lemma 2. Let \( \zeta \in \partial D \). If \( |g(\zeta)| = 1 \) and \( g'(\zeta) \neq \infty \) exist then
\[
|g'(\zeta)| = \sum_{\gamma \in \Gamma} |g'(\zeta)| .
\]
Conversely, if this sum converges then \( |g(\zeta)| = 1 \) and \( g'(\zeta) \neq \infty \) exist.

Cargo [3] has shown that, in the above case,
\[
g(z) \to g(\zeta) \text{ as } z \to \zeta \text{ in every oricycle at } \zeta .
\]

Proof of Theorem 1. (i) \( \Rightarrow \) (ii): It is sufficient to show that only countably many points on \( \partial F \cap \partial D \) can be \( \Gamma \)-equivalent to some other point on \( \partial F \cap \partial D \). Let \( \zeta, \zeta' \in \partial F \cap \partial D \) and \( \zeta = \gamma(\zeta') \) for some \( \gamma \in \Gamma, \gamma \neq \iota \). Since \( F \) is n.e. (= non-euclidean) convex and contains a disk around 0, it is easy to see that the radial segments \([0, \zeta] \) and
[0, ζ'] lie in F. Hence the n.e. segment from γ(0) to γ(ζ') = ζ lies in γ(F'), and we deduce that the n.e. bisector of [0, γ(0)] also ends at ζ. There exist only countably many such bisectors and thus only countably many such points ζ.

(ii) ⇒ (iii): It is easy to deduce from (ii) [19, p. 514] that Γ' is of convergence type. Furthermore, (ii) implies

\[ \beta \leq \sum_{\gamma \in \Gamma} \text{mes } \gamma(B) = \sum_{\gamma \in \Gamma} \int_B |\gamma'(\xi)| \, |d\xi| = \int_B \left( \sum_{\gamma \in \Gamma} |\gamma'(\xi)| \right) |d\xi|. \]

Hence the sum (2.2) converges almost everywhere on B, and Lemma 2 shows that g'(ζ) ≠ ∞ exists almost everywhere on B and therefore almost everywhere on \( \cup_{\gamma \in \Gamma} \gamma(B) \). It follows from (ii) that this is a disjoint union and that it has measure \( \geq \beta \). We remark that, by (2.4) and (2.2),

\[ \int_B |g'(\xi)| \, |d\xi| = \sum_{\gamma \in \Gamma} \text{mes } \gamma(B). \]

(iii) ⇒ (iv): It is sufficient to show that ζ ∈ ∂D is an oricyclic point if \( |g(\xi)| = 1 \) and \( g'(\xi) \neq \infty \) exist. Lemma 2 shows that, under these conditions,

\[ \sum_{\gamma \in \Gamma} |\gamma'(\xi)| = |g'(\xi)| < \infty. \]

If \( \varepsilon > 0 \) it follows that, for some finite subset \( \Gamma_0 = \Gamma_0(\varepsilon) \) of Γ,

\[ \frac{1 - |\alpha|^2}{|\xi - \alpha|^2} = |\gamma'(\xi)| < \varepsilon \quad \text{for } \gamma \in \Gamma \setminus \Gamma_0 \]

where we use the notation (1.1). This is our assertion because \( a = \gamma^{-1}(0) \) and because all oricycles at ζ have the form

\[ \left\{ \frac{1 - |z|^2}{|\xi - z|^2} = \text{Re} \frac{\xi + z}{\xi - z} \geq \varepsilon \right\} \quad (0 < \varepsilon < \infty). \]

(iv) ⇒ (i): This assertion follows at once from Lemma 1.

**Proof of Lemma 1.** Let ζ be oricyclic and not one of the countably many points where \( |\gamma_1'(\xi)| = |\gamma_2'(\xi)| \) for some \( \gamma_1 \neq \gamma_2 \). Then, by definition, every oricycle (2.7) contains only finitely many points \( a = \gamma^{-1}(0) \) (\( \gamma \in \Gamma \)). Hence (2.6) holds for some finite set \( \Gamma_0 = \Gamma_0(\varepsilon) \). It follows that, for \( \gamma \in \Gamma \setminus \Gamma_0 \),

\[ |\gamma'(z)| = \frac{1 - |\alpha|^2}{1 - |\alpha z|^2} \leq \varepsilon \frac{1}{|z| - 1/|\alpha|} \quad (z \in D). \]

Let \( \delta > 0 \) and let S be the Stolz angle (2.1). Since \( |1/\alpha| > 1 \) it is easy to deduce geometrically from (2.8) that \( |\gamma'(z)| < 1 \) for \( z \in S \) and \( \gamma \in \Gamma \setminus \).
\( I_0' \) if \( \varepsilon > 0 \) is sufficiently small. By the above property of \( \zeta \) we can choose \( \varphi(\delta) \) in (2.1) so small that \( |\gamma'_1(z)| \neq |\gamma'_2(z)| \) for \( z \in S \) and distinct \( \gamma_1, \gamma_2 \in I_0' \). Then there exists a unique \( \varphi \in I_0' \) such that

\[ |\gamma'(z)| < |\varphi'(z)| \quad (z \in S) \]

for \( \gamma \in I_0', \gamma \neq \varphi \). Since \( |\varphi'(z)| \geq \varepsilon'(z) = 1 \) this relation holds for all \( \gamma \in I', \gamma \neq \varphi \), and it follows that \( S \subseteq \varphi^{-1}(F) \).

3. Groups of accessible type and Riemann surfaces

We give first a characterization in terms of simply connected domains.

**Theorem 2.** The Fuchsian group \( \Gamma \) is of accessible type if and only if there exists a simply connected domain \( G \subset D \) containing no two \( \Gamma \)-equivalent points, such that \( \partial G \cap \partial D \) has positive harmonic measure relative to \( G \).

If \( z = \psi(s) \) maps \( \{ |s| < 1 \} \) conformally onto \( G \) and if \( A \) is the set of points \( e^{i\theta} \) where the angular limit \( \psi(e^{i\theta}) \) exists and satisfies \( |\psi(e^{i\theta})| = 1 \), then the last condition of Theorem 2 means that \( \text{mes } A > 0 \).

**Proof.** (a) Let \( \Gamma \) be of accessible type. Then we choose the normal fundamental domain \( F \) as \( G \). Since \( \partial F \) is a rectifiable Jordan curve and since \( \text{mes } (\partial F \cap \partial D) > 0 \) by Theorem 1 (i), it follows from Riesz' theorem [5, p. 50] that \( \text{mes } A > 0 \).

(b) Conversely, let the condition of the theorem be satisfied and let \( e^{i\theta} \in A \). Since \( |\psi(e^{i\theta})| = 1 > |\psi(s)| \), it is clear that \( \text{arg } (\psi(s) - \psi(e^{i\theta})) \) is bounded in \( s \in D \). Hence it follows from McMillan's twist point theorem [11, Th. 1] [16, p. 326] that the angular derivative \( \psi'(e^{i\theta}) \) exists and is \( \neq 0, \infty \) on a set \( A_0 \subset A \) with \( \text{mes } A_0 = \text{mes } A > 0 \). Another result of McMillan [11, Th. 2 (iii)] [16, p. 328] then shows that \( B_0 = \psi(A_0) \subset \partial D \) has positive measure. By a simple property of the angular derivative [16, p. 303], the domain \( G \) is tangential to \( \partial D \) at every \( \zeta \in B_0 \). This implies that the sets \( \gamma(B_0) \) (\( \gamma \in \Gamma' \)) are disjoint because the domains \( \gamma(G) \) are disjoint by the hypothesis of the theorem. Hence \( B_0 \) satisfies condition (ii) of our Theorem 1 and \( \Gamma \) is therefore of accessible type.

We turn now to necessary and sufficient (conformally invariant) conditions in terms of the Riemann surface \( D \setminus \Omega \) obtained by identifying \( \Gamma \)-equivalent points. We assume that \( \Gamma \) has no elliptic elements, so that \( D \) is (conformally equivalent to) the universal covering surface of \( D \setminus \Omega \). Then Theorem 2 states that \( \Gamma \) is of accessible type if and only if \( D \setminus \Omega \) contains a simply connected domain \( H \) such that \( \partial H \cap \partial (D \setminus \Omega) \) has positive harmonic measure.

In the next criterion, we allow multiply connected domains. The suf-
ficiency proof is based on a modification of the Lusin–Privalov construction due to McMillan [11]. We denote by \( A(E) \) the linear measure (one-dimensional Hausdorff measure) of \( E \subset C \).

**Theorem 3.** The Fuchsian group \( \Gamma \) without elliptic elements is of accessible type if and only if there exists a domain \( A \subset D \setminus \Gamma \) with the following properties:

1. there is a conformal map \( h \) of some plane domain \( H \) onto \( A \);
2. there is a set \( E \subset \partial H \) with \( A(E) > 0 \) such that, for every \( w \in E \), the interior \( T(w) \) of some equilateral triangle with apex \( w \) lies in \( G \);
3. if \( w_n \in H \), \( w_n \to w \in E \) \( (n \to \infty) \) then the points \( h(w_n) \in A \) have no limit point in \( D \setminus \Gamma \).

**Proof.** (a) Let \( \Gamma \) be of accessible type and let \( \psi(s) \) map \( D \) conformally onto \( F \). Since \( \partial F \) is a rectifiable Jordan curve and since \( \text{mes} (\partial F \cap \partial D) > 0 \), we can find a set \( A_0 \subset \psi^{-1}(\partial F \cap \partial D) \) with \( \text{mes} A_0 > 0 \) such that the angular derivative \( \psi'(s) \neq \infty \) exists for all \( s \in A_0 \) [5, p. 51], [16, p. 320]. Hence \( F \) is tangential to \( \partial D \) at each point \( w \in E = \psi(A_0) \) so that \( F \) contains a triangle, and \( \text{mes} A_0 > 0 \) implies \( A(E) = \text{mes} E > 0 \).

Since \( F \) contains no two \( \Gamma \)-equivalent points the projection \( h \) of \( D \) onto \( D \setminus \Gamma \) maps \( F \) (one-to-one) conformally onto some domain \( A \subset D \setminus \Gamma \). Finally let \( w_n \in F \), \( w_n \to w \in E \). We have \( |w_n| \leq |\gamma(w_n)| \) \( (\gamma \in \Gamma) \) by the definition of the normal fundamental domain. Since \( |w| = 1 \) it follows that \( (h(w_n)) \) has no limit point in \( D \setminus \Gamma \).

(b) Conversely, let the condition of the theorem be satisfied. We may assume that the triangle \( T(w) \) has the rational angle \( \alpha(w) \) at \( w \) and that its base lies on the (oriented) line \( L(w) \) of rational inclination and rational distance from \( 0 \). Since \( \{ (\alpha(w), L(w)) : w \in E \} \) is countable and since \( A(E) > 0 \), there exists \( E_0 \subset E \) with \( A(E_0) > 0 \) such that

\[
\alpha(w) = \alpha_0, \quad L(w) = L_0 \quad \text{for} \quad w \in E_0.
\]

The union of the domains \( T(w) \) \( (w \in E_0) \) has a connected component \( H_0 \) such that \( A(E_0 \cap \partial H_0) > 0 \). It follows from (3.1) that \( H_0 \) is simply connected and that \( A(\partial H_0) > 0 \). Let \( \varphi(s) \) map \( D \) conformally onto \( H_0 \). Then \( \varphi(s) \) is continuous in \( \bar{D} \), and

\[
A_0 = \{ e^{i\theta} : \varphi(e^{i\theta}) \in E_0 \cap \partial H_0 \} \subset \partial D
\]

satisfies \( \text{mes} A_0 > 0 \) [16, p. 322].

Since \( H_0 \subset H \) by property (2), we see from (1) that \( h(H_0) \) is a simply connected subdomain of \( A \subset D \setminus \Gamma \). Since \( \Gamma \) contains no elliptic elements the inverse \( p^{-1} \) of the projection maps \( h(H_0) \) conformally onto some simply connected domain \( G_0 \) containing no two \( \Gamma \)-equivalent points, and \( \varphi = p^{-1} \circ h \circ \varphi \) maps \( D \) conformally onto \( G_0 \). It follows from property
that \( \psi(A_0) \subset \partial D \). Hence \( \partial G_0 \cap \partial D \) has harmonic measure \( \geq \operatorname{mes} A_0 > 0 \), and we conclude from Theorem 2 that \( \Gamma \) is of accessible type.

We construct now an example. Let \( L_1 \) be an open arc on \( \partial D \); we allow \( L_1 = \emptyset \). We choose a countable set \( P \subset D \) with

\[
\overline{P} \cap \partial D = L_0 = \partial D \setminus L_1
\]

such that, at each \( w \in L_0 \), the symmetric Stolz angle of opening \( \pi / 2 \) contains infinitely many points of \( P \). Let \( \Gamma' \) be the Fuchsian group associated with the domain \( G = D \setminus P \). Thus \( G \) is conformally equivalent to \( D \setminus \Gamma \). Hence the projection map is an automorphic function \( f(z) \) \( (z \in D) \) with \( f(D) = G \) which is thus non-constant and bounded. In particular, it follows that \( \Gamma' \) is of convergence type. We may assume that \( f(0) = 0 \).

By Fatou's theorem the angular limit \( f(\zeta) \) exists for almost all \( \zeta \in \partial D \). We set

\[
(3.2) \quad E_j = \{ \zeta \in \partial D : f(\zeta) \in L_j \} \quad (j = 1, 2).
\]

Since all angular limits \( f(\zeta) \) lie on \( \partial G = L_0 \cup L_1 \cup P \) and since \( P \) has zero capacity, it follows [14, p. 209] that \( \operatorname{mes} E_0 + \operatorname{mes} E_1 = 2\pi \).

We show now that

\[
(3.3) \quad \operatorname{mes} \left[ \bigcup_{\gamma \in \Gamma} \partial \gamma(F) \cap E_j \right] = 0.
\]

Otherwise there would exist \( \gamma \in \Gamma \) such that \( \operatorname{mes} (E_0 \cap \partial \gamma(F)) > 0 \). Let \( \varphi(s) \) map \( D \) conformally onto \( \gamma(F) \). Since \( \partial \gamma(F) \) is rectifiable it follows from Riesz' theorem that \( A_0 = \varphi^{-1}(E_0 \cap \partial \gamma(F)) \) has positive measure. Now \( \psi(s) = f(\varphi(s)) \) maps \( D \) conformally onto \( f(\partial \gamma(F)) = f(F) \subset G \). It follows from (3.2) for every \( e^{i\theta} \in A_0 \) that \( \psi(s) \) tends to a limit on \( L_0 \) as \( s \to e^{i\theta} \) along a suitable arc. Since \( \psi \) is a bounded function it follows that the angular limit exists and satisfies \( \psi(e^{i\theta}) \in f(E_0) = L_0 \), in particular \( |\psi(e^{i\theta})| = 1 \). As in the proof of Theorem 2 we therefore deduce from McMillan's twist point theorem that \( f(F) \), and thus \( G \), is tangential to \( \partial D \) at some point of \( L_0 \). But this is false by our choice of \( P = D \setminus G \). Thus (3.3) has been proved.

If we choose \( L_1 = \emptyset \) then \( \operatorname{mes} E_0 = 2\pi \). It follows from (3.3) that \( \operatorname{mes} (\partial F \cap \partial D) = 0 \). Hence we have obtained (compare Tsuji [19, p. 515):

Example 1. There is a Fuchsian group not of accessible type for which there exists a non-constant bounded automorphic function and which is therefore of convergence type.

Let now \( L_1 \) be an arc of length \( \epsilon \) and let \( \omega(w) \) be the harmonic
measure of $L_1$ at $w$ relative to $D$. Then $\omega(f(z))$ is bounded and harmonic in $D$ and has, by (3.2), the angular limit 0 on $E_0$ and 1 on $E_1$. Therefore

$$\epsilon = 2\pi \omega(0) = 2\pi \omega(f(0)) = \int_{E_1} |d^*\zeta| = \text{mes } E_1.$$ 

Using (3.3) and the fact that $\text{mes } E_0 + \text{mes } E_1 = 2\pi$ we deduce that

$$\text{mes} \left[ \partial D \cap \bigcup_{\gamma \in \Gamma} \partial \gamma(F) \right] \leq \text{mes } E_1 = \epsilon.$$

Hence we have shown:

Example 2. For every $\epsilon > 0$, there is a Fuchsian group of the second kind (thus of accessible type) such that

$$\sum_{\gamma \in \Gamma} \text{mes} \left[ \partial D \cap \partial \gamma(F) \right] \leq \epsilon.$$ 

In particular the limit set of $\Gamma$ has measure $\geq 2\pi - \epsilon$.

4. Groups of Widom type

Let $\Gamma$ be a Fuchsian group of convergence type for which 0 is not an elliptic fixed point. We set

$$u(z) = \sum_{\gamma \in \Gamma} |\gamma'(z)| \quad (z \in \tilde{D}).$$

Then $u(z) > 1$ because $t \in \Gamma$. It follows from (1.5) that the Green’s function $g(z)$ satisfies

$$|g'(z)| = \left| g(z) \sum_{\gamma \in \Gamma} \gamma'(z) \right| \leq u(z) \quad (u \in D).$$

Theorem 4. The following three conditions are equivalent:

(i) $g'(z)$ is of bounded characteristic;

(ii) $\int_{\partial D} \log u(z) |dz| < \infty$;

(iii) there exists a character-automorphic function $g^*(z)$ with $g^*(0) \neq 0$ such that

$$|g^*(z)| \leq \frac{|g'(z)|}{u(z)} \leq 1 \quad (z \in D).$$

If (i) holds then we can choose $g^*$ as the inner factor of $g'$, so that

$$g'(z) = g^*(z) \exp \left[ \frac{1}{2\pi} \int_{\partial D} \frac{\zeta + z}{\zeta - z} \log u(\zeta) |d\zeta| \right] \quad (z \in D).$$
We say that $I'$ is of Widom type if it satisfies the above equivalent conditions; we shall describe the relation of our definition with Widom's work in Section 5. We deduce first a consequence of (iii):

**Theorem 5.** Let $I'$ be of Widom type and let $g^*(z)$ be the inner factor of $g'(z)$. If $v$ is any character of $I'$ and if $h(z)$ is analytic and bounded in $D$ then

$$f(z) = \frac{g(z)}{g'(z)} \sum_{\gamma \in I'} v(\gamma) g^*(\gamma z) h(\gamma z) \frac{\gamma'(z)}{\gamma(z)}$$

is analytic in $D$ and satisfies $f(\gamma(z)) = v(\gamma)f(z)$ ($\gamma \in I'$) and

$$\sup_{z \in D} |f(z)| \leq \sup_{z \in D} |h(z)|, \quad f(0) = g^*(0)h(0).$$

Thus (4.4) defines a bounded linear operator from $H^\infty$ into $H^\infty(I', v)$; compare [7]. If $v^*$ is the character associated with $g^*$ then we can write (4.4) as

$$f(z) = \frac{g(z)g^*(z)}{g'(z)} \sum_{\gamma \in I'} v(\gamma) v^*(\gamma) h(\gamma(z)) \frac{\gamma'(z)}{\gamma(z)}.$$

Metzger and Rajeswara Rao [12] have shown that this Poincaré theta series is $\neq 0$ if $h(z) \neq 0$ is a polynomial.

We only mention that (4.4) defines a bounded linear operator from the Hardy space $H^p$ into $H^p(I', v)$ for every $p \geq 1$ and that

$$\|f\|_p \leq \|h\|_p \quad (1 \leq p \leq \infty);$$

compare Earle–Marden [7] and Widom [21].

**Proof of Theorem 5.** Since $g(0) / \gamma(0) = 0$ for $\gamma \neq \iota$ and $= g'(0)$ for $\gamma = \iota$, we have $f(0) = g^*(0)h(0)$ because $v(i) = 1$. If $|h(z)| \leq M$ for $z \in D$ then, by (4.6), (4.1) and (iii),

$$|f(z)| \leq M \left| \frac{g^*(z)}{g'(z)} \right| \sum_{\gamma \in I'} \left| \frac{g(z)}{\gamma(z)} \right| |\gamma'(z)| \leq M \left| \frac{g^*(z)}{g'(z)} \right| u(z) \leq M.$$

In particular, we see that the series (4.4) converges absolutely and that $f(z)$ is analytic in $D$. For $\varphi \in I'$, we obtain from (4.4) and (1.5) that

$$f(\varphi(z)) \varphi'(z) = \frac{g(z)}{g'(z)} \sum_{\gamma \in I'} v(\gamma) g^*(\gamma \circ \varphi(z)) h(\gamma \circ \varphi(z)) \frac{\varphi'(\varphi(z))}{\gamma(\varphi(z))} = \frac{g(z)}{g'(z)} \sum_{\chi \in I'} v(\varphi) v(\chi) g^*(\chi(z)) h(\chi(z)) \frac{\chi'(z)}{\chi(z)} = v(\varphi)f(z).$$

**Proof of Theorem 4.** (a) Suppose that (i) holds. Since, by (2.2), $|g'(\xi)| = u(\xi)$ for almost all $\xi \in \partial D$, it follows [6, p. 17] that (ii) holds. Furthermore we can write [6, p. 25]
On the Green’s function of Fuchsian groups

419

(4.7) \[ g'(z) = g^*(z) \, w(z) \quad (z \in \mathcal{D}) \]

where the inner factor is \( g^*(z) \) and where the outer factor is given by the exponential in (4.3) because \( |g'(\zeta)| = u(\zeta) \) for almost all \( \zeta \). Hence

(4.8) \[
\log |w(z)| = \frac{1}{2\pi} \int_{\partial \mathcal{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \log u(\zeta) \, |d\zeta|.
\]

It follows from a well-known identity and from \( u(\zeta) = u(\gamma(\zeta)) \, |\gamma'(\zeta)| \) that, for \( \gamma \in \Gamma' \),

\[
\log |w(z)| = \frac{1}{2\pi} \int_{\partial \mathcal{D}} \frac{1 - |\gamma(z)|^2}{|\gamma(\zeta) - \gamma(z)|^2} \, |\gamma'(\zeta)| \, |\log u(\gamma(\zeta)) + \log |\gamma'(\zeta)|| \, |d\zeta|.
\]

If we substitute \( \zeta^* = \gamma(\zeta) \) and use the Poisson integral formula to evaluate the contribution from the second summand, we see that

\[
\log |w(z)| = \log |g(\gamma(z))| + \log |\gamma'(z)|.
\]

It follows that \( w(z) \) is character-automorphic, hence also \( g^*(z) \).

We write now \( \Gamma' = \{ \gamma_k : k = 1, 2, \ldots \} \) and

(4.9) \[ v_n(z) = \log \sum_{k=1}^{n} |\gamma_k'(z)| \quad (n = 1, 2, \ldots). \]

Computation shows that the Laplacian is

\[
\Delta v_n = -e^{-2\nu} \left| \sum_{k=1}^{n} \frac{|\gamma_k'|}{|\gamma_k|} \frac{|\gamma_k'|}{|\gamma_k|} \right| + e^{-\nu} \sum_{k=1}^{n} \frac{|\gamma_k'|^2}{|\gamma_k|^2}.
\]

Hence we obtain from Schwarz’s inequality that \( \Delta v_n \geq 0 \). Therefore \( v_n(z) \) is subharmonic in \( \mathcal{D} \), and it follows from (4.9) and (4.8) that

\[
v_n(z) \leq \frac{1}{2\pi} \int_{\partial \mathcal{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \, \nu_n(\zeta) \, |d\zeta| \leq \log |w(z)| \quad (z \in \mathcal{D}).
\]

If we let \( n \to \infty \) we obtain that \( \log u \leq \log |w| \) and thus, by (4.7), that \( u \leq |w| = |g'/g^*| \). Hence (iii) holds.

(b) Suppose now that (ii) holds. It is easy to deduce from (1.1) that \( |\gamma'(r \, z)| \leq 4 \, |\gamma'(z)| \) for \( |z| = 1, \, 0 \leq r < 1 \). Hence \( u(r \, z) \leq 4 \, u(z) \) by (4.1). Therefore it follows from (4.2) that, for \( 0 \leq r < 1 \),

\[
\frac{1}{2\pi} \int_{\partial \mathcal{D}} \log |g'(r \, z)| \, |dz| \leq \log 4 + \frac{1}{2\pi} \int_{\partial \mathcal{D}} \log u(z) \, |dz| < \infty.
\]

Thus (i) holds. This proof is due to Ahern and Clark [1, p. 118].
(c) Suppose finally that (iii) holds. Then
\[ |g^*(z)| \leq 1, \quad \left| \frac{g^*(z)}{g'(z)} \right| \leq \frac{1}{u(z)} \leq 1 \quad (z \in D) \]
so that \( g'(z) \) is the quotient of two bounded analytic functions and therefore of bounded characteristic [14, p. 189].

**Theorem 6.** If there exists a measurable set \( B \subseteq \partial D \) containing no two \( \Gamma \)-equivalent points such that, with \( l(\gamma) = \text{mes} \gamma(B) \),

\[ \sum_{\gamma \in \Gamma} l(\gamma) = 2\pi, \quad \sum_{\gamma \in \Gamma} l(\gamma) \log \frac{2\pi}{l(\gamma)} < \infty \]

then \( \Gamma \) is of Widom type.

**Proof.** We shall verify that condition (ii) of Theorem 4 is satisfied. It follows from the inequality between the geometric and arithmetic means that
\[
\exp \left( \frac{1}{l(\gamma)} \int_{\gamma(B)} \log u(z) \, |dz| \right) \leq \frac{1}{l(\gamma)} \int_{\gamma(B)} u(z) \, |dz| = \frac{1}{l(\gamma)} \int_{D} u(\zeta) \, |d\zeta| = \frac{2\pi}{l(\gamma)}
\]
where we have used (2.5). Hence, by (4.11),
\[
\int_{D} \log u \, |dz| = \sum_{\gamma \in \Gamma} \int_{\gamma(B)} \log u \, |dz| \leq \sum_{\gamma \in \Gamma} l(\gamma) \log \frac{2\pi}{l(\gamma)} < \infty.
\]

**Remark 1.** The conditions (4.10) may be related to Carleson sets [4]. These are closed sets \( E \subseteq \partial D \) for which
\[
\sum_{n} l_n = 2\pi, \quad \sum_{n} l_n \log \frac{2\pi}{l_n} < \infty
\]
where \( l_n \) are the lengths of the open arcs of which \( \partial D \setminus E \) is composed. The Carleson sets are the zero sets on \( \partial D \) of analytic functions with boundary values in \( \text{Lip} \, \alpha \) for some \( \alpha > 0 \). Their zero sets \( A \) in \( \overline{D} \) satisfy
\[ \int_{\partial D} \log \frac{1}{\text{dist}(z, A)} \, |dz| < \infty \]
as Taylor and Williams [18] have shown (I want to thank Dr. J. Stegbuchner for this reference). Since
\[
|\gamma'(z)| = \frac{1 - |a|^2}{|z - a|^2} \leq \frac{1 - |\gamma^{-1}(0)|^2}{\text{dist}(z, A)^2} \quad (\gamma \in \Gamma, \ z \in \partial D)
\]
it is clear that

\[(4.12) \quad A = \{ \gamma(0) : \gamma \in \Gamma \} \text{ satisfies (4.11) } \Rightarrow \Gamma \text{ is of Widom type.} \]

It will be proved in a forthcoming paper in the Michigan Mathematical Journal that \( \Gamma \) is of Widom type if the limit points of \( \Gamma \) form a Carleson set and if \( \Gamma \) has no elliptic elements.

Remark 2. In a manner similar to Theorem 6, one can show that, for \( 0 < p < 1 \),

\[(4.13) \quad \sum_{\gamma \in \Gamma} l(\gamma) = 2\pi, \quad \sum_{\gamma \in \Gamma} (\gamma')^{1-p} < \infty \Rightarrow g' \in H^p. \]

If \( \Gamma \) is finitely generated and of the second kind, Beardon [2] has proved that

\[\sum_{\gamma \in \Gamma} |\gamma'(0)|^{1-p} < \infty \quad \text{for some } p = p(\Gamma) > 0.\]

It is easily seen that \( |\gamma'(z)| \leq \text{const} \cdot |\gamma'(0)| \) holds on the free sides of \( F \), hence on \( B = \partial F \cap \partial D \) except for the parabolic vertices. Hence (4.13) shows that \( g' \in H^p \).

**Theorem 7.** If \( \Gamma \) is of Widom type then

\[(4.14) \quad \sum_{\gamma \in \Gamma} |\gamma'(0)| \log \frac{1}{|\gamma'(0)|} < \infty. \]

In a similar manner we can show that \( g' \in H^p \) implies \( \sum |\gamma'(0)|^{1-p} < \infty \). This estimate is stronger than the estimate of Ahern and Clark [1, p. 120] for general Blaschke products.

**Proof.** Let \( B = \partial F \cap \partial D \). There exists \( \alpha \) \((0 < \alpha < 1)\) such that \( B_0 = \{ z \in B : u(z) > e^\alpha \} \) has positive measure because \( u(z) > 1 \).

Since \( u(\gamma(z)) |\gamma'(z)| = u(z) \) we see that

\[(4.15) \quad \sum_{\gamma \in \Gamma} \int_{B_0} \left( z + \log \frac{1}{|\gamma'(z)|} \right) |\gamma'(z)| \, |dz| \leq \sum_{\gamma \in \Gamma} \int_{B_0} \log \frac{u(z)}{|\gamma'(z)|} \cdot |\gamma'(z)| \, |dz|

= \sum_{\gamma \in \Gamma} \int_{\gamma(B_0)} \log u(\zeta) \, |d\zeta| \leq \int_{\partial D} \log u(\zeta) \, |d\zeta| .\]

We set \( \xi(t) = t \left[ z + \log \left( 1/t \right) \right] \ (0 < t \leq 1) \). There is a unique \( t_0 \) with \( 0 < t_0 < 1 \) and \( \xi(t_0) = \xi(1) = \alpha \). It is easily verified that \( \xi(t_1) < \xi(t_2) \) for \( t_1 < t_0, \ 0 < t_1 \leq t_2 \leq 1 \). Since \( (1/4) |\gamma'(0)| \leq |\gamma'(z)| \leq 1 \) for \( z \in B \) and since \( (1/4) |\gamma'(0)| = t_0 \) for all but finitely many \( \gamma \in \Gamma \), we deduce that

\[\frac{1}{4} |\gamma'(0)| \left( z + \log \frac{4}{|\gamma'(0)|} \right) \leq |\gamma'(z)| \left( z + \log \frac{1}{|\gamma'(z)|} \right) .\]
We integrate over \( B_0 \). Since \( \text{mes } B_0 > 0 \) the assertion (4.14) follows from (4.15).

5. The inner factor of the derivative

Let \( \Gamma \) be a Fuchsian group of convergence type without elliptic elements, so that \( D \) is conformally equivalent to the universal covering surface of \( D / \Gamma \).

We need some results about the Green's function. For \( \zeta \in D \), we define the Green's function with respect to \( \zeta \) by

\[
g(z, \zeta) = \prod_{\gamma \in \Gamma} \left[ \frac{\gamma(z) - \zeta}{1 - \overline{\zeta} \gamma(z)} e^{-i\theta(\gamma)} \right], \quad \theta(\gamma) = \arg \frac{\gamma(0) - \zeta}{1 - \overline{\zeta} \gamma(0)}.
\]

It is character-automorphic and satisfies \( g(0, \zeta) > 0 \), \( |g(z, \zeta)| < 1 \) and \( g(z, 0) = g(z) \). We easily see that

\[
|g(z, \zeta)| = |g(\zeta, z)| \quad (z, \zeta \in D).
\]

In particular \( g(0, \zeta) = |g(\zeta)| \).

Let now \( \zeta \in \partial D \) be a parabolic fixed point of \( \Gamma \). Its stabilizer \( \Gamma_{\zeta} = \{ \varphi \in \Gamma : \varphi(\zeta) = \zeta \} \) consists of the elements

\[
g_n(z) = \frac{2z + in\beta(\zeta - z)}{2\zeta + in\beta(\zeta - z)} \quad (n = 0, \pm 1, \ldots)
\]

for some \( \beta = \beta(\zeta) > 0 \). Let \( R_{\zeta} \) denote a complete set of right coset representatives of \( \Gamma \) with respect to \( \Gamma_{\zeta} \). Thus we can write \( \Gamma \) as the disjoint union

\[
\Gamma = \bigcup_{\gamma \in R_{\zeta}} (\Gamma_{\zeta} \circ \gamma).
\]

Using the sin-product one can show that

\[
|g(z, r \zeta)| \to \exp \left[-\frac{2\pi}{\beta(\zeta)} \sum_{\gamma \in R_{\zeta}} \text{Re} \frac{\zeta + \gamma(z)}{\zeta - \gamma(z)} \right]
\]

as \( r \to 1 - 0 \), locally uniformly in \( D \). Hence we are led to define the Green's function with respect to the parabolic fixed point \( \zeta \) by

\[
g(z, \zeta) = \exp \left[-\frac{2\pi}{\beta(\zeta)} \sum_{\gamma \in R_{\zeta}} \left(\frac{\zeta + \gamma(z)}{\zeta - \gamma(z)} - i \text{Im} \frac{\zeta + \gamma(0)}{\zeta - \gamma(0)}\right) \right].
\]

This function is character-automorphic and satisfies \( 0 < |g(z, \zeta)| < 1 \) and \( g(0, \zeta) > 0 \). It follows from (5.2) and (5.5) that \( |g(r \zeta)| = g(0, r \zeta) \to g(0, \zeta) \) as \( r \to 1 - 0 \). Hence the angular limit \( g(\zeta) \) satisfies
On the Green's function of Fuchsian groups 423

(5.7) \[ |g(\zeta)| = g(0, \zeta) = \exp \left[ -\frac{2\pi}{\beta(\zeta)} \sum_{y \in \mathbb{R}} \frac{1 - |y(0)|^2}{|\zeta - y(0)|^2} \right]. \]

Since \( g'(z) / g(z) \) is of the form (1.5) we can write [8, p. 111]

(5.8) \[ (\zeta - z)^2 \frac{g'(z)}{g(z)} = \sum_{n=m}^{\infty} a_n \exp \left[ -\frac{2\pi n \zeta + z}{\beta \zeta - z} \right] \quad (a_m \neq 0), \]
a power series in the 'local uniformizer' \( \exp [-(2\pi / \beta)(\zeta + z)/(\zeta - z)] \).
The number \( m \) is the multiplicity of \( \zeta \).

The open set \( \{ z \in D : |g(z)| < r \} \quad (0 < r < 1) \) is invariant under \( \Gamma \).
Let \( G(r) \) be the component of 0 and let \( \Gamma(r) = \{ \gamma \in \Gamma : \gamma(G(r)) = G(r) \} \) be the stabilizer of \( G(r) \).

Theorem 8. Let \( \Gamma \) be a Fuchsian group of convergence type without elliptic elements. Then the following three conditions are equivalent:

(i) \( \Gamma \) is of Widom type;

(ii) the first Betti number \( b(r) \) of \( G(r) / \Gamma(r) \) satisfies

\[ \int_{0}^{1} b(r) r^{-1} \, dr < \infty; \]

(iii) \( \partial G(r) \cap \partial D \) consists of only finitely many equivalence classes of parabolic fixed points, and

\[ \prod_{k} |g(z_k)| > 0 \]

where \( z_k \) denotes a full system of non-equivalent zeros of \( g'(z) \) in \( D \) and of non-equivalent parabolic fixed points on \( \partial D \), each with proper multiplicity.

If \( \Gamma \) is of Widom type then the inner factor of \( g'(z) \) is given by

(5.9) \[ g^*(z) = \prod_{k} g(z, z_k). \]

The first Betti number of the Riemann surface \( G(r) / \Gamma(r) \) is the rank of the first singular homology group, in other words the maximal number of linearly independent elements in the abelianized group \( \Gamma(r) \). H. Widom [21, p. 305] proved that

(ii) \( \Leftrightarrow H^2(\Gamma, \nu) \neq \{ \text{const}\} \) for every character \( \nu \) of \( \Gamma \).

His results were expressed in terms of cross-sections of unitary line bundles which become character-automorphic functions by uniformization. We shall only need the following easier result:

Lemma 3 (Widom [21, p. 312]). We have
\[
\exp \int_0^1 b(r) r^{-1} \, dr = \sup_v \inf \{ \|f\|_\infty : f \in H^\infty(\Gamma, v), \ |f(0)| = 1 \} \]

where \( v \) runs through all characters of \( \Gamma \).

Proof of Theorem 8. (i) \( \Rightarrow \) (ii). Choosing \( h(z) \equiv 1 \) in Theorem 5 we obtain a function \( f \in H^\infty(\Gamma, v) \) with \( |f(z)| < 1, \ f(0) = g^* (0) \). Hence it follows from Lemma 3 that

\[
\exp \int_0^1 b(r) r^{-1} \, dr \leq \frac{1}{|g^*(0)|} < \infty.
\]

(ii) \( \Rightarrow \) (iii). It follows from (ii) that \( b(r) < \infty \) for every \( r < 1 \). Hence \( G(r) / \Gamma(r) \) is a compact bordered surface with at most finitely many punctures. The border components of \( G(r) \) have to lie in \( D \) (and not on \( \partial D \) because \( |g(z)| = 1 \) for almost all \( z \in \partial D \)); the punctures correspond to parabolic fixed points of \( \Gamma \), and of these there are only finitely many equivalence classes.

Furthermore \( b(r) \) is the number of equivalence classes of critical points and parabolic fixed points for which \( |g(z_k)| < r \). Hence

\[
\log \prod_k |g(z_k)| = \int_0^1 (\log r) \, db(r) = -\int_0^1 b(r) r^{-1} \, dr > -\infty.
\]

Thus (iii) holds.

We need a lemma to complete the proof. Let \( w_r(z) \) map \( D \) conformally onto the simply connected domain \( G(r) \) such that \( w_r(0) = 0, \ w_r'(0) > 0 \). Then

\[
\Phi(r) = \{ \varphi = w_r^{-1} \circ \gamma \circ w_r : \gamma \in \Gamma(r) \}
\]

is a Fuchsian group in \( D \).

Lemma 4. The Green's function \( g_r(z) \) of \( \Phi(r) \) with respect to 0 satisfies \( g_r(z) = r^{-1} g(w_r(z)) \).

Proof. The function \( r^{-1} g(w_r(z)) \) is character-automorphic with respect to \( \Phi(r) \) and is bounded by 1. Since the Blaschke product \( g_r(z) \) has the same zeros \( \varphi(0) \ (\varphi \in \Phi(r)) \) we see that

\[
q(z) = r^{-1} g(w_r(z)) / g_r(z)
\]

satisfies \( 0 < |q(z)| < 1 \). If \( D_0 \) is a sufficiently small disk around 0 then the disks \( \gamma(D_0) \ (\gamma \in \Gamma') \) are disjoint and \( |q(z)| > \alpha > 0 \) outside these disks. Since \( q(z) \neq 0 \) it follows from the minimum principle that

\[
|q(z)| > \alpha' > 0 \quad \text{for} \ z \in D.
\]
Finally it follows from (5.11) that $|q(z)| = 1$ for almost all $z \in \mathbb{D}$. Hence $q(z)$ is a bounded inner function, and its representation [6, p. 24] shows that (5.12) is impossible unless $|q(z)| = 1$ and therefore $q(z) = 1$.

(iii) $\Rightarrow$ (i). We conclude from (iii) and (5.7) that

$$\prod_{k} g(0, z_{h}) = \prod_{k} |g(z_{h})| > 0.$$ 

Hence it follows from the choice of $z_{h}$ that the functions

$$(5.13)\quad \tilde{g}(z) = \prod_{k} g(z, z_{h}), \quad h(z) = \tilde{g}(z) / g'(z)$$

are analytic in $D$. Let now $\zeta$ be a parabolic fixed point of multiplicity $m$. We see from (5.13), (5.6) and (5.8) that

$$(5.14)\quad |h(z)| \leq \frac{|g(z, \zeta)|^{m}}{|g'(z)|} = O(|\zeta - z|^{2}) \quad (z \rightarrow \zeta).$$

We consider again the group $\Phi(r)$ ($0 < r < 1$) defined by (5.10). Let $\Gamma(r)$ denote its normal fundamental domain. Let $\xi$ be a parabolic fixed point of $\Phi(r)$. Then $\xi = \varpi(\xi)$ is a parabolic fixed point of $\Gamma(r)$ and hence of $\Gamma$. Since some oricycle at $\xi$ belongs to $\varpi(r)$, the mapping function has a finite non-zero angular derivative $\varpi'(\xi)$ by a theorem of Carathéodory [16, p. 308]. Hence we conclude from (5.14) that

$$(5.15)\quad h(w_{r}(z)) / w_{r}(z) = O(|\xi - z|^{2}) \quad \text{as } z \rightarrow \xi \text{ in every angle.}$$

We consider now the subharmonic function

$$(5.16)\quad u_{r}(z) = \left| \frac{r h(w_{r}(z))}{w_{r}'(z)} \right| \sum_{\varphi \in \varpi(r)} |\varphi'(z)|.$$ 

Since [19, p. 517] [17, p. 636]

$$\sum_{\varphi \in \varpi(r)} |\varphi'(z)| = O((1 - |z|)^{-1}) \quad (|z| \rightarrow 1)$$

it follows from (5.15) and (5.16) that $u_{r}(z) \rightarrow 0$ as $z \rightarrow \xi$, $z \in \varpi(r)$. Hence we conclude from (iii) and (5.16) that $u_{r}(z)$ is continuous in $\varpi(r)$ and that $u_{r}(\xi) = 0$ for all parabolic fixed points $\xi$.

Since $u_{r}(\varphi(z)) = u_{r}(z)$ ($\varphi \in \varpi(r)$) we deduce that the subharmonic function $u_{r}(z)$ attains its maximum on the free sides of $\varpi(r)$ where, by Lemma 4 and by (5.13),

$$u_{r}(z) = \left| \frac{r h(w_{r}(z))}{w_{r}'(z)} \right| g_{r}'(z) = |h(w_{r}(z)) g'(w_{r}(z))| = |\tilde{g}(w_{r}(z))| \leq 1.$$ 

Hence $u_{r}(z) \leq 1$ for $z \in D$ and therefore, by (5.16),
\begin{equation}
(5.17) \quad \sum_{\varphi \in \Phi(r)} |\varphi'(z)| \leq \left| \frac{w'_r(z)}{r h(w_r(z))} \right| \quad (z \in D).
\end{equation}

We keep \( z \in D \) fixed and let \( r \to 1 - 0 \). Since the left-hand side of (5.17) contains only non-negative terms and since \( w_r(z) \to z \), \( w'_r(z) \to 1 \) we see from (5.10) and (5.13) that

\begin{equation}
(5.18) \quad u(z) = \sum_{\gamma \in I} |\gamma'(z)| \leq \frac{1}{|h(z)|} = \left| \frac{g'(z)}{g(z)} \right| \quad (z \in D).
\end{equation}

Hence condition (iii) of Theorem 4 is satisfied, so that \( I' \) is of Widom type.

To prove (5.9) we write the inner factor of \( g'(z) \) in the form [6, p. 24]

\begin{equation}
(5.19) \quad g^*(z) = g_0(z) \exp \left( -\frac{1}{2\pi} \int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right)
\end{equation}

where \( g_0(z) \) is a Blaschke product and \( \mu \) is a non-negative singular measure because \( |g^*| \leq 1 \). It follows from (5.1) and (5.13) that the contribution to \( g^*(z) \) from the zeros \( z_k \in D \) is equal to \( g_0(z) \). We see from (5.8) that \( \mu(\{\zeta\}) = 2\pi m(\zeta) / \beta(\zeta) \) where \( m(\zeta) \) is the multiplicity of the parabolic fixed point \( \zeta \). Hence (5.6) and (5.13) show that the contribution to \( g^*(z) \) from the parabolic fixed points is cancelled by a corresponding term in (5.19), and it follows that \( |g^*(z)| \leq |\tilde{g}(z)| \).

On the other hand, we obtain from (5.18) that, for \( 0 < \varrho < 1 \)

\[ \log \left| \frac{\tilde{g}(0)}{g^*(0)} \right| = \frac{1}{2\pi \varrho} \int_{|\zeta|=\varrho} \log \left| \frac{\tilde{g}}{g^*} \right| |dz| \leq \frac{1}{2\pi \varrho} \int_{|\zeta|=\varrho} \log \left| \frac{g'}{g^u} \right| |dz|. \]

Since \( |g'(z)| = u(z) \) and \( |g^*(z)| = 1 \) for almost all \( z \in \partial D \) and since \( u(0) \leq 4u(z) \) for \( z \in \partial D \), it is easy to show that the last integral tends to \( 0 \) as \( \varrho \to 1 - 0 \). It follows that \( |\tilde{g}(0)| / g^*(0) | \leq 1 \) and hence from \( |g^*| \leq |\tilde{g}| \) that \( |g^*| = |\tilde{g}| \), \( g^* = \tilde{g} \).

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On the Green's function of Fuchsian groups

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