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ON THE COUNTING FUNCTION FOR THE a-VALUES OF A MEROMORPHIC FUNCTION

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1. Introduction and results

1. We use the usual notation of the Nevanlinna theory. We let Σ denote the Riemann sphere. For any function f in the plane let $n(r) = \sup_{a \in \Sigma} n(r, a)$ be the maximum number of roots of the equation $f(z) = a$ in $|z| \leq r$, and let $A(r)$ be the average value of $n(r, a)$ as a moves over the Riemann sphere. Hayman [2] has proved that

$$1 \leq \liminf_{r \to \infty} n(r) / A(r) \leq e . \tag{1}$$

We shall consider the following problem of Hayman [3, Problem 1.16]: Can e in (1) be replaced by any smaller quantity and in particular by 1 ? We shall show by an example that e here cannot be replaced by 1 .

Theorem 1. *Let* $h(z) = 4(z-1)/(3z)$, $g_n(w) = 1 - (1/w)^{2^n}$, *and*

$$f(z) = \prod_{p=1}^{\infty} \frac{g_{2p-1}(h(-z/3^{2p-1}))}{g_{2p}(h(z/3^{2p}))} .$$

Then f *satisfies the condition* $\liminf_{r\to\infty} n(r)/A(r) \geq 80/79$.

2. Next we shall consider the following problem of Erdös (Hayman [1, Problem 1.25]): Does there exist a meromorphic function such that for every pair of distinct values a, b we have $\limsup_{r\to\infty} n(r, a)/n(r, b) = \infty$? We prove the following theorems.

Theorem 2. *There exists a meromorphic functton* f *such that for every pair* $a, b, a \neq b$, *we have* $\limsup_{r\to\infty} n(r, a)/n(r, b) = \infty$.

Theorem 3. *There exists an entire function* f *such that for every pair of distinct finite values* a *and* b, $\limsup_{r\to\infty} n(r, a)/n(r, b) = \infty$.

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3. If $a \in \Sigma$ and $b \in \Sigma$, the distance between a and b is defined to be the length of the shorter great circle arc on Σ joining a and b. This distance is denoted by $\delta(a, b)$. If $E \subset [1, \infty)$, we denote the logarithmic measure of E by $m_1(E) = \int_E dt / t$. If $E_r = E \cap [1, r]$, by the lower logarithmic density of E we mean $\liminf_{r \to \infty} m_1(E_r) / \log r$. Hayman and Stewart [4] have proved the following

Theorem A. *If f is meromorphic in $|z| < \infty$ and $\varepsilon > 0$, there exists a set E of r-values having positive lower logarithmic density on which $n(r) < (1 + \varepsilon) e A(r)$.*

Miles [5] has proved the following

Theorem B. *There exist absolute constants $K < \infty$ and $C \in (0, 1)$ such that if f is any nonconstant meromorphic function in $|z| < \infty$, there exists $E \subset [1, \infty)$ having lower logarithmic density at least C with the property that, if $\varepsilon > 0$, there exists $r_0 = r_0(\varepsilon)$ such that if $a_1, \ldots, a_q$ are elements of Σ with $\delta(a_i, a_j) \geq \varepsilon$ for $i \neq j$, then*

$$\sum_{j=1}^{q} |n(r, a_j) / A(r) - 1| < K$$

for all $r \in E$, $r > r_0(\varepsilon)$.

We shall show by an example that the characterization of the set E in Theorem A and in Theorem B is the best possible in the sense that the exceptional set of r-values may have positive lower logarithmic density.

Theorem 4. *Let $s > 10$ be an integer. The function*

$$f(z) = \prod_{n=1}^{\infty} (1 - z \exp\{-(2s)^n\})^{(-s)^n}$$

satisfies the condition $n(r, 0) / A(r^{1+1/(5s)}) > s / 2$ in a set B having lower logarithmic density at least $(2s)^{-4}$.

4. For $B \subset \Sigma$, we denote $n(r, B) = \sup_{w \in B} n(r, w)$. Rickman [6] has proved the following result.

Theorem C. *Given $M > 1$, there exists $K > 1$ such that if f is meromorphic in the plane with at least one asymptotic value a and B is a compact subset of Σ not containing the point a, then*

$$\limsup_{r \to \infty} n(r, B) / A(K r) \leq M. \tag{2}$$

We shall show by an example that if $M < 9 / 5$, the constant K in (2) cannot be replaced by 1.

Theorem 5. *Let $t_0 = 100$ and for $n \geq 1$ let $t_n = 4^{t_{n-1}}$ and $\log r_n = (t_n / t_{n-1}) \log (6 / 5)$. The entire function*

$$f(z) = \prod_{n=1}^{\infty} (1 - z / r_n)^{t_n}$$

satisfies the condition $\limsup_{r\to\infty} n(r, 0) / A(7r/6) \geq 9/5$.

Furthermore, we shall show that the set B in (2) cannot be replaced by Σ.

Theorem 6. *Let* $t_1 = 100$ *and* $t_n = 2^{t_{n-1}}$ *for* $n \geq 2$, *and let* $\log r_n = t_n$. *The entire function*

$$f(z) = \prod_{n=1}^{\infty} (1 + z / r_n)^{t_n}$$

satisfies the condition $\limsup_{r\to\infty} n(r) / A(Kr) = \infty$ *for every constant* $K \geq 1$.

2. Proofs

5. *Proof of Theorem* 1. We denote $r_n = 3^n$. The function h maps the circle $|z - 16/7| = 12/7$ onto the circle $|w| = 1$. Therefore we see easily that

$$f(z) \to A = \prod_{p=1}^{\infty} \frac{1 - (3/4)^{2^{2p-1}}}{1 - (3/4)^{2^{2p}}}$$

as $z \to \infty$ outside the union of the discs

$$C_n = \{ z : |z - 16(-1)^n r_n / 7| < 25 r_n / 14 \}.$$

Let $n_p(r, a)$ be the number of roots of the equation $f(z) = a$ in $C_p \cap \{ z : |z| \leq r \}$. We denote by D_ε the union of the discs $|w| < \varepsilon$, $|w - A| < \varepsilon$ and $|w| > 1/\varepsilon$. Let $\varepsilon > 0$. It follows from Rouché's theorem that $n_p(5 r_p, a) = 2^p$ for all large values of p and all $a \notin D_\varepsilon$.

From the properties of the function g_n we see that if p is sufficiently large, say $p \geq p_\varepsilon$, then

$$n_{2p}(3 r_{2p}, \infty) < (19/20) 2^{2p} \tag{3}$$

and if $a \notin D_\varepsilon$, then

$$n_{2p-1}(r, a) - n_{2p-1}(r, 0) < \varepsilon 2^{2p-1} \tag{4}$$

and

$$n_{2p}(r, a) - n_{2p}(r, \infty) < \varepsilon 2^{2p} \tag{5}$$

for every $r > 0$. Let $p > p_\varepsilon$ and $r_{2p} \leq r < r_{2p+1} = 3 r_{2p}$. Then we have

$$n(r, 0) = \sum_{k=1}^{2p-2} 2^k + n_{2p-1}(r, 0) + n_{2p+1}(r, 0) + 2^{2p}$$

and for $w \notin D_\varepsilon$

$$n(r\,,w) \;=\; \sum_{k=1}^{2p-2} 2^k + n_{2p-1}(r\,,w) + n_{2p+1}(r\,,w) + n_{2p}(r\,,w)\,.$$

Now it follows from (4) and (5) that $n(r\,,w) \le n(r\,,0) - 2^{2p} + n_{2p}(r\,,\infty) + 4\,\varepsilon\,2^{2p}$ and we see from (3) that $n(r\,,w) \le n(r\,,0)\,(1 - 1\,/\,80 + 4\,\varepsilon)$ for $w \notin D_\varepsilon$. If $r_{2p-1} \le r < r_{2p}$, we see in the same manner as above that $n(r\,,w) \le n(r\,,\infty)\,(79\,/\,80 + 4\,\varepsilon)$ for $w \notin D_\varepsilon$. Therefore we have $A(r) \le n(r)\,(79\,/\,80 + 4\,\varepsilon + 30\,\varepsilon^2)$ for all large values of r and we get $\liminf_{r\to\infty} n(r)\,/\,A(r) \ge 80\,/\,79$. Theorem 1 is proved.

6. *Proof of Theorem* 2. Let $q_1\,, q_2, \ldots$ be the sequence of all rational numbers on the segment $[0\,,2\pi]$. We denote by $\varepsilon_n(z)$ a function satisfying the condition $|\varepsilon_n(z)| < 1\,/\,n$. We choose a sequence t_n of positive integers such that $\lim_{n\to\infty} t_{n+1}\,/\,t_n = \infty$ and set

$$f(z) \;=\; z \prod_{n=1}^{\infty} f_n(z)$$

where

$$f_n(z) \;=\; \prod_{k=1}^{t_n} \frac{1 - z\,/\,r_{n,k}}{1 - z\,/\,(r_{n,k} + \delta_n)}\,.$$

Here $r_{n,k} = r_n \exp\{i\,k\,/\,(n^2\,t_n)\}$, $k = 1, \ldots, t_n$, $|\delta_n| = 1$ and $\arg \delta_n = \arg A_n - q_n$ where

$$A_n \;=\; \prod_{p=1}^{n-1} \prod_{k=1}^{t_p} (1 + \delta_p\,/\,r_{p,k})\,.$$

We assume that $r_n \to \infty$ as $n \to \infty$ so rapidly that

$$f(z) \;=\; (1 + \varepsilon_n(z))\,r_n\,A_n(r_{n,k} - z)\,/\,\delta_n \tag{6}$$

in every $C_{n,k}$: $|z - r_{n,k}| < r_n^{-1/2}$, $n \ge 2$, that there exists a finite limit $\lim_{n\to\infty} A_n = A \ne 0$, and that $f(z) \to \infty$ as $z \to \infty$ outside the union of the discs $C_{n,k}$.

Let $w \ne \infty$. It follows from Rouché's theorem that if n is sufficiently large, f takes the value w exactly once in every $C_{n,k}$. We choose an increasing sequence n_s such that $\lim_{s\to\infty} \arg \delta_{n_s} = \pi$. For large values of s we have

$$n(r_{n_s} - 1\,/\,2\,, w\,, f) \;=\; 1 + \sum_{k=1}^{n_s-1} t_k \;<\; 2\,t_{n_s-1}$$

and $n(r_{n_s} - 1\,/\,2\,,\infty\,, f) > t_{n_s}$. This implies that

$$\limsup_{r\to\infty} n(r\,,\infty)\,/\,n(r\,,w) \;=\; \infty\,.$$

Similarly, choosing the sequence n_s such that $\lim_{s\to\infty} \arg \delta_{n_s} = 0$, we see that $\limsup_{r\to\infty} n(r, w) / n(r, \infty) = \infty$.

Let a and b be finite, $a \neq b$, and $|a| \geq |b|$. For large values of n we choose $a_{n,k}$, $b_{n,k} \in C_{n,k}$ such that $f(a_{n,k}) = a$ and $f(b_{n,k}) = b$. It follows from (6) that

$$r_{n,k} - w_{n,k} = \delta_n w / ((1 + \varepsilon_n(w_{n,k})) A_n r_n) \tag{7}$$

for $w = a, b$. We denote

$$d_n = \frac{|a| - |a - b|^2 / (32 |a|)}{|A_n| r_n}.$$

We choose an increasing sequence n_s such that $\lim_{s\to\infty} q_{n_s} = \arg a$. Then $\lim_{s\to\infty} \arg (r_{n_s,k} - a_{n_s,k}) = 0$ and we see from (7) that $|a_{n_s,k}| < r_{n_s} - d_{n_s} < |b_{n_s,k}|$ for all large values of s. This implies that

$$\limsup_{r\to\infty} n(r, a) / n(r, b) = \infty.$$

If the sequence n_s is chosen such that $\lim_{s\to\infty} q_{n_s} = \pi + \arg a$, then we have $|b_{n_s,k}| < r_{n_s} + d_{n_s} < |a_{n_s,k}|$ for all large values of s. Therefore $\limsup_{r\to\infty} n(r, b) / n(r, a) = \infty$. This completes the proof of Theorem 2.

7. *Proof of Theorem* 3. Let q_n and $\varepsilon_n(z)$ be as in the proof of Theorem 2. We choose a sequence t_n of positive integers such that $\lim_{n\to\infty} t_n / t_{n-1}^4 = \infty$. For each n we choose δ_n, $1 < \delta_n < 1 + 1/n$, such that the polynomial

$$g(z) = \prod_{k=1}^{t_n^2} (1 - z / b_k)$$

where $b_k = \exp\{2 \pi i k / t_n^2\}$, $k = 1, \ldots, t_n$, and $b_k = \delta_n \exp\{2 \pi i k / t_n^2\}$ for $k = t_n + 1, \ldots, t_n^2$, satisfies the condition $g(z) = (1 + \varepsilon_{t_n}(z)) (1 - z^{t_n^2})$ in

$$\{ z : \pi / t_n^2 \leq \arg z \leq 2 (t_n + 1/2) \pi / t_n^2, \quad 1/2 \leq |z| \leq 2 \}.$$

We set

$$f(z) = \prod_{n=1}^{\infty} f_n(z) \text{ and } f_n(z) = \prod_{k=1}^{t_n^2} (1 - z / r_{n,k})$$

where $r_{n,k} = r_n \exp\{2 (p_n + k) \pi i / t_n^2\}$, $k = 1, \ldots, t_n$, and $r_{n,k} = r_n \delta_n \exp\{2 (p_n + k) \pi i / t_n^2\}$ for $k = t_n + 1, \ldots, t_n^2$. We denote

$$A_n = \prod_{s=1}^{n-1} \prod_{k=1}^{t_s^2} (-r_{s,k})^{-1} \text{ and } s_n = \sum_{s=1}^{n-1} t_s^2.$$

Here p_n is the smallest positive integer such that $\arg A_n z^{s_n} = q_n$ for some

z satisfying the condition $2\pi p_n / t_n^2 \leq \arg z < 2\pi (p_n + 1) / t_n^2$. We assume that $r_n \to \infty$ as $n \to \infty$ so rapidly that $f(z) \to \infty$ as $z \to \infty$ outside the union of the discs $C_{n,k}: |z - r_{n,k}| < 1 / r_n$, and for every $n \geq 2$, $r_n |A_n| > 1$ and $r_n (\delta_n - 1) > 1$. Furthermore, we may assume that

$$f(z) / f_n(z) = (1 + \varepsilon_{4n}(z)) A_n z^{s_n} \tag{8}$$

in $r_n / 2 < |z| < 2 r_n$ and

$$\begin{aligned} f_n(z) &= (1 + \varepsilon_{t_n}(z)) (1 - (z / r_n)^{t_n^2}) \\ &= t_n^2 (1 + \varepsilon_{4n}(z)) (r_{n,k} - z) / r_{n,k} \end{aligned} \tag{9}$$

in $C_{n,k}$, $1 \leq k \leq t_n$, $n \geq 2$.

Let $a \neq \infty$. If n is sufficiently large, say $n \geq n_0$, there exists exactly one point $a_{n,k} \in C_{n,k}$ such that $f(a_{n,k}) = a$. It follows from (8) and (9) that

$$r_{n,k} - a_{n,k} = a (1 + \varepsilon_n(a_{n,k})) / (|A_n| r_n^{s_n - 1} t_n^2 \exp \{i q_n\}) \tag{10}$$

for $n > n_0$, $k = 1, \ldots, t_n$.

Let a and b be finite, $a \neq b$. As in the proof of Theorem 2, we see from (10) that there exist arbitrarily large values of n such that for some ϱ_n, $r_n - 1 / r_n < \varrho_n < r_n + 1 / r_n$, we have $n(\varrho_n, a, f) > t_n$ and $n(\varrho_n, b, f) < 2 t_{n-1}^2$. Therefore $\limsup_{r \to \infty} n(r, a) / n(r, b) = \infty$, and Theorem 3 is proved.

8. *Proof of Theorem* 4. We denote $r_n = \exp \{(2s)^n\}$ and $\varrho_n = r_n^{1+1/(2s)}$. Let $n \geq 8$ be even. Let $|z| = \varrho_n$. We have $\log |1 - z / r_{n-1}| \geq \log r_n - 2 / r_n$ and $\log |1 - z / r_{n-2}| \leq (1 + 1 / (2s)) \log r_n$. Therefore we get

$$\log \left| \prod_{m=1}^{n-1} (1 - z / r_m)^{(-s)^m} \right| \leq -s^{n-1} (1 - 2 / s) \log r_n .$$

Furthermore, we have $\log |(1 - z / r_n)^{s^n}| \leq (3 s^{n-1} / 5) \log r_n$ and

$$\log \left| \prod_{m=n+1}^{\infty} (1 - z / r_m)^{(-s)^m} \right| \leq 1 .$$

Combining these estimates we see that $|f(z)| < 1 / r_n$ on the circle $|z| = \varrho_n$, if n is even, $n \geq 8$. Similarly, if $n \geq 9$ is odd then $|f(z)| > r_n$ on $|z| = \varrho_n$.

Let $p \geq 5$. It follows from Rouché's theorem that $n(\varrho_{2p}, a) = n(\varrho_{2p}, \infty)$ for $|a| \geq 1 / r_{2p}$, and if $|a| < 1 / r_{2p}$ then $n(\varrho_{2p}, a) \leq n(\varrho_{2p+1}, a) = n(\varrho_{2p+1}, 0)$. Therefore

$$A(\varrho_{2p}) \leq n(\varrho_{2p}, \infty) + n(\varrho_{2p}, 0) / r_{2p} < 2 s^{2p-1}$$

and we see that $n(r_{2p}, 0) > (s / 2) A(\varrho_{2p})$. This implies that for all

$$r \in B = \bigcup_{p=5}^{\infty} [r_{2p}, r_{2p}^{1+1/(5s)}]$$

we have $n(r, 0) / A(r^{1+1/(5s)}) > s / 2$. Clearly B has lower logarithmic density at least $(2 s)^{-4}$. Theorem 4 is proved.

9. *Proof of Theorem* 5. Let $n \geq 4$. We have $|f(z)| < 1 / r_n$ on $|z - r_n| = 3 r_n / 4$ and $|f(z)| > r_n$ on the circles $|z - r_n| = 11 r_n / 12$ and $|z| = r_n / 12$. Then it follows from Rouché's theorem that $n(r_n / 12, a) = n(r_n / 12, 0)$ for $|a| < r_n$. The function $\log f(z)$, $\arg f(11 r_n / 6) = 0$, is analytic in

$$D_n = \{ z : 3 r_n / 4 \leq |z - r_n| \leq 11 r_n / 12 , \operatorname{Re} z \geq r_n \}$$

and if $3 r_n / 4 \leq y \leq 11 r_n / 12$, then $\arg f(r_n + i y) > \pi t_n / 2$ and $\arg f(r_n - i y) < - \pi t_n / 2$. Therefore f takes every value a satisfying $1 / r_n < |a| < r_n$ at least $t_n / 2$ times in D_n. The disc $|z| \leq 7 r_n / 6$ does not contain any point of D_n, and we see that

$$n(7 r_n / 6, a) < t_n / 2 + \sum_{k=1}^{n-1} t_k$$

for $1 / r_n < |a| < r_n$. We have $M(2 r_n, f) < \min_{|z|=r_n^2} |f(z)|$ and therefore $n(2 r_n, a) < 2 t_n$ for any $a \in \Sigma$. Combining these estimates we get

$$A(7 r_n / 6) < t_n / 2 + \sum_{k=1}^{n-1} t_k + 4 t_n / r_n .$$

Because $n(r_n, 0) > t_n$, we see now that $n(r_n, 0) > (9 / 5) A(7 r_n / 6)$ for $n \geq 4$. Theorem 5 is proved.

10. *Proof of Theorem* 6. Let $K \geq 1$. Let $n \geq 4$ be so large that $s_n = r_n / (8 K) > 9 r_{n-1}$. We denote by D the bounded domain bounded by the lines $L_1 : \operatorname{Re} z = s_n$, $L_2 : \operatorname{Re} z = 3 s_n$, $L_3 : \operatorname{Im} z = s_n$, and $L_4 : \operatorname{Im} z = - s_n$. The boundary of D is denoted by Γ. The function $\log f(z)$, $\arg f(1) = 0$, is regular in $\operatorname{Re} z > 0$, and we see that $|\arg f(z)| > s_n t_n / (2 r_n)$ on $(L_3 \cup L_4) \cap \Gamma$, $|f(z)| < f(2 s_n)$ on $L_1 \cap \Gamma$, and $|f(z)| > f(2 s_n)$ on $L_2 \cap \Gamma$. Therefore $\log f(z)$ takes in D all values $\log f(2 s_n) + i y$, $|y| \leq s_n t_n / (2 r_n)$. This implies that

$$n(4 s_n) \geq t_n / (32 \pi K) . \tag{11}$$

We see easily that $f(z) / z \to \infty$ as $z \to \infty$ outside the union of the discs $C_n : |z + r_n| < r_n / 12$. If n is large then $n(r_n / 2, a) = n(r_n / 2, 0) < 2 t_{n-1}$ for $|a| < r_n$. Because $M(r_n, f) < \min_{|z|=r_n^2} |f(z)|$, we have $n(r_n, a) \leq n(r_n^2, 0) < 2\, t_n$ for every $a \in \Sigma$. Therefore

$$A(r_n / 2) < 2\, t_{n-1} + 8\, t_n / r_n < 3\, t_{n-1} \tag{12}$$

for all large values of n. Combining (11) and (12), we see that $\limsup_{r\to\infty} n(r) / A(K r) = \infty$. Theorem 6 is proved.

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