WEIERSTRASS DIVISION
WITH
QUASIANALYTIC BOUNDARY VALUES

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1. Introduction

Let \( g: \mathbb{R}^+ \to \mathbb{R}^+ \) (\( \mathbb{R}^+ \) is the set of nonnegative real numbers) be a convex increasing function such that \( g(0)=0 \) and \( t^{-1}g(t) \to +\infty \) as \( t \to +\infty \). Define a sequence \( \{M_n\}_{n \in \mathbb{Z}^+} \) (\( \mathbb{Z}^+ \) is the set of nonnegative integers) by \( M_n = \exp g(n) \), \( n \in \mathbb{Z}^+ \).

We assume \( g \) grows fast enough to ensure that \( M_n \geq n! \), \( n \in \mathbb{Z}^+ \).

Let \( \Omega \) be a domain in \( \mathbb{C}^k \) with 0 a point on its boundary \( \partial \Omega \). We denote by \( \mathcal{A}_k = \mathcal{A}_k(\{M_n\}, \Omega) \) the set of germs at 0 of complex-valued Whitney \( C^\infty \) functions \( f \) on \( \bar{\Omega} \) (the closure of \( \Omega \) which are analytic in \( \Omega \) and satisfy the following growth conditions on their derivatives: for each \( r > 0 \) sufficiently small that \( f \) is represented by a function on \( \bar{\Omega} \cap A_k(r) \) \( (A_k(r) = \{z \in \mathbb{C}^k : |z_j| = r, 1 \leq j \leq k\}) \) there exist constants \( A \) and \( B \), which depend in general on both \( f \) and \( r \) but not on \( n \in \mathbb{Z}^+ \), such that for all \( n \in \mathbb{Z}^+ \),

\[
\sup_{z \in (\mathbb{Z}^+)^k, |z| = n, z \in \partial \Omega \cap A_k(r)} |D^s f(z)| \leq AB^nM_n,
\]

(1.1) \( D^s = D_z^s = \frac{\partial |z|^s}{\partial z_1^{a_1} \cdots \partial z_k^{a_k}} \), where \( z = (z', z_k) = (z_1, \ldots, z_k) \) are coordinates on \( \mathbb{C}^k \), \( a = (a_1, \ldots, a_k) \in (\mathbb{Z}^+)^k \), and \( |z| = a_1 + \cdots + a_k \). We assume that \( \mathcal{A}_k \) is quasianalytic in the sense of Denjoy and Carleman:

\[
f \in \mathcal{A}_k \quad \text{and} \quad D^s f(0) = 0 \quad \text{for all} \quad a \in (\mathbb{Z}^+)^k \quad \text{imply} \quad f = 0 \in \mathcal{A}_k.
\]

Before going on, we remark that by the use of the logarithmic convexity of the sequence \( \{M_n\} \), it is not difficult to show that \( \mathcal{A}_k \) is a local algebra with maximal ideal \( m_k = \{f \in \mathcal{A}_k : f(0) = 0\} \). The quasianalyticity assumption is independent of the dimension \( k \). If the sequence \( \{M_n\} \) satisfies certain additional hypotheses, then \( \mathcal{A}_k \) is closed under composition whenever the composition makes sense, and \( \mathcal{A}_k \) is also closed under differentiation. For a more complete discussion, see [2].

In this paper we consider a quasianalytic local algebra \( \mathcal{A}_k(\{M_n\}) \). We show a Weierstrass—Malgrange—Mather type division theorem does not hold in \( \mathcal{A}_k(\{M_n\}) \).
if \( k \geq 2 \), \( A_k([n!]) \) is a proper subset of \( A_k(M_n) \), and \( b\Omega \) is \( C^2 \) smooth and strongly Levi pseudoconvex at 0. If, however, \( b\Omega \) is Levi pseudoflat at 0, we prove a generic division theorem holds in \( A_k(M_n) \), \( k \geq 2 \). We further show in this case that division is possible in \( A_k(M_n) \) by every regular element of \( \mathcal{O}_k \), the local algebra of germs at 0 of analytic functions. (The case in which \( b\Omega \) is pseudoconcave at 0 is trivial, since in this case \( A_k(M_n) \) reduces to \( \mathcal{O}_k \). See L. Hörmander, [4].)

2. Preliminaries

The following proposition is any easy consequence of the closed graph theorem:

**Proposition 2.1.** Let \( E \) be a Banach space and \( F = \bigcup_{n=1}^{+\infty} F_n \) be an inductive limit of Banach spaces. If \( T : E \to F \) is a continuous linear map, then there exists a positive integer \( n_0 \) such that \( T(E) \subseteq F_{n_0} \). \( \square \)

We will apply this proposition to estimate the derivatives of the quotient and remainder when we divide by a fixed regular element \( f \in A_k(M_n) \). The result will be that the growth of the derivatives of the element we are dividing by \( f \) determines the growth of the derivatives of the quotient and remainder.

For positive integers \( v \) and \( N \), let

\[
A_{k,v,N} = \left\{ f : f \text{ is a Whitney } C^\infty \text{ function on } \Omega \cap A_k(1/v) \right\}
\]

and

\[
\bigcup_{n \in \mathbb{Z}^+} \sup_{a \in (A^*)^n, |a| \leq n} |D^a f(z)|/N^n M_n < +\infty.
\]

Note that for all positive integers \( v \) and \( N \), \( A_{k,v,N} \) is a Banach space, and the inductive limit \( \bigcup_{v,N=1}^{+\infty} A_{k,v,N} \) may be identified with \( A_k(M_n) \).

Fix \( f \in A_k(M_n) \), which is regular in \( z_0 \) of order \( d \). (This means \( f(0) = \partial f(0)/\partial z_k = \ldots = \partial^{d-1} f(0)/\partial z_k^{d-1} = 0 \), while \( \partial^d f(0)/\partial z_k^d \neq 0 \).) Let \( v_0 \) be the smallest positive integer such that \( f \) is represented by a function on \( A_k(1/v) \) for all \( v \geq v_0 \). We define a map

\[
(g, r_1, \ldots, r_d) \to g = f + \sum_{j=1}^{d} r_j z_k^{-j},
\]

\[
\bigcup_{v=v_0}^{+\infty} \left( A_{k,v,N} \oplus \bigoplus_{N=1}^{+\infty} A_{k-1,v,N} \right) \to \bigcup_{v=v_0}^{+\infty} A_{k,v,N}.
\]

This map is continuous, for its restriction to each direct summand \( A_{k,v,N} \oplus (\bigoplus_{N=1}^{+\infty} A_{k-1,v,N}) \) is continuous. The assumption that the \( A_k(M_n) \) are quasianalytic implies the map is injective. The map is surjective if and only if division by \( f \) is pos-
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Given positive integers \( v \equiv v_0 \) and \( N \), Proposition 2.1 implies that there exist positive integers \( v' \equiv v_0 \) and \( N' \), as well as a constant \( A \), such that for each \( g \in A_{k,v,N} \), there exist \( q \in A_{k,v',N'} \), and \( r_j \in A_{k-1,v',N'} \), \( 1 \leq j \leq d \), which satisfy

\[
g = f_q + \sum_{j=1}^{d} r_j z_k^{d-j},
\]
and which also satisfy the estimates

\[
sup_{n \in \mathbb{Z}^+} \sup_{\beta \in (\mathbb{Z}^+)^N, |eta| \leq n, z \in \Omega \cap A_{k-1}(1/v)} |D^\beta r_j(z')|/(N')^n M_n^n \\ \leq A \sup_{n \in \mathbb{Z}^+} \sup_{\beta \in (\mathbb{Z}^+)^N, |eta| \leq n, z \in \Omega \cap A_k(1/v)} |D^\beta g(z)|/N^n M_n^n,
\]

\[
1 \leq j \leq d, \text{ and } \]

\[
sup_{n \in \mathbb{Z}^+} \sup_{\beta \in (\mathbb{Z}^+)^N, |eta| \leq n, z \in \Omega \cap A_k(1/v)} |D^\beta q(z)|/(N')^n M_n^n \\ \leq A \sup_{n \in \mathbb{Z}^+} \sup_{\beta \in (\mathbb{Z}^+)^N, |eta| \leq n, z \in \Omega \cap A_k(1/v)} |D^\beta g(z)|/N^n M_n^n.
\]

We summarize these results as a lemma.

\[\text{Lemma 2.2. Let } f \in \mathcal{A}_k(\{M_n\}) \text{ be regular in } z_k \text{ of order } d, \text{ and suppose that division by } f \text{ is possible within } \mathcal{A}_k(\{M_n\}). \text{ Let } v_0 \text{ be the smallest positive integer such that } f \text{ is represented by a function on } A_k(1/v) \text{ for all } v \equiv v_0. \text{ Given positive integers } v \equiv v_0 \text{ and } N, \text{ there exist positive integers } v' \equiv v_0 \text{ and } N', \text{ as well as a constant } A, \text{ such that for each } g \in A_{k,v,N}, \text{ there exist } q \in A_{k,v',N'} \text{ and } r_j \in A_{k-1,v',N'}, \text{ } 1 \leq j \leq d, \text{ which satisfy equation (2.1) and estimates (2.2) and (2.3).} \]

We will need one technical lemma, which we now state. The proof may be found in [2].

\[\text{Lemma 2.3. Let } \lambda(a) = \sup_{n \in \mathbb{Z}^+} |a^n|/M_n \text{ for } a \in \mathbb{C}, \text{ and suppose there exist } \varepsilon > 0, A > 0, \text{ and } C > 0 \text{ such that} \]

\[
\exp(\varepsilon a) \leq C \lambda(a), \quad a \in \mathbb{R}, \quad a > A.
\]

\[\text{Then there exist } \alpha > 0 \text{ and } \beta > 0 \text{ such that} \]

\[
M_n \leq \alpha \beta^n n!, \quad n \in \mathbb{Z}^+. \]

\[\square\]
3. The case of a strongly pseudoconvex boundary

Let \( \Omega \subseteq \mathbb{C}^k \) be a domain with \( C^2 \) smooth boundary \( \partial \Omega \), and assume \( \Omega \) is strongly pseudoconvex at \( 0 \in \partial \Omega \). Then there is an open neighborhood \( U \) of \( 0 \) in \( \mathbb{C}^k \) and a \( C^2 \) smooth function \( \varphi : U \rightarrow \mathbb{R} \) with the following properties:

\[
\varphi(0) = 0, \\
\text{grad } \varphi(0) \neq 0, \quad \text{and} \\
\text{the Levi form } (\partial^2 \varphi(0)/\partial z_i \partial \overline{z}_j)_{1 \leq i, j \leq k} \text{ is strictly positive definite.}
\]

After analytic change of coordinates in \( \mathbb{C}^k \), we may assume \( \varphi \) has the form

\[
\varphi(z) = \text{Im } z_k + \sum_{j=1}^k c_j |z_j|^2 + O(|z|^3),
\]

where \( c_j > 0 \) is constant, \( 1 \leq j \leq k \). (A proof of this fact may be found in Hörmander, [4].)

Let \( \mathcal{A}_k(\{n!\}) \) be a proper subset of \( \mathcal{A}_k(\{M_n\}) \), \( k \geq 2 \), a quasianalytic local algebra as defined in the Introduction. Set \( z' = (z_1, \ldots, z_{k-1}) \) and \( f(z) = f(z', z_k) = -z_k^2 + z_1 \). Then \( f \) is an analytic Weierstrass polynomial of degree two in \( z_k \). For \( a \in \mathbb{C} \), set \( g(z, a) = e^{iaz_k} \). Note that for each \( a \in \mathbb{C} \), \( g \in \mathcal{O}_k \subseteq \mathcal{A}_k(\{M_n\}) \). Suppose it were possible to write for each \( a \in \mathbb{C} \)

\[
(3.1) \quad g(z) = \text{Im } z_k + \sum_{j=1}^k c_j |z_j|^2 + O(|z|^3),
\]

where \( c_j > 0 \) is constant, \( 1 \leq j \leq k \). (A proof of this fact may be found in Hörmander, [4].)

Now consider only values of \( a \in \mathbb{R} \) with \( a < 0 \). If \( z \in \Omega \) and \( |z| \) is sufficiently small, it follows from equation (3.1) that \( \text{Im } z_k \leq 0 \). Thus, if \( a \in \mathbb{C} \), \( \varphi(2, a) = g(2, a) \in \mathcal{A}_k(\{M_n\}) \). Since the roots of \( f(z', z_k) = 0 \) are \( z_k = \pm i \sqrt{z_1} \), it would follow from equation (3.2) that

\[
(3.3) \quad r_1(z', a) = i(e^{a \sqrt{z_1}} - e^{-a \sqrt{z_1}})/2 \sqrt{z_1}.
\]

Thus, for each \( a < 0 \), \( g = g(z, a) \in \mathcal{A}_{k,1,1} \). If we apply Lemma 2.2, it follows that there exist \( e_1 > 0 \) and \( A_1 > 0 \), both independent of \( a \), such that

\[
(3.4) \quad \sup_{z' \in \mathcal{D} \cap \mathcal{A}_{k-1}(e_1)} |r_1(z', a)| \leq A_1 \lambda(a), \quad a < 0.
\]
Let $\varepsilon > 0$, $z' = (\varepsilon, 0, \ldots, 0)$, and $z_k = -i\varepsilon$. If $\varepsilon$ is chosen sufficiently small, then $z' \in A_{k-1}(\varepsilon)$ and $\varphi(z) = -\varepsilon + c_1 \varepsilon^2 + c_k \varepsilon^d + O(\varepsilon^3) < 0$, so that $z \in \overline{\Omega}$. Thus estimate (3.4) yields

$$|e^a \sqrt{\varepsilon} - e^{-a} \sqrt{\varepsilon}/2 \sqrt{\varepsilon} | \equiv A_1 \lambda(a), \quad a < 0.$$  

Since $(e^a \sqrt{\varepsilon} - e^{-a} \sqrt{\varepsilon})/2 \sqrt{\varepsilon}$ is asymptotic to $e^{-a} \sqrt{\varepsilon}/2 \sqrt{\varepsilon}$, inequality (3.5) implies there exist constants $A > 0$ and $K > 0$, both independent of $a$, such that

$$e^a \sqrt{\varepsilon} \equiv K \lambda(a), \quad a > A.$$  

Inequality (3.6) together with Lemma 2.3 now imply the existence of constants $\alpha > 0$ and $\beta > 0$ such that

$$M_n \equiv \alpha \beta^n n!, \quad n \in \mathbb{Z}^+.$$  

This implies $\mathcal{A}_k(\{M_n\}) = \mathcal{A}_k(\{n!\})$, contrary to assumption. We conclude the Weierstrass division theorem does not generalize to $\mathcal{A}_k(\{M_n\})$ when $\mathcal{A}_k(\{M_n\}) \nsubseteq \mathcal{A}_k(\{n!\})$ and $k \equiv 2$. Indeed, we have shown that it isn't always possible to divide in $\mathcal{A}_k(\{M_n\})$ by Weierstrass polynomials from $\mathcal{C}_k(\{z_k\})$.

### 4. The case of a pseudoflat boundary

Let $\Omega \subseteq \mathbb{C}^k$, $k \equiv 2$, be a product domain with 0 a member of the pseudoflat part of $\partial \Omega$. Thus, let $U_1 \subseteq \mathbb{C}$ be any plane domain with $0 \in \partial U_1$, let $U_j = \{z \in \mathbb{C}: |z| < 1\}$ be the open unit disc in the plane for $2 \equiv j \equiv k$, and let $\Omega = U_1 \times U_2 \times \ldots \times U_k$. Let $\mathcal{A}_k(\{M_n\})$ be a quasianalytic local algebra. We show in this section that a generic division theorem holds in $\mathcal{A}_k(\{M_n\})$. We also show that division is possible in $\mathcal{A}_k(\{M_n\})$ by every regular element of $\mathcal{C}_k$.

By a generic monic polynomial in $z_k$ of degree $d$ we mean an element in $\mathbb{C}[z_k]$ of the form $P_d(z_k, \lambda) = z_k^d + \sum_{j=1}^d \lambda_j z_k^{d-j}$, where $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$.

**Theorem 4.1.** (Generic division theorem for $\mathcal{A}_k(\{M_n\})$) Let $P_d = P_d(z_k, \lambda)$ be a generic polynomial in $z_k$ of degree $d$. For each $g \in \mathcal{A}_k(\{M_n\})$, there exists $\varepsilon > 0$ such that if $\lambda \in \Delta_d(\varepsilon)$, then there exist unique elements $q = q(z, \lambda) \in \mathcal{A}_k(\{M_n\})$ and $r_j = r_j(z', \lambda) \in \mathcal{A}_{k-1}(\{M_n\})$, $1 \equiv j \equiv d$, such that

$$g = P_d q + \sum_{j=1}^d r_j z_k^{d-j}. \quad (4.1)$$

Furthermore, all the germs in equation (4.1) are defined for $(z, \lambda) \in (\overline{\Omega} \cap \Delta_d(\varepsilon)) \times \Delta_d(\varepsilon)$ and are analytic in $(z, \lambda)$ on $(\overline{\Omega} \cap \Delta_d(\varepsilon)) \times \Delta_d(\varepsilon)$.

**Proof.** Choose $0 < r < 1$ so that the germ $g$ is defined on $\overline{\Omega} \cap \Delta_d(r)$. Let $0 < \delta < r$. By Cauchy's integral formula, if $z \in \overline{\Omega} \cap \Delta_d(\delta/2)$, then

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta| = \delta} \frac{g(z', \zeta)}{\zeta - z_k} d\zeta. \quad (4.2)$$
Observe that

\[ P_j(\zeta, \lambda) = \zeta^j + \sum_{i=1}^{j} \lambda_i \zeta^{j-i} = \zeta^j \left( \sum_{i=1}^{j-1} \lambda_i \zeta^{j-1-i} \right) + \lambda_j \]

\[ = \zeta P_{j-1}(\zeta, \lambda) + \lambda_j, \]

and so

\[ -\lambda_j = \zeta P_{j-1}(\zeta, \lambda) - P_j(\zeta, \lambda). \]

Thus

\[ P_d(\zeta, \lambda) - P_d(z_k, \lambda) = \zeta P_{d-1}(\zeta, \lambda) + \lambda_d - \sum_{j=1}^{d} \lambda_j z_k^{d-j} - z_k^d \]

\[ = \zeta P_{d-1}(\zeta, \lambda) + \sum_{j=1}^{d-1} (-\lambda_j) z_k^{d-j} - z_k^d \]

\[ = \zeta P_{d-1}(\zeta, \lambda) + \sum_{j=1}^{d-1} \left[ (\zeta P_{j-1}(\zeta, \lambda) - P_j(\zeta, \lambda)) z_k^{d-j} - z_k^d \right] \]

\[ = \sum_{j=1}^{d} \zeta P_{j-1}(\zeta, \lambda) z_k^{d-j} - \sum_{j=0}^{d-1} P_j(\zeta, \lambda) z_k^{d-j} \]

\[ = \left( \sum_{j=1}^{d} P_{j-1}(\zeta, \lambda) z_k^{d-j} \right)(\zeta - z_k). \]

Adding \( P_d(z_k, \lambda) \) to both sides of the identity we have obtained, viz.,

\[ P_d(\zeta, \lambda) - P_d(z_k, \lambda) = \left( \sum_{j=1}^{d} P_{j-1}(\zeta, \lambda) z_k^{d-j} \right)(\zeta - z_k), \]

and dividing through by \( P_d(\zeta, \lambda)(\zeta - z_k) \), we obtain

(4.3)

\[ \frac{1}{\zeta - z_k} = \frac{P_d(z_k, \lambda)}{P_d(\zeta, \lambda)(\zeta - z_k)} + \sum_{j=1}^{d} \frac{P_{j-1}(\zeta, \lambda)}{P_d(\zeta, \lambda)} z_k^{d-j}. \]

Now choose \( s > 0 \) such that \( \lambda \in \Delta_d(s) \) implies that the roots of \( P_d(z_k, \lambda) \) are contained in \( \Delta_4(\delta/2) \). If \( z \in \mathbb{Q} \cap \Delta_k(\delta/2) \) and \( \lambda \in \Delta_d(s) \), substitution of expression (4.3) for \( 1/(\zeta - z_k) \) into equation (4.2) yields

\[ g(z) = \left[ \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{g(\zeta', \zeta)}{P_d(\zeta, \lambda)(\zeta - z_k)} d\zeta \right] P_d(z_k, \lambda) \]

\[ + \sum_{j=1}^{d} \left[ \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{g(\zeta', \zeta) P_{j-1}(\zeta, \lambda)}{P_d(\zeta, \lambda)} d\zeta \right] z_k^{d-j}. \]
Thus, we get an equation of the form (4.1) with

\[
q(z, \lambda) = \frac{1}{2\pi i} \int_{|\zeta| = \delta} \frac{g(z', \zeta)}{P_d(\zeta, \lambda)(\zeta - z_k)} \, d\zeta
\]

and

\[
r_j(z', \lambda) = \frac{1}{2\pi i} \int_{|\zeta| = \delta} \frac{g(z', \zeta) P_{j-1}(\zeta, \lambda)}{P_d(\zeta, \lambda)} \, d\zeta, \quad 1 \leq j \leq d.
\]

Let \( \varepsilon = \min(\delta/2, s) \), and note \( |P_d(\zeta, \lambda)| \geq C > 0 \) and \( |\zeta - z_k| \geq \varepsilon > 0 \) for \( |\zeta| = \delta \) and \( \lambda \in \Delta_d(\varepsilon) \). We may thus differentiate under the integral sign in equation (4.4) and obtain that \( q(z, \lambda) \) is analytic in \( (z, \lambda) \) on \( (\Omega \cap \Delta_k(\varepsilon)) \times \Delta_d(\varepsilon) \). Also, since \( g(z) \) represents an element of \( \mathcal{A}_k(\{M_n\}) \) on \( \overline{\Omega} \cap \Delta_k(\delta) \), there exist \( A_1 > 0 \) and \( B_1 > 0 \) such that for all \( n \in \mathbb{Z}^+ \),

\[
\sup_{z \in (Z^*)^n, |z| \leq n, (z', \lambda) \in \Omega \cap \Delta_d(\delta)} |D^2 g(z)| \leq A_1 B_1^3 M_n.
\]

Thus,

\[
\sup_{z \in (Z^*)^n, |z| \leq n, (z', \lambda) \in \Omega \cap \Delta_d(\delta)} |D^2 g(z', \zeta)| \leq A_1 B_1^3 M_n.
\]

Since \( |P_d(\zeta, \lambda)| \) and \( |\zeta - z_k| \) are bounded away from 0 for \( |\zeta| = \delta \) and \( (z, \lambda) \in (\overline{\Omega} \cap \Delta_k(\varepsilon)) \times \Delta_d(\varepsilon) \), it follows that there exist \( A > 0 \) and \( B > 0 \), both independent of \( \zeta \) with \( |\zeta| = \delta \) and \( \lambda \in \Delta_d(\varepsilon) \), such that

\[
\sup_{z \in (Z^*)^n, |z| \leq n, (z', \lambda) \in \Omega \cap \Delta_k(\delta)} |D^2 [g(z', \zeta)/P_d(\zeta, \lambda)(\zeta - z_k)]| \leq AB^a M_n.
\]

We may thus differentiate under the integral sign in equation (4.4), estimate in a straightforward manner, and obtain that if \( \lambda \in \Delta_d(\varepsilon) \), then \( q = q(z, \lambda) \) represents an element of \( \mathcal{A}_k(\{M_n\}) \) on \( \overline{\Omega} \cap \Delta_k(\delta) \). A similar argument shows that if \( \lambda \in \Delta_d(\varepsilon) \), then \( r_j = r_j(z', \lambda) \) in equation (4.5) represents an element of \( \mathcal{A}_k - 1(\{M_n\}) \) on \( \overline{\Omega} \cap \Delta_k - 1(\varepsilon) \) which is analytic in \( (z', \lambda) \) on \( (\Omega \cap \Delta_k - 1(\varepsilon)) \times \Delta_d(\varepsilon) \) for \( 1 \leq j \leq d \).

Finally, to prove uniqueness, suppose that

\[
g = P_d q + \sum_{j=1}^{d} r_j z_k^{-d-j}
\]

\[
= P_d \tilde{q} + \sum_{j=1}^{d} \tilde{r}_j z_k^{-d-j},
\]

where \( q = q(z, \lambda) \), \( \tilde{q} = \tilde{q}(z, \lambda) \in \mathcal{A}_d(\{M_n\}) \) and \( r_j = r_j(z', \lambda) \), \( \tilde{r}_j = \tilde{r}_j(z', \lambda) \in \mathcal{A}_d - 1(\{M_n\}) \) for \( 1 \leq j \leq d \) and for each \( \lambda \in \mathbb{C}^d \) which is sufficiently small. Then for some \( \varepsilon > 0 \) and all \( (z, \lambda) \in (\overline{\Omega} \cap \Delta_k(\varepsilon)) \times \Delta_d(\varepsilon) \),

\[
\sum_{j=1}^{d} (r_j(z', \lambda) - \tilde{r}_j(z', \lambda)) z_k^{d-j} = P_d(z_k, \lambda)(\tilde{q}(z, \lambda) - q(z, \lambda)).
\]
Theorem 4.2. Let $f = f(z) \in \mathcal{O}_k$, $k \geq 2$, be regular in $z_k$ of order $d$. Then we may divide by $f$ in $\mathcal{A}_k(\{M_n\})$.

Proof. Since $f \in \mathcal{O}_k$ is regular in $z_k$ of order $d$, we may apply the Weierstrass preparation theorem in $\mathcal{O}_k$ to write

$$f = uP,$$

where $u \in \mathcal{O}_k$ is a unit and $P \in \mathcal{O}_{k-1}[z_k]$ is a Weierstrass polynomial in $z_k$ of degree $d$. Let $g \in \mathcal{A}_k(\{M_n\})$. If we can perform the division

$$g = Pq' + r',$$

where $q' \in \mathcal{A}_k(\{M_n\})$ and $r' \in \mathcal{A}_{k-1}(\{M_n\})[z_k]$, then we can obtain the division

$$g = fq + r$$

by taking $q = u^{-1}q' \in \mathcal{A}_k(\{M_n\})$ and $r = r' \in \mathcal{A}_{k-1}(\{M_n\})[z_k]$. Thus, we may assume $f = P$.

Choose a polydisc $\Delta_k(r)$ such that the germ $P$ is defined on $\Delta_k(r)$ and the germ $g$ is defined for $z \in \Omega \cap \Delta_k(r)$. Since $P$ is a Weierstrass polynomial in $z_k$, we can find numbers $\delta_j$ with $0 < \delta_j < r$, $1 \leq j \leq k$, such that $P(z) \neq 0$ if $|z_k| = \delta_k$ and $|z_j| \leq \delta_j$, $1 \leq j \leq k-1$. Let $\Delta_k(\delta) = \{z \in \mathbb{C}^k : |z_j| < \delta_j \text{ for } 1 \leq j \leq k\}$. For $z \in \Omega \cap \Delta_k(\delta)$, define

$$q(z) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_k} \frac{g(z', \zeta)}{P(z', \zeta)} \frac{d\zeta}{\zeta - z_k}$$

and

$$r(z) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_k} \frac{g(z', \zeta)}{P(z', \zeta)} \frac{P(z', \zeta) - P(z', z_k)}{\zeta - z_k} \frac{d\zeta}{\zeta - z_k}.$$

By the Cauchy integral theorem, if $z \in \Omega \cap \Delta_k(\delta)$, then

$$P(z)q(z) + r(z) = \frac{1}{2\pi i} \int_{|\zeta| = \delta_k} \frac{g(z', \zeta)}{\zeta - z_k} d\zeta = g(z).$$

Since $P$ is a Weierstrass polynomial in $z_k$ of degree $d$, $r$ is a polynomial in $z_k$ of degree at most $d-1$. We may differentiate under the integral signs in equations (4.6) and (4.7) and see that $q$ and $r$ are analytic in $z$ on $\Omega \cap \Delta_k(\delta)$. Since $g$ represents an element of $\mathcal{A}_k(\{M_n\})$ on $\Omega \cap \Delta_k(\delta)$, there exist $A_1 > 0$ and $B_1 > 0$ such that for all $n \in \mathbb{Z}^+$,

$$\sup_{x \in \Omega \cap \Delta_k(\delta)} |D^s g(z)| \leq A_1 B_1^n M_n.$$
Thus for all $\zeta$ with $|\zeta| = \delta_k$ and all $n \in \mathbb{Z}^+$,

$$\sup_{z \in (\mathbb{Z}^+)^{k-1}, |z| \leq n, z \in B \cap \Delta_{k-1}(\delta)} |D^k_z g(z', \zeta)| \equiv A_1 B^n M_n.$$ 

Since $P(z', \zeta) \neq 0$ for $|\zeta| = \delta_k$ and $|z_j| < \delta_j$, $1 \leq j \leq k-1$, $|P(z', \zeta)| \equiv C > 0$ for $|\zeta| = \delta_k$ and $|z_j| < \delta_j/2$. Let $\varepsilon = \min \left\{ \frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}, \ldots, \frac{\varepsilon_k}{2} \right\}$. It follows that there exist $A > 0$ and $B > 0$, both independent of $\zeta$ with $|\zeta| = \delta_k$, such that for all $n \in \mathbb{Z}^+$,

$$\sup_{z \in (\mathbb{Z}^+)^{k-1}, |z| \leq n, z \in B \cap \Delta_{k-1}(\varepsilon)} |D^k_z g(z', \zeta)| \equiv AB^n M_n.$$

We may now differentiate under the integral sign in equation (4.6) and estimate in a straightforward manner to see that $g$ represents an element of $\mathcal{A}_k(\{M_n\})$ on $\overline{D} \cap \Delta_k(e)$. A similar argument shows that $r$, given by equation (4.7), represents an element of $\mathcal{A}_{k-1}(\{M_n\})[z_k]$ on $\overline{D} \cap \Delta_k(e)$.

To prove uniqueness, suppose

$$g = Pq + r = P\bar{q} + \bar{r}.$$ 

Then for some polydisc $\Delta_k$ and all $z \in \overline{D} \cap \Delta_k$,

$$r(z) - \bar{r}(z) = P(z)(\bar{q}(z) - q(z)).$$

For $z'$ sufficiently small, $P(z', z_k)$ has exactly $d$ zeros, while $r(z) - \bar{r}(z)$ is a polynomial in $z_k$ of degree at most $d-1$. Hence $\bar{r} = r$ and $\bar{q} = q$. \(\square\)

**Corollary 4.3.** Let $k \geq 2$, $P \in \mathcal{O}_{k-1}[z_k]$ be a Weierstrass polynomial, and $f \in \mathcal{A}_k(\{M_n\})$. If $fP \in \mathcal{A}_{k-1}(\{M_n\})[z_k]$ is a polynomial, then $f$ is a polynomial, $f \in \mathcal{A}_{k-1}(\{M_n\})[z_k]$.

**Proof.** Since $fP$ and $P$ are polynomials in $z_k$ over $\mathcal{A}_{k-1}(\{M_n\})$, we may apply the algebraic division theorem for polynomials to write

$$fP = Pq + r,$$

where $q, r \in \mathcal{A}_{k-1}(\{M_n\})[z_k]$ and $r$ has degree less than the degree of $P$. By the uniqueness part of Theorem 4.2, $r = 0$ and $f = q$. Thus $f \in \mathcal{A}_{k-1}(\{M_n\})[z_k]$, as desired. \(\square\)

In closing, we mention one final application of Theorem 4.2. Let $R$ be a commutative ring with unit and $M$ be an $R$-module. $M$ is a flat $R$-module if for every exact sequence of $R$-modules $A \rightarrow B \rightarrow C$, the tensored sequence

$$A \otimes_R M \rightarrow B \otimes_R M \rightarrow C \otimes_R M$$

is also exact. It is possible to use Theorem 4.2 to establish that $\mathcal{A}_k(\{M_n\})$ is a flat ring extension of $\mathcal{O}_k$, $k \geq 2$. The details are so similar to those found in Nagel [7], however, that we choose to omit them.
References


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