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ELEMENTS OF LIPSCHITZ TOPOLOGY

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1. Introduction

1.1. A map f of a metric space (X, d) into a metric space (Y, d') is said to be a *Lipschitz map* if there is a constant L such that

$$d'(f(x), f(y)) \leqq Ld(x, y) \tag{1.2}$$

for all x, y in X. If every $x \in X$ has a neighborhood U such that $f|U$ is Lipschitz, f is said to be *locally Lipschitz* (abbreviated LIP). If X is compact, every LIP map of X is Lipschitz. We also let LIP denote the category of metric spaces and LIP maps. An isomorphism in the category LIP is called a *lipeomorphism*. Thus a lipeomorphism is a bijective map f such that both f and f^{-1} are LIP.

Lipschitz topology can be defined as the study of those properties of metric spaces which are invariant under lipeomorphisms. We shall be particularly interested in *Lipschitz manifolds:* metric spaces which are locally lipeomorphic to a euclidean space.

Let us compare LIP with three important categories: TOP=topological spaces and continuous maps, PL=polyhedra and PL maps, and DIFF=smooth manifolds and smooth maps. Without an essential loss of generality, we may assume

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that each polyhedron and each smooth manifold is embedded in a euclidean space. Then they are metric spaces, and PL and smooth maps are LIP. Thus we have the diagram

$$\begin{matrix} \mathrm{DIFF} \searrow & & \\ & \mathrm{LIP} \to \mathrm{TOP} \\ \mathrm{PL} \nearrow & & \end{matrix}$$

where each arrow is a forgetful functor. Alternatively, we could use locally metric spaces, see 3.4.

There is an extensive literature on the topology of manifolds in the categories TOP, PL, and DIFF. Lipschitz manifolds have been mentioned in some papers, especially in Whitehead [31], but no systematic treatment seems to be published so far.

In this paper we give the basic geometric tools needed in Lipschitz topology. These include the cone construction, extension and approximation of maps, general position and collaring. We also give some results concerning LIP embedding, the Schönflies problem, and the LIP Hauptvermutung. For example, we show that for $n \neq 4$, R^n has a unique LIP structure.

1.3. *Notation and terminology*. Our set-theoretical and topological notations are fairly standard. We let $A \setminus B$ denote the set-theoretical difference of A and B, and $\complement A$ is the complement of A in a given space. A singleton $\{x\}$ is usually written simply as x. The symbols ∂A and $\operatorname{int} A$ are used, somewhat ambiguously, for both manifolds and subspaces. A map is always continuous, a function need not be.

We let R^n denote the euclidean n-space, and R^n_+ is the closed upper half space $\{x \in R^n | x_n \geqq 0\}$. The standard orthogonal basis for R^n is written as $(e_1, \ldots, e_n)$. If $p<n$, we identify R^p with the subspace $R^p \times 0$ of R^n by $(x_1, \ldots, x_p) = (x_1, \ldots, x_p, 0, \ldots, 0)$. We shall use the euclidean norm $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ and the euclidean distance $d(x, y) = |x-y|$ in R^n. Given two sets A, B in R^n, AB is their rectilinear join. Given two vectors x, y in R^n, we let $x \cdot y$ denote their inner product, and

$$\operatorname{ang}(x, y) = \arccos \frac{x \cdot y}{|x|\,|y|}, \quad \operatorname{ac}(x, y) = \arccos \frac{|x \cdot y|}{|x|\,|y|}$$

are the angle and the acute angle between x and y, with the special conventions $\operatorname{ang}(x, y) = \pi$, $\operatorname{ac}(x, y) = \pi/2$, if $x=0$ or $y=0$.

When we are dealing with a metric space X, the letter d will stand for the metric of X. If we consider maps of X into another space Y, we shall use d' for the metric of Y. The distance between two non-empty sets A, B in X will be written as $d(A, B)$, and the diameter of A by $d(A)$ with $d(\emptyset)=0$. If S is a set and if $f, g\colon S \to X$ are functions, then $d(f, g) = \sup\{d(f(x), g(x)) | x \in S\}$ is the distance between f and g. For $a \in X$ and $r>0$, we let $B(a, r)$ denote the open ball $\{x | d(x, a) < r\}$. If $X = R^n$, we may use the notation $B^n(a, r)$ for $B(a, r)$ and the abbreviations $B^n(r) = B^n(0, r)$, $B^n = B^n(1)$. For spheres, we write $S^{n-1}(a, r) = \partial B^n(a, r) = \{x \in R^n | \, |x-a| = r\}$, $S^{n-1}(r) =$

$=S^{n-1}(0, r)$, and $S^{n-1}=S^{n-1}(1)$. The unit cube in R^n is $I^n=[-1, 1]^n$. Thus I^1 is different from the standard interval $I=[0, 1]$. We shall also use the symbol I for a general indexing set if there is no danger of misunderstanding. $\mathbf{Z}$ is the set of integers and $\mathbf{N}$ is the set of positive integers.

The definitions for a Lipschitz map, a LIP map (locally Lipschitz map) and a lipeomorphism were given in 1.1. The *Lipschitz constant* $\operatorname{lip} f$ of a Lipschitz map $f\colon X\to Y$ is the smallest number $L\geqq 0$ satisfying the condition (1.2). If $\operatorname{lip} f\leqq L$, f is said to be *L-Lipschitz.* If f is bijective and if both f and f^{-1} are L-Lipschitz, f is an *L-lipeomorphism.* If this is true for some $L\geqq 0$, f is a *strong lipeomorphism.* If $f\colon X\to Y$ is injective and f defines a lipeomorphism $f_1\colon X\to fX$, f is a LIP *embedding.* If f_1 is a strong lipeomorphism, f is a *Lipschitz embedding.* If every x in X has a neighborhood U such that $f|U$ is a LIP embedding, f is a LIP *immersion.* Two metrics d, d' in a space X are LIP *equivalent* or *Lipschitz equivalent* if $\mathrm{id}\colon (X, d)\to (X, d')$ is a lipeomorphism or a strong lipeomorphism, respectively.

We remark that Whitehead [31] has used the term "Lipschitz map" for a LIP map and the term "regular Lipschitz map" for a LIP immersion.

2. Basic properties of Lipschitz and LIP maps

2.1. We begin by stating some algebraic properties for the class of Lipschitz maps. The straightforward proofs will be omitted.

2.2. Lemma. *Suppose that $f\colon X\to Y$ and $g\colon Y\to Z$ are maps between metric spaces. If f is L_1-Lipschitz and g is L_2-Lipschitz, then gf is L_1L_2-Lipschitz. If f and g are* LIP, *then gf is* LIP. □

2.3. Lemma. *Let X be a metric space, let $f, g\colon X\to R^n$ and $\varphi, \psi\colon X\to R^1$ be Lipschitz, and let $c\in R^1$. Then $f+g$, cf, $|f|$, $\max(\varphi, \psi)$, and $\min(\varphi, \psi)$ are Lipschitz. If f and φ are bounded, then φf is Lipschitz. If, in addition, φ is bounded away from zero, then f/φ is Lipschitz.* □

2.4. Corollary. *Let X be a metric space, let $f, g\colon X\to R^n$ and $\varphi, \psi\colon X\to R^1$ be* LIP, *and let $c\in R^1$. Then $f+g$, cf, $|f|$, $\max(\varphi, \psi)$, $\min(\varphi, \psi)$, and φf are* LIP. *If $\varphi(x)\neq 0$ for all x, then f/φ is* LIP. □

2.5. Lemma. *Let X be a metric space and let $\emptyset\neq A\subset X$. Then the map $x\mapsto d(x, A)$ is 1-Lipschitz. If A and B are disjoint closed sets in X, there is a* LIP *map $f\colon X\to[0, 1]$ such that $A=f^{-1}(0)$, $B=f^{-1}(1)$.*

Proof. The first assertion is elementary and well-known (but extremely useful). To prove the second assertion, we may set $f(x)=d(x, A)/(d(x, A)+d(x, B))$ if $A\neq\emptyset\neq B$. If $B=\emptyset$ and $\emptyset\neq A\neq X$, choose $y\in X\setminus A$ and set $f(x)=d(x, A)/2(d(x, A)+d(x, y))$. If $B=\emptyset$ and $A=X$, set $f(x)=0$ for all x. If $A=B=\emptyset$, set $f(x)=1/2$ for all x. The case where $A=\emptyset\neq B$ is treated similarly. □

2.6. *Cone construction.* Let $a \in R^n$ and $Q \subset R^n$ be such that a and Q are *independent*, or equivalently, aQ is a *cone*. This means that $a \notin Q$ and no ray from a meets Q in more than one point. Each point $x \in aQ$ can be written as $x = \lambda a + \mu y$ with $y \in Q$, $0 \leqq \lambda \leqq 1$, $\lambda + \mu = 1$. This representation is unique for $x \neq a$. Given a map $f\colon Q \to R^p$ and a point $b \in R^p$, the *b-cone of f with vertex a* is the function $\bar{f}\colon aQ \to R^p$ defined by $\bar{f}(\lambda a + \mu y) = \lambda b + \mu f(y)$. In general, $\bar{f}$ need not be continuous. However, if Q is compact, then $\bar{f}$ is continuous. This is one reason why cones work best for compact sets. Observe also that a cone of a compact set is always compact, while a cone of a locally compact set is not usually locally compact at the vertex a. Even if Q is compact, a cone of a Lipschitz map $f\colon Q \to R^p$ need not be Lipschitz. For example, let $Q = \{(x, y) | 0 \leqq x \leqq 1, y = x^2\} \subset R^2$, $a = (1, 0)$, $b = 0 \in R^1$, and $f\colon Q \to R^1$ the constant map $f(x, y) = 1$. Then for $z_1 = (\varepsilon, 0)$ and $z_2 = (\varepsilon, \varepsilon^2)$ we have $|\bar{f}(z_1) - \bar{f}(z_2)|/|z_1 - z_2| = 1/\varepsilon$, and hence $\bar{f}$ is not Lipschitz.

2.7. Definition. Let $Q \subset R^n$ be compact, and let $a \in R^n$ be such that aQ is a cone. We say that a and Q are *Lipschitz independent* and aQ is a *Lipschitz cone* if for every Lipschitz map $f\colon Q \to R^p$ and for every $b \in R^p$, the b-cone $\bar{f}\colon aQ \to R^p$ of f is Lipschitz.

2.8. *Radial projection.* Let $a \in R^n$, and let $Q_1, Q_2 \subset R^n \setminus a$. Suppose that each ray from a through Q_1 meets Q_2 in exactly one point. Then the *radial projection* from Q_1 into Q_2 with center a is the unique function $f\colon Q_1 \to Q_2$ such that $f(x)$ belongs to the ray from a through x. If Q_2 is compact, then f is continuous but not necessarily LIP.

2.9. We are going to establish several equivalent conditions for Lipschitz independence. We first introduce some notation. If $a \in R^n$, $Q \subset R^n \setminus a$, $b \in Q$, we set

$$\beta(Q, a, b) = \lim_{r \to 0} \inf \{\mathrm{ac}(b - a, x - y) \,|\, x, y \in Q \cap B^n(b, r)\}. \tag{2.10}$$

It is easy to see that

$$\beta(Q, a, b) = \lim_{r \to 0} \inf \{\mathrm{ac}(x - a, x - y) \,|\, x, y \in Q \cap B^n(b, r)\}. \tag{2.11}$$

For $x, y \in R^n$ we set $s(x, y) = \{\lambda x + \mu y | \lambda + \mu = 1, \lambda\mu \leqq 0\}$. If $x = y$, then $s(x, y) = \{x\}$. If $x \neq y$, then $s(x, y)$ is the line through x and y less the open segment between x and y. For a set $Q \subset R^n$ we write $s(Q) = \cup \{s(x, y) | x \in Q, y \in Q\}$.

2.12. Lemma. *Let $p\colon R^n \setminus 0 \to S^{n-1}$ be defined by $p(x) = x/|x|$. Then $|p(x) - p(y)|^2 \leqq |x - y|^2/|x|\,|y|$ and $|p(x) - p(y)|\,|y| \leqq 2|x - y|$ for all x and y. Hence p is* LIP.

Proof. Using elementary estimates, we obtain $|x|\,|y|\,|p(x) - p(y)|^2 = 2|x|\,|y| - 2x \cdot y \leqq |x|^2 + |y|^2 - 2x \cdot y = |x - y|^2$ and $|p(x) - p(y)|\,|y| = \big|y - |y|x/|x|\big| \leqq |y - x| + \big||x| - |y|\big| \leqq 2|x - y|$. $\square$

2.13. Theorem. *Let aQ be a cone in R^n with Q compact. Then the following conditions are equivalent:*

(1) *aQ is a Lipschitz cone.*

(2) *If $c\colon Q \to R^1$ is the constant map $c(x)=1$, the 0-cone $\bar{c}\colon aQ \to R^1$ of c is Lipschitz.*

(3) $a \notin \overline{s(Q)}$.

(4) $\inf \{\beta(Q, a, b) | b \in Q\} > 0$.

(5) *$\beta(Q, a, b) > 0$ for all $b \in Q$.*

(6) *If $p\colon R^n \setminus a \to S^{n-1}$ is the map $p(x) = (x-a)/|x-a|$, $p|Q$ is a Lipschitz embedding.*

(7) *For every set $Q_1 \subset R^n$ such that $a \notin \bar{Q}_1$ and such that each ray from a through Q_1 meets Q in exactly one point, the radial projection $f\colon Q_1 \to Q$ is Lipschitz.*

Proof. We shall prove the implications (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6)⇒(1) and (6)⇒(7)⇒(6). We may assume that $a=0$. Choose positive numbers r and R such that $Q \subset B^n(R) \setminus B^n(r)$.

(1)⇒(2): Trivial.

(2)⇒(3): If (3) is false, there are points $x, y \in Q$ and $z \in s(x, y)$ with $|z| = d(0, s(x, y)) < (L+1/r)^{-1}$ where $L = \operatorname{lip} \bar{c}$. Now $|z| < r$ implies $x \neq z \neq y$. Hence z and $x-y$ are orthogonal. We may assume $|z-x| < |z-y|$. Letting u be the orthogonal projection of x on y, we obtain

$$\frac{|\bar{c}(x)-\bar{c}(u)|}{|x-u|} = \frac{|y|-|u|}{|y|\,|x-u|} = \frac{|y-z|}{|y|\,|z|} \geqq \frac{1}{|z|} - \frac{1}{|y|} > L,$$

which is a contradiction.

(3)⇒(4): If (4) is false, it follows from (2.11) that there are sequences of points $x_j \in Q$, $y_j \in Q$ such that $0 < |x_j - y_j| \to 0$ and $\alpha_j = \operatorname{ac}(x_j, x_j - y_j) \to 0$. For large j we have $|x_j - y_j| < r/\sqrt{2}$ and $\alpha_j < \pi/4$. Then $|x_j - y_j| < |x_j| \cos \alpha_j$, which implies $d(0, s(x_j, y_j)) = |x_j| \sin \alpha_j \leqq R \sin \alpha_j \to 0$. Hence $0 \in \overline{s(Q)}$.

(4)⇒(5): Trivial.

(5)⇒(6): By 2.12, $p|Q$ is Lipschitz. We must show that the inverse $q\colon pQ \to Q$ of $p|Q$ is LIP. Let $z \in pQ$. Then there is an open neighborhood U of $q(z)$ in Q and a positive number β such that $\operatorname{ac}(x, x-y) > \beta$ for all $x, y \in U$. Since $Q \setminus U$ is compact, the set $V = pU = pQ \setminus p[Q \setminus U]$ is an open neighborhood of z in pQ. If $x, y \in V$ and $\alpha = \operatorname{ac}(q(x), q(x)-q(y))$, then

$$|q(x)-q(y)| \leqq \frac{|q(y)|}{\sin \alpha} |x-y| \leqq \frac{R}{\sin \beta} |x-y|.$$

Hence q is LIP.

(6)⇒(1): Let $f\colon Q \to R^p$ be Lipschitz, let $b \in R^p$, and let $\bar{f}\colon aQ \to R^p$ be the b-cone of f. We must show that $\bar{f}$ is Lipschitz. We may assume that $b=0$. Let $q\colon pQ \to Q$ be the inverse of $p|Q$. Then $g = fq\colon pQ \to R^p$ is Lipschitz. Letting $\bar{g}$

and $\bar{p}$ denote the 0-cones of g and $p|Q$, respectively, with vertex 0, we have $\bar{f}=\bar{g}\bar{p}$. Hence it is sufficient to show that $\bar{p}$ and $\bar{g}$ are Lipschitz. Choose $L>0$ and $M>0$ such that q and g are L-Lipschitz and $|f(z)|\leqq M$ for all $z\in Q$. For $x, y\in 0Q\setminus 0$ we obtain

$$|\bar{p}(x)-\bar{p}(y)| = \frac{\big||q(p(y))|x-|q(p(x))|y\big|}{|q(p(x))|\,|q(p(y))|}$$

$$\leqq \frac{|q(p(y))|\,|x-y|}{r^2}+\frac{\big||q(p(y))|-|q(p(x))|\big|\,|y|}{r^2}$$

$$\leqq \frac{R}{r^2}|x-y|+\frac{L}{r^2}|p(x)-p(y)|\,|y|.$$

By 2.12, this implies $|\bar{p}(x)-\bar{p}(y)|\leqq(R+2L)r^{-2}|x-y|$. Since $\bar{p}$ is continuous, this also holds for $x=0$. Hence $\bar{p}$ is Lipschitz.

To show that $\bar{g}$ is Lipschitz let $x, y\in 0(pQ)$. If $x\neq 0\neq y$, we obtain $|\bar{g}(x)-\bar{g}(y)|=$ $=\big||x|g(p(x))-|y|g(p(y))\big|\leqq|x|\,|g(p(x))-g(p(y))|+|x-y|\,|g(p(y))|$. By 2.12, this implies $|\bar{g}(x)-\bar{g}(y)|\leqq(2L+M)|x-y|$. Since $\bar{g}$ is continuous, this also holds for $x=0$. Hence $\bar{g}$ is Lipschitz.

(6)$\Rightarrow$(7): The radial projection $f\colon Q_1\to Q$ can be written as $f=q(p|Q_1)$ where p is as in (6) and $q\colon pQ\to Q$ is the inverse of $p|Q$. By 2.12, $p|Q_1$ is Lipschitz. Hence f is Lipschitz.

(7)$\Rightarrow$(6): Trivial, since the inverse $q\colon pQ\to Q$ of $p|Q$ is a radial projection. □

2.14. Corollary. *If aQ is a Lipschitz cone and $Q_1\subset Q$ is compact, then aQ_1 is a Lipschitz cone.*

Proof. This follows, for example, from (2) of 2.13. □

2.15. *Extended cones.* If aQ is a cone in R^n, the corresponding *extended cone* $aQ\infty$ is defined as $\{\lambda a+\mu x|x\in Q,\ \lambda+\mu=1,\ \mu\geqq 0\}$. It consists of all rays from a through Q. If $f\colon Q\to R^p$ is a map and if $b\in R^p$, the extended b-cone of f with vertex a is the function $f^*\colon aQ\infty\to R^p$ defined by $f^*(\lambda a+\mu x)=\lambda b+\mu f(x)$.

2.16. Theorem. *If aQ is a Lipschitz cone, then every extended cone of every Lipschitz map $f\colon Q\to R^p$ is Lipschitz.*

Proof. We may assume that $a=0$ and $f^*(a)=0$. Since $f^*|aQ$ is the cone of f, it is L-Lipschitz for some L. Let $x, y\in aQ\infty$. Choosing $t>0$ so that tx and ty lie in aQ, we obtain $|f^*(x)-f^*(y)|=|t^{-1}f^*(tx)-t^{-1}f^*(ty)|\leqq L|x-y|$. □

2.17. Theorem. *Suppose that $A\subset R^n$ is a compact convex set and that $a\in\operatorname{int} A$. Then $a(\partial A)$ is a Lipschitz cone.*

Proof. Choose $r>0$ such that $B^n(a, r)\subset A$. Using the notation of 2.9 we see that $s(x, y)\cap B^n(a, r)=\emptyset$ for all $x, y\in\partial A$. The theorem follows from 2.13 (3). □ The result is well-known [16].

2.18. Theorem. *Let $P\subset R^n$ be a polyhedron. Then every* PL *map f: $P\to R^p$ is* LIP.

Proof. The theorem is obviously true if $\dim P=0$, and we proceed by induction on $\dim P$. Assume that the theorem is true for $\dim P<m$ and suppose $\dim P=m$. Let $a\in P$. Then there is an n-cube A, centered at a, such that $P\cap\partial A=Q$ is compact, $P\cap A=aQ$, and $f|P\cap A$ is a cone of $f|Q$. Since Q is a polyhedron with $\dim Q<m, f|Q$ is Lipschitz. By 2.17 and 2.14, aQ is a Lipschitz cone. Hence $f|P\cap A$ is Lipschitz. □

2.19. Theorem. *Let aQ be a cone in R^n such that Q is a compact polyhedron. Then aQ is a Lipschitz cone.*

Proof. Let c: $Q\to R^1$ be the constant map $c(x)=1$. Since c is PL, its 0-cone $\bar{c}$: $aQ\to R^1$ is also PL. By 2.18, $\bar{c}$ is Lipschitz. By 2.13 (2), aQ is a Lipschitz cone. □

2.20. *Standard mistakes.* The well-known standard mistake of PL topology is a radial projection of a compact polyhedron onto another. It is usually not PL. By 2.13 and 2.19, such a map is always Lipschitz. A standard mistake of another kind will be considered in 2.43.

2.21. *Unions.* Let $A\cup B$ be a metric space. Let f: $A\cup B\to Y$ be a function such that $f|A$ and $f|B$ are LIP. Without any additional condition, f need not be LIP or even continuous. If $A\setminus B$ and $B\setminus A$ are separated, that is, $\overline{A\setminus B}\cap$ $\cap(B\setminus A)=\emptyset=(A\setminus B)\cap\overline{A\setminus B}$, then f is continuous. In this case we say that $A\cup B$ is a *proper union* of A and B. If A and B are both open or both closed in $A\cup B$, then $A\cup B$ is a proper union. In particular, the union of two compact sets is always proper.

Even if A and B are compact, f need not be LIP. For example, let A be the parabolic arc $\{(x, y)\in R^2|0\leqq x\leqq 1,\ \ y=x^2\}$, let $B=\{(x, y)|(x, -y)\in A\}$, and let f: $A\cup B\to R^1$ be defined by $f(x, y)=x$ for $x\in A$ and by $f(x, y)=-x$ for $(x, y)\in B$. Then $f|A$ and $f|B$ are Lipschitz, but f is not.

We say that $A\cup B$ is a LIP *proper union* of A and B if $A\cup B$ is a proper union and if a function f: $A\cup B\to Y$ is LIP whenever $f|A$ and $f|B$ are LIP. A related concept has been considered by Wilker [32].

We shall give some characterizations for LIP proper unions. Before that, we give a simple but useful result, which can often be directly used to prove that a map is Lipschitz.

2.22. Lemma. *Suppose that $A\cup B$ is a metric space, that $E\subset A\cup B$, and that $C\geqq 1$ is a constant such that for every pair of points $a\in A\cap E$, $b\in B\cap E$, there* *is* $c \in A \cap B$ *such that* $d(a, c)+d(c, b) \leqq Cd(a, b)$. *If* $f\colon A \cup B \to Y$ *is a function such that* $f|A$ *and* $f|B$ *are L-Lipschitz, then* $f|E$ *is CL-Lipschitz.*

Proof. For $a \in A \cap E$, $b \in B \cap E$, we obtain $d'(f(a), f(b)) \leqq d'(f(a), f(c)) + d'(f(c), f(b)) \leqq Ld(a, c)+Ld(c, b) \leqq CLd(a, b)$. □

2.23. Theorem. *Let* $A \cup B$ *be a proper union with* $A \cap B \neq \emptyset$. *Define* $u\colon A \cup B \to R^1$ *by* $u(x)=d(x, A \cap B)$ *for* $x \in A$ *and by* $u(x)=0$ *for* $x \in B$. *Then* $A \cup B$ *is a* LIP *proper union if and only if* u *is* LIP.

Proof. The necessity of the condition is clear. Conversely, assume that u is LIP. Let $f\colon A \cup B \to Y$ be a map such that $f|A$ and $f|B$ are LIP, and let $x \in A \cup B$. If $x \in A \setminus B$, there is a neighborhood U of x in $A \cup B$ with $U \subset A$. Then $f|U$ is LIP. Next suppose $x \in A \cap B$. Choose $r>0$ such that for $U=B(x, 2r)$, $f|A \cap U$, $f|B \cap U$, and $u|U$ are Lipschitz. Let $a \in A \cap B(x, r)$ and $b \in B \cap B(x, r)$, $a \neq b$. Pick a point $c \in A \cap B \cap U$ such that $d(a, c)<d(a, A \cap B)+d(a, b)$. Then $d(a, c)+ +d(c, b) \leqq 2d(a, c)+d(a, b) \leqq 2(u(a)-u(b))+3d(a, b) \leqq (2 \operatorname{lip}(u|U)+3)d(a, b)$. By 2.22, $f|B(x, r)$ is Lipschitz. □

2.24. Theorem. *Let* $A \cup B$ *be a proper union with* $A \cap B \neq \emptyset$. *Then* $A \cup B$ *is a* LIP *proper union if and only if*

$$\limsup_{r \to 0} \left\{ \frac{d(a, A \cap B)}{d(a, B \setminus A)} \middle| a \in (A \setminus B) \cap B(x, r) \right\} < \infty$$

for all $x \in \overline{A \setminus B} \cap \overline{B \setminus A} \subset A \cap B$.

Proof. Suppose that $A \cup B$ is a LIP proper union. Let $x \in \overline{A \setminus B} \cap \overline{B \setminus A}$. By 2.23, there are $r>0$ and $L \geqq 1$ such that for $U=B(x, 2r)$, $u|U$ is L-Lipschitz. Let $a \in (A \setminus B) \cap B(x, r)$ and $b \in (B \setminus A) \cap U$. Then $d(a, A \cap B)=|u(a)-u(b)| \leqq \leqq Ld(a, b)$. Hence $d(a, A \cap B) \leqq Ld(a, B \setminus A)$, and the limit is finite.

Conversely, assume that the limit is finite for all x. Let $x \in \overline{A \setminus B} \cap \overline{B \setminus A}$. There is a neighborhood U of x and a constant $L \geqq 1$ such that $d(a, A \cap B) \leqq Ld(a, B \setminus A)$ for all $a \in (A \setminus B) \cap U$. Then $|u(a)-u(b)| \leqq Ld(a, b)$ for all $a \in U \cap A$ and $b \in B \cap U$. Thus $u|U$ is Lipschitz, and the theorem follows from 2.23. □

2.25. We next turn to the case where A and B are subsets of R^n with $A \cap B \neq \emptyset$. For $a \in A$, $b \in B$, we set $\alpha(A, B, a, b)=\sup \{\operatorname{ang}(a-y, b-y) | y \in A \cap B\}$. The *intersection angle* of A and B at a point $x \in A \cap B$ is defined by

$$\alpha(A, B, x) = \liminf_{a, b \to x} \alpha(A, B, a, b).$$

For example, if A and B are line segments ax and bx with $x=A \cap B$, then $\alpha(A, B, x)$ is the ordinary angle between A and B. Note that $\alpha(A, B, 0)=0$ for the example in 2.21.

2.26. Theorem. *Let* $A \cup B$ *be a proper union of* $A, B \subset R^n$. *If* $\alpha(A, B, x)>0$ *for all* $x \in A \cap B$, *then* $A \cup B$ *is a* LIP *proper union.*

Proof. We may assume $A \cap B \neq \emptyset$. Let $u\colon A \cup B \to R^1$ be the map given in 2.23, and let $x \in A \cap B$. It suffices to show that u is Lipschitz in a neighborhood of x. Set $\varepsilon = \alpha(A, B, x)/2$ and choose a neighborhood U of x such that $\alpha(A, B, a, b) > \varepsilon$ for all $a \in (A \setminus B) \cap U$ and $b \in (B \setminus A) \cap U$. For such a pair a, b choose $y \in A \cap B$ such that $\operatorname{ang}(a-y, b-y) > \varepsilon$. Then $|u(a)-u(b)|/|a-b| \leqq |a-y|/|a-b| \leqq 1/\sin \varepsilon$. □

2.27. Remark. The converse of 2.26 is not true. A counterexample is given in 3.10 (4).

2.28. Remark. Sometimes the LIP properness of a union $A \cup B$ depends only on the space $A \cup B$ and not on the sets A, B. For example, let $A \cup B$ be a proper union such that $A \cup B$ is a convex subset of R^n. Then $\alpha(A, B, x) = \pi$ for all $x \in A \cap B$, and the LIP properness follows from 2.26.

We shall next consider this phenomenon in a more general situation.

2.29. *Quasiconvexity.* Let X be a metric space. Given a pair of points $a, b \in X$, we let $C(a, b, X)$ denote the infimum of all numbers $C \geqq 1$ such that there is a rectifiable path γ in X joining a and b with length $l(\gamma) \leqq C d(a, b)$. If no such path exists, we set $C(a, b, X) = \infty$. For $A \subset X$ we write $C(A, X) = \sup \{C(a, b, X) \mid a \in A, b \in A\}$ and $C(X) = C(X, X)$. If $C(X) < \infty$, we say that X is *quasiconvex*, and X is *C-quasiconvex* if $C(X) \leqq C$. If each point of X has a neighborhood U such that $C(U, X) < \infty$, X is *locally quasiconvex.*

For example, a convex set in a normed vector space is 1-quasiconvex. The spheres S^n are $(\pi/2)$-quasiconvex $(n \geqq 1)$. A quasiconvex space is always locally quasiconvex. The arc $\{(x, y) \in R^2 \mid 0 \leqq x \leqq 1, |y| = x^2\}$ is not locally quasiconvex.

2.30. Lemma. *Every open subset of a locally quasiconvex metric space is locally quasiconvex.*

Proof. Let V be open in a locally quasiconvex space X. For $x \in V$ choose a neighborhood U such that $C(U, X) = C < \infty$. Next choose $r > 0$ such that $B(x, (2C+3)r) \subset U \cap V$. Let W be the ball $B(x, r)$. If $a, b \in W$, there is a path γ joining a and b in X such that $l(\gamma) \leqq (C+1) d(a, b)$. If y is any point on $\operatorname{im} \gamma$, then $d(a, y) \leqq l(\gamma) \leqq 2(C+1)r$. Hence $\operatorname{im} \gamma \subset V$, which implies $C(W, V) \leqq C+1$. □

2.31. *Retracts.* A subset A of a metric space X is a *Lipschitz retract* of X if there is a Lipschitz map $r\colon X \to A$ with $r|A = \mathrm{id}$. If A is a Lipschitz retract of a neighborhood of A, then A is a *Lipschitz neighborhood retract* of X. Similarly, we define the concepts LIP *retract* and LIP *neighborhood retract.*

2.32. Lemma. *A Lipschitz retract of a quasiconvex space is quasiconvex. A* LIP *neighborhood retract of a locally quasiconvex space is locally quasiconvex.*

Proof. Suppose that X is C-quasiconvex and that $r\colon X \to A$ is an L-Lipschitz retraction. Given $a, b \in A$ and $\varepsilon > 0$, there is a path γ joining a and b in X with

$l(\gamma)\leqq(C+\varepsilon)d(a, b)$. Then $r\gamma$ joins a and b in A, and $l(r\gamma)\leqq L(C+\varepsilon)d(a, b)$. Thus A is LC-quasiconvex.

Next assume that $r\colon V\to A$ is a LIP retraction of a neighborhood V of A. Let $x\in A$ and choose an open neighborhood U of x in X such that $U\subset V$ and lip $(r|U)=L<\infty$. By 2.30, there is a neighborhood $W\subset U$ of x with $C(W, U)=C<\infty$. Arguing as in the first part of the proof we see that $C(A\cap W, A)\leqq LC$. Hence A is locally quasiconvex. □

2.33. Lemma. *If X is compact, connected, and locally quasiconvex, then X is quasiconvex.*

Proof. Since X is connected and locally quasiconvex, each pair of points $x, y\in X$ can be joined by a rectifiable path in X. Let $\delta(x, y)$ be the infimum of the lengths of all such paths. Then δ is a metric in X. Since X is locally quasiconvex, the identity map id: $(X, d)\to(X, \delta)$ is LIP. Since X is compact, it is Lipschitz, whence X is quasiconvex. □

2.34. Theorem. *Every compact connected polyhedron in R^n is quasiconvex. Every polyhedron in R^n is locally quasiconvex.*

Proof. It suffices to prove the first assertion. Let $P\subset R^n$ be a compact connected polyhedron. Then P is a PL retract of a regular neighborhood N of P. By 2.18, P is a Lipschitz neighborhood retract of R^n. The theorem follows from 2.32 and 2.33. □

2.35. Theorem. *Suppose that X is C-quasiconvex and that $\mathscr{A}$ is a cover of X such that for each $x\in X$, the star* st $(x, \mathscr{A})=\cup\{A|x\in A\in\mathscr{A}\}$ *of x is a neighborhood of x. (For example, $\mathscr{A}$ is a locally finite closed cover of X or $\mathscr{A}$ is a family of two sets with a proper union.) If $f\colon X\to Y$ is a function such that $f|A$ is L-Lipschitz for all $A\in\mathscr{A}$, then f is CL-Lipschitz.*

Proof. Let $a, b\in X$, and let $\varepsilon>0$. Choose a path $\gamma\colon I\to X$ with $\gamma(0)=a$, $\gamma(1)=b$, $l(\gamma)\leqq(C+\varepsilon)d(a, b)$. For every $s\in I$, choose an open interval neighborhood $U(s)$ in I such that $\gamma U(s)\subset\mathrm{st}(\gamma(s), \mathscr{A})$. Choose a subcover $\{U(s_i)|1\leqq i\leqq k\}$ of $\{U(s)|s\in I\}$ which has no proper subcover. We may assume that $s_i<s_{i+1}$ and $\emptyset\neq U(s_i)\cap U(s_{i+1})\subset(s_i, s_{i+1})$ for all $i\in\{1, \dots, k-1\}$. Choosing numbers $s_i'\in U(s_i)\cap\cap U(s_{i+1})$ and relabeling the sequence $(0, s_1, s_1', \dots, s_{k-1}', s_k, 1)$ we obtain numbers $0=t_0<\dots<t_{2k}=1$ such that for $x_i=\gamma(t_i)$, $\{x_{i-1}, x_i\}$ is contained in some member of $\mathscr{A}$. Thus $d'(f(x_{i-1}), f(x_i))\leqq Ld(x_{i-1}, x_i)$, which implies

$$d'(f(a), f(b))\leqq L\sum_{i=1}^{2k} d(x_{i-1}, x_i)\leqq Ll(\gamma)\leqq(C+\varepsilon)Ld(a, b). \qquad \square$$

2.36. Theorem. *Suppose that X is locally quasiconvex and that $\mathscr{A}$ is a point-finite cover of X such that for each $x\in X$,* st $(x, \mathscr{A})$ *is a neighborhood of x. If $f\colon X\to Y$ is a function such that $f|A$ is* LIP *for all $A\in\mathscr{A}$, then f is* LIP.

Proof. Let $x \in X$, and let $A_1, \ldots, A_k$ be the members of $\mathscr{A}$ containing x. Choose an open neighborhood U of x and a positive number L such that $U \subset \mathrm{st}(x, \mathscr{A}) = A_1 \cup \ldots \cup A_k$ and such that $f|U \cap A_i$ is L-Lipschitz for all i. Since f is clearly continuous, it follows that also $f|U \cap \bar{A}_i$ is L-Lipschitz for all i. By 2.30, there is a neighborhood $V \subset U$ of x with $C(V, U) = C < \infty$. We show that $f|V$ is CL-Lipschitz. Let $a, b \in V$ and $\varepsilon > 0$. Choose a path γ joining a and b in U with $l(\gamma) \leqq (C+\varepsilon) d(a, b)$. We can find numbers $0 = t_0 < \ldots < t_s = 1$ such that for each $i \in \{1, \ldots, s\}$, $\{\gamma(t_{i-1}), \gamma(t_i)\}$ is contained in some of the sets $U \cap \bar{A}_i$. Hence $d'(f(\gamma(t_{i-1})), f(\gamma(t_i))) \leqq L d(\gamma(t_{i-1}), \gamma(t_i))$, which implies

$$d'(f(a), f(b)) \leqq L \sum_{i=1}^{s} d(\gamma(t_{i-1}), \gamma(t_i)) \leqq L l(\gamma) \leqq (C+\varepsilon) L d(a, b). \qquad \square$$

2.37. Corollary. *If $A \cup B$ is locally quasiconvex and a proper union, then $A \cup B$ is a* LIP *proper union.* $\square$

2.38. Theorem. *If A and B are polyhedra with a proper union, then $A \cup B$ is a* LIP *proper union.*

Proof. Since a proper union of polyhedra is obviously a polyhedron, the theorem follows from 2.34 and 2.37. $\square$

2.39. *Cartesian products.* Let (X, d) and (X', d') be metric spaces. Then the distance between points (x, x') and (y, y') in $X \times X'$ can be defined in three natural ways: $d(x, y) + d'(x', y')$, $(d(x, y)^2 + d'(x', y')^2)^{1/2}$, or $\max(d(x, y), d'(x', y'))$. Letting d_1, d_2, d_3 denote the corresponding metrics of $X \times X'$, we have $d_3 \leqq d_2 \leqq$ $\leqq d_1 \leqq 2d_3$. Hence these metrics are Lipschitz equivalent. From the point of view of LIP topology, it makes no difference which metric we use.

It is readily seen that cartesian products of LIP maps are again LIP. Moreover, a map (f, f') of a metric space Z into $X \times X'$ is LIP if and only if f and f' are LIP.

2.40. *Maps of $X \times R^1$.* Let X be a metric space. We shall later make use of maps $f: X \times R^1 \to X \times R^1$ of the following type: Let α_i and β_i be LIP maps $X \to R^1$, $i = 0, \ldots, k$, such that $\alpha_0 < \ldots < \alpha_k$ and $\beta_0 < \ldots < \beta_k$. Then $f(x, t) = (x, r_x(t))$ where r_x is the PL homeomorphism of R^1 onto R^1 which maps $[\alpha_{i-1}(x), \alpha_i(x)]$ affinely onto $[\beta_{i-1}(x), \beta_i(x)]$ and is a translation on $(-\infty, \alpha_0(x)]$ and on $[\alpha_k(x), \infty)$.

We shall show that f is a lipeomorphism of $X \times R^1$ onto itself. Since f^{-1} is obtained by changing the roles of α_i and β_i, it suffices to show that f is LIP. For this, it is sufficient to prove that the map r: $X \times R^1 \to R^1$, defined by $r(x, t) = r_x(t)$, is LIP.

Set $A_i = \{(x, t) | \alpha_{i-1}(x) \leqq t \leqq \alpha_i(x)\}$ for $1 \leqq i \leqq k$ and $A_0 = \{(x, t) | t \leqq \alpha_0(x)\}$, $A_{k+1} = \{(x, t) | t \geqq \alpha_k(x)\}$. For $(x, t) \in A_1$ we have $r(x, t) = (1-u)\beta_0(x) + u\beta_1(x)$

where $u=(t-\alpha_0(x))/(\alpha_1(x)-\alpha_0(x))$. Using 2.4 we see that $r|A_1$ is LIP. A similar proof shows that $r|A_i$ is LIP for all i.

To show that r is LIP in $X\times R^1$ we use 2.22. For example, let us show that $r|A_1\cup A_2$ is LIP. Since the question is local, we may assume that $\operatorname{lip}\alpha_1=L<\infty$. Let $a=(x,s)\in A_1$ and $b=(y,t)\in A_2$. Then $c=(x,\alpha_1(x))\in A_1\cap A_2$. Using the metric d_1 of 2.39 in $X\times R^1$ we have $d_1(a,c)=\alpha_1(x)-s$ and $d_1(c,b)=d(x,y)+|t-\alpha_1(x)|$. If $t\geqq\alpha_1(x)$, we obtain $d_1(a,c)+d_1(c,b)=d_1(a,b)$. If $t<\alpha_1(x)$, then $t\geqq\alpha_1(y)\geqq\alpha_1(x)-Ld(x,y)$, which implies $d_1(a,c)+d_1(c,b)\leqq t-s+(2L+1)d(x,y)\leqq(2L+1)d_1(a,b)$. Hence $r|A_1\cup A_2$ is LIP.

2.41. Lemma. *Let C be a $(p+q+1)$-simplex which is the join of a p-simplex A and a q-simplex B, $p\geqq 0$, $q\geqq 0$. Let $\pi\colon A\times B\times I\to C$ be the map $\pi(x,y,t)=(1-t)x+ty$. Then π is* LIP *and defines a lipeomorphism $\pi_0\colon A\times B\times(0,1)\to C\setminus(A\cup B)$.*

Proof. Using an auxiliary affine homeomorphism, we may assume that $C\subset R^{p+q+1}=R^p\times R^q\times R^1$, $A\subset R^p\times 0\times 0$, $B\subset 0\times R^q\times 1$. Choose $R>0$ such that $C\subset B^{p+q+1}(R)$. Let $z_1=(x_1,y_1,t_1)$ and $z_2=(x_2,y_2,t_2)\in A\times B\times I$. Then

$$\begin{aligned}|\pi(z_1)-\pi(z_2)| &\leqq (1-t_1)|x_1-x_2|+|t_1-t_2|\,|x_2|+t_1|y_1-y_2|+|t_1-t_2|\,|y_2|\\ &\leqq (2+2R)|z_1-z_2|.\end{aligned}$$

Hence π is LIP.

Let $0<\delta<1/2$, and let $z_1, z_2\in A\times B\times(\delta,1-\delta)$. Then $\delta|x_1-x_2|\leqq|(1-t_1)x_1-(1-t_2)x_2|+|t_2-t_1|\,|x_2|\leqq(1+R)|\pi(z_1)-\pi(z_2)|$, and similarly $\delta|y_1-y_2|\leqq(1+R)|\pi(z_1)-\pi(z_2)|$, which implies $|z_1-z_2|\leqq(1+(2+2R)/\delta)|\pi(z_1)-\pi(z_2)|$. Hence π_0 is a lipeomorphism. □

2.42. *Stretching maps.* We shall consider the stretching process between dual skeletons, which is often used in an engulfing argument. Suppose that K is a finite simplicial complex and that L, M are disjoint subcomplexes of K such that every simplex of K is the join of a simplex in L and a simplex in M. Let X be the set of all $(x,y)\in|L|\times|M|$ such that xy lies in a simplex of K, and let $\alpha,\beta\colon X\to(0,1)$ be LIP maps. Choose $t_1, t_2\in(0,1)$ such that $\operatorname{im}\alpha\cup\operatorname{im}\beta\subset(t_1,t_2)$. For each $(x,y)\in X$ we let $\omega_{xy}\colon I\to I$ denote the PL homeomorphism which maps the points 0, t_1, $\alpha(x,y)$, t_2, 1 to the points 0, t_1, $\beta(x,y)$, t_2, 1, and is affine on the remaining intervals. Let $h\colon|K|\to|K|$ be the unique bijective function such that $h||L|\cup|M|=\mathrm{id}$ and $h((1-t)x+ty)=(1-\omega_{xy}(t))x+\omega_{xy}(t)y$ for all $(x,y)\in X$. We shall prove that $h\colon|K|\to|K|$ is a lipeomorphism.

Since h^{-1} is obtained from h by changing the roles of α and β, it suffices to show that h is LIP. By 2.38, it suffices to show that $h|C$ is LIP for every $C\in K$. Write $C=AB$ with $A\in L$, $B\in M$. We may assume that $A\neq\emptyset\neq B$. Let $\pi\colon A\times B\times I\to C$ be the map of 2.41. Since C is convex, it suffices to show that h is LIP in each of the sets $C_i=\pi D_i$, $i=1,2$, where $D_1=A\times B\times([0,t_1]\cup[t_2,1])$ and $D_2=A\times B\times$

$\times[t_1, t_2]$. Since $h|C_1=\mathrm{id}$, it is LIP. Furthermore, we may write $h|C_2=\pi_0 g\pi_0^{-1}|C_2$ where π_0 is the lipeomorphism of 2.41, and $g(x, y, t)=(x, y, \omega_{xy}(t))$. It follows from 2.40 that g is LIP. Hence h is LIP.

The stretching process will be used in 8.4 in the following situation. K has a subcomplex K_1 such that $|K|$ and $|K_1|$ are concentric n-cubes, L is the 2-skeleton of K, and M is the dual $(n-3)$-skeleton of a derived subdivision of K. Given open neighborhoods U and V of $|L|$ and $|M|$, we need a lipeomorphism $h\colon |K|\to|K|$ such that $|K_1|\subset hU\cup V$ and $h|\partial|K|=\mathrm{id}$. For this, we first choose $\varepsilon\in(0, 1/2)$ such that for all $(x, y, t)\in X\times[0, \varepsilon]$, $(1-t)x+ty\in U$ and $tx+(1-t)y\in V$. Then choose a LIP map $\lambda\colon |K|\to[\varepsilon, 1/2]$ such that $\lambda(x)=\varepsilon$ for $x\in|K_1|$ and $\lambda(x)=1/2$ for $x\in\partial|K|$, and set $\alpha(x, y)=(\lambda(x)+\lambda(y))/2$, $\beta(x, y)=1-\alpha(x, y)$. Then the stretching map $h\colon |K|\to|K|$ has the desired properties.

2.43. Remark. Suppose that the maps α and β of 2.42 are constants, $\alpha\neq\beta$. It is sometimes stated that the corresponding homeomorphism $h\colon |K|\to|K|$ is PL. However, this is usually not true, for example if $|K|$ is a triangle abc, $|L|=a$, $|M|=bc$. We may call this map a *standard mistake of the second kind.* A PL homeomorphism is easily obtained as follows: Let $f\colon K\to I$ be the simplicial map which maps L into 0 and M into 1. Choose deriveds K_1, K_2 of K near L [25, p. 32] such that $f^{-1}[0, \alpha]$ and $f^{-1}[0, \beta]$ are the underlying polyhedra of the derived neighborhoods $N(L, K_1)$ and $N(L, K_2)$, respectively. Then the canonical simplicial isomorphism $\varphi\colon K_1\to K_2$ maps $N(L, K_1)$ onto $N(L, K_2)$.

3. Lipschitz manifolds

3.1. Definition. A *Lipschitz n-manifold* (or a LIP n-manifold) is a separable metric space M such that every point $x\in M$ has a closed neighborhood U lipeomorphic to I^n.

3.2. Remarks. (1) Recall that a connected paracompact topological manifold has a countable base. Hence, separability is not a restriction for connected manifolds.

(2) Since I^n is lipeomorphic to $\bar{B}^n$, the pair (U, x) in 3.1 is lipeomorphic to either $(I^n, 0)$ or (I^n, e^n). It follows that each point of a LIP manifold has an open neighborhood lipeomorphic to either R^n or R^n_+.

(3) The boundary ∂M of a LIP n-manifold is either empty or a LIP $(n-1)$-manifold.

(4) If M and N are LIP manifolds, so is $M\times N$.

(5) A LIP manifold is locally quasiconvex.

3.3. *Atlases.* There is an alternative way to define a LIP manifold based on atlases. Let M be a Hausdorff space. A LIP *atlas* on M is a family of charts (U_i, h_i) where the sets U_i form an open cover of M, h_i maps U_i homeomorphically onto a set U_i' which is open either in R^n or in R^n_+, and $h_i h_j^{-1}$ defines a lipeomorphism of $h_j[U_i \cap U_j]$ onto $h_i[U_i \cap U_j]$ for all i and j. Two LIP atlases are called equivalent if their union is a LIP atlas. Then a LIP manifold can be defined as a pair consisting of M and an equivalence class of LIP atlases. The concept of a LIP map between LIP manifolds is then defined in the obvious way using charts.

If M is a LIP manifold in the sense of 3.1, it has a natural LIP atlas consisting of all lipeomorphisms $h\colon U \to U'$ such that U is open in M and U' is open in R^n or in R^n_+. Moreover, the two definitions of LIP maps of M and into M are consistent. One can show that a paracompact LIP manifold is lipeomorphic to a metric space. Hence, for second countable spaces, the atlas definition is not essentially more general than 3.1. This can be proved in several ways. For example, it will follow from our embedding theorem 4.2. But it is also a special case of a metrization theorem for locally metric spaces (Theorem 3.5 and Remark 3.7), which is our next goal. Moreover, Weller [30] has proved the result (stated for closed manifolds) using a method somewhat similar to the proof of 3.5.

3.4. *Local metrics.* A *local metric* on a Hausdorff space X is a family of metric spaces (U_i, d_i) such that the sets U_i form an open cover of X, d_i is compatible with the topology of U_i, and for each pair i, j of indices, the restrictions of d_i and d_j are LIP equivalent on $U_i \cap U_j$. Two local metrics on X are called LIP equivalent if their union is a local metric. Cf. Whitehead [31, p. 166]. A *locally metric space* is a pair consisting of a Hausdorff space X and an equivalence class of local metrics on X. Each metric space defines a locally metric space in the obvious way. We shall show that each paracompact locally metric space can be obtained in this way.

Every LIP atlas $(U_i, h_i)_{i \in I}$ of a manifold M defines a local metric on M consisting of pairs (U_i, d_i) where $d_i(x, y) = |h_i(x) - h_i(y)|$. Moreover, equivalent LIP atlases define LIP equivalent local metrics. Hence every LIP manifold in the atlas sense can be regarded as a locally metric space. The same is true for abstract PL and DIFF manifolds.

3.5. Theorem (Metrization). *On a paracompact Hausdorff space, every local metric is* LIP *equivalent to a metric.*

Proof. Suppose that $\delta = (U_i, d_i)_{i \in I}$ is a local metric on a paracompact Hausdorff space X. We may assume that the cover $(U_i)_{i \in I}$ is locally finite. We first show that δ is LIP equivalent to a local metric $\delta' = (U_j', d_j')_{j \in J}$ of X such that for every pair $j, k \in J$, d_j' and d_k' are Lipschitz (not only LIP) equivalent in $U_j' \cap U_k'$. There is an open cover $(V_i)_{i \in I}$ of X with $\overline{V}_i \subset U_i$ for all i. For every $x \in X$, choose an open neighborhood $W(x)$ such that (1) $W(x) \subset V_i$ if $x \in V_i$, (2) $W(x) \subset U_i$ if $x \in U_i$, (3) $W(x) \cap \overline{V}_i = \emptyset$ if $x \notin \overline{V}_i$, (4) d_i and d_j are Lipschitz equivalent in $W(x)$ if $x \in U_i \cap U_j$. Next choose a locally finite open refinement $(U_j')_{j \in J}$ of the cover $(W(x))_{x \in X}$. For every $j \in J$, choose $x_j \in X$ and $i(j) \in I$ such that $U_j' \subset W(x_j) \subset V_{i(j)}$. Then $d_j' = d_{i(j)} | U_j' \times U_j'$ is a metric for U_j', compatible with the topology of U_j'. Suppose that U'_j meets U'_k. Then $W(x_k)$ meets $V_{i(j)}$, which implies $x_k \in \overline{V}_{i(j)}$. Thus $x_k \in U_{i(j)} \cap \cap U_{i(k)}$, whence d'_j and d'_k are Lipschitz equivalent in $W(x_k)$, and hence in $U'_j \cap U'_k$. Thus the local metric $\delta' = (U'_j, d'_j)_{j \in J}$ has the desired property. Obviously δ and δ' are LIP equivalent.

Changing notation, we assume that the original local metric $\delta = (U_i, d_i)_{i \in I}$ has the above property. Choose again an open cover $(V_i)_{i \in I}$ of X with $\overline{V}_i \subset U_i$. We define a function $d: X \times X \to R^1$ as follows: For $x, y \in X$, let $P(x, y)$ be the set of all finite sequences of the form $\pi = (x_0, \dots, x_k; i_1, \dots, i_k)$ such that $x_0 = x, x_k = y$, and $\{x_{j-1}, x_j\} \subset V_{i_j}$. For such π, set

$$s(\pi) = \sum_{j=1}^{k} d_{i_j}(x_{j-1}, x_j).$$

If $P(x, y) = \emptyset$, we set $d(x, y) = 1$, otherwise

$$d(x, y) = \min\big(1, \inf\{s(\pi) | \pi \in P(x, y)\}\big).$$

It is easy to see that d is a pseudometric for the set X.

For the rest of the proof we fix the following notation: Let $x \in X$ and choose $i \in I$ such that $x \in V_i$. Choose an open neighborhood $V \subset V_i$ of x such that $J_1 = \{j \in I | V \text{ meets } U_j\}$ is finite and for every $j \in J_1$, either $V \subset U_j$ or $V \cap V_j = \emptyset$. Set $J_2 = \{j \in J_1 | V \subset U_j\}$. There is $b > 1$ such that $b^{-1} d_k(u, v) \leqq d_j(u, v) \leqq b d_k(u, v)$ for all j, k in J_1 and for all u, v in $U_j \cap U_k$. Choose $r \in (0, 1)$ such that the ball $B_{d_j}(x, r) = \{y \in U_j | d_j(x, y) < r\}$ is contained in V for all $j \in J_2$.

We must show that d is compatible with the topology of X (hence a metric) and LIP equivalent to δ. If $y, z \in V_j$, then $d(y, z) \leqq s(y, z; j) = d_j(y, z)$. Hence id: $X \to (X, d)$ is continuous, and id: $(U_i, d_i) \to (U_i, d)$ is LIP. We next show that id: $(X, d) \to X$ is continuous. Let W be a neighborhood of x in X. Replacing V by $V \cap W$, we may assume $V \subset W$. We claim that $d(y, x) < r/b$ implies $y \in W$. Since $r/b < 1$, there is $\pi = (x_0, \dots, x_k; i_1, \dots, i_k) \in P(x, y)$ such that $s(\pi) < r/b$. Then $x_0 = x \in V$. Suppose inductively that $x_i \in V$ for all $i \leqq q$. Then $i_{q+1} \in J_2$ and $x_i \in U_{i_{q+1}}$ for all $i \leqq i_{q+1}$. Hence $d_{i_{q+1}}(x_{q+1}, x) \leqq b s(\pi) < r$, and thus $x_{q+1} \in V$, which implies $y = x_k \in V \subset W$.

It remains to show that id: $(U_j, d) \to (U_j, d_j)$ is LIP for every $j \in I$. It suffices to show that this map is Lipschitz in a neighborhood of x for every $j \in I$ such that $x \in U_j$. Since δ is a local metric, we may assume $j = i$. The set $B = \bigcap \{B_{d_j}(x, r/2) | j \in J_2\}$ is a neighborhood of x in U_i. We shall show that $d_i(y, z) \leqq 2bd(y, z)$ for $y, z \in B$. Since $(y, z; i) \in P(y, z)$, $d(y, z) < 1$. Let $\pi = (y_0, \dots, y_k; i_1, \dots, i_k) \in P(y, z)$. If $y_\nu \in V$ for all ν, then

$$d_i(y, z) \leqq \sum_{\nu=1}^{k} d_i(y_{\nu-1}, y_\nu) \leqq b s(\pi). \tag{3.6}$$

If $y_\nu \notin V$ for some ν, let q be the least ν with this property. Then $i_q \in J_2$, whence $d_{i_q}(y_q, x) \geqq r$. Thus $d_{i_q}(y_q, y) > r/2$, which implies $d_i(y, z) < r < 2d_{i_q}(y_q, y) \leqq 2bs(\pi)$. Together with (3.6), this implies $d_i(y, z) \leqq 2bd(y, z)$. □

3.7. Remark. Suppose that M is a second countable Hausdorff space with a LIP atlas $\mathscr{A}$. Then it has a natural local metric as in 3.4. Since M is paracompact, it follows from 3.5 that this local metric is LIP equivalent to a metric d on X. If $\mathscr{A}$ is replaced by an equivalent LIP atlas $\mathscr{A}'$, we obtain a metric d' which is LIP equivalent to d. To get a full equivalence with the atlas definition, we should define a LIP manifold as a pair consisting of a space X and a LIP equivalence class of metrics of X such that for some (and hence for each) d in the class, (X, d) is locally lipeomorphic to I^n. However, we prefer to use the conceptually simpler Definition 3.1.

3.8. *Terminology.* A LIP *n-ball* is a metric space lipeomorphic to $\bar{B}^n$. A LIP *n-sphere* is a metric space lipeomorphic to S^n. A subset N of a LIP manifold M is a LIP *submanifold* of M if it is a LIP manifold in the metric inherited from M. If N is a LIP q-submanifold of a LIP n-manifold M with $N\subset \operatorname{int} M$, we say that N is *locally* LIP *flat* at a point $x\in \operatorname{int} N$ if x has a neighborhood U in M such that $(U, U\cap N)$ is lipeomorphic to $(V, V\cap R^q)$ for some V open in R^n. At a point $x\in\partial N$, $(U, U\cap N)$ should be lipeomorphic to $(V, V\cap R^q_+)$. One can obviously choose $V=R^n$.

3.9. Theorem. (Cf. [16]) *Let A be a compact convex set in R^n. Then A is a locally* LIP *flat* LIP *ball. In fact, there is a strong lipeomorphism of R^n which maps A onto $\bar{B}^q$, $q=\dim A$.*

Proof. Let T be the affine subspace of R^n spanned by A. Choose an interior point $v\in A$ in the topology of T. By an auxiliary isometry, we may assume $T=R^q$ and $v=0$. Then A is the cone $0(\partial A)$. By 2.17, this cone is a Lipschitz cone, and hence $p(x)=x/|x|$ defines a lipeomorphism $p\colon \partial A\to S^{q-1}$. By 2.16, the extended cone $p^*\colon R^q\to R^q$ of p is a strong lipeomorphism, and so is $f=p^*\times \mathrm{id}\colon R^n\to R^n$. Since $fA=\bar{B}^q$, the theorem is proved. □

3.10. Examples. (1) By a LIP *arc* we mean a LIP 1-ball. Unlike PL and DIFF arcs, a LIP arc in R^3 need not be locally LIP flat. For example, the construction of the Fox—Artin arc, given in [26, pp. 61—62], can be modified so as to yield a LIP arc, which is not even locally TOP flat at the end points. We do not know whether a locally TOP flat LIP arc in R^n is always locally LIP flat.

(2) A LIP arc is always rectifiable. Indeed, if f is an L-Lipschitz map of I onto A, then the length of A is at most L. It is not difficult to show that a metric space which is a TOP arc is a LIP arc if and only if it is quasiconvex. For example, the arc $\{(x, y)\in R^2 | 0\leqq x\leqq 1,\ |y|=x^2\}$ is not a LIP arc. See also Katětov [15, 3.9].

(3) As a rectifiable arc, a LIP arc $A\subset R^n$ has a tangent at almost every point. Hence it pierces an $(n-1)$-disk at almost every point. On the other hand, there is a TOP arc in R^3 which pierces no disk (Bing [2]). This is so wild that no homeomorphism of R^3 maps it onto a LIP arc.

(4) Although a LIP arc $A\subset R^n$ has a tangent at almost every point, its tangential behavior may be fairly complicated at certain points. Let $f\colon R^2\to R^2$ be

the homeomorphism defined by $f(r, \varphi)=(r, \varphi+\log r)$ in polar coordinates. A computation shows that the derivative of f is bounded in $R^2\setminus 0$. Hence f is Lipschitz. Since $f^{-1}(r, \varphi)=(r, \varphi-\log r)$, it similarly follows that f^{-1} is Lipschitz. Thus the image C of the segment joining $(-1, 0)$ and $(1, 0)$ is a locally LIP flat LIP arc, which consists of two logarithmic spirals and the origin. The origin divides C into two closed LIP subarcs A, B. The union $C=A\cup B$ is LIP proper, although the intersection angle $\alpha(A, B, 0)$ is zero. Hence the converse of 2.26 is not true. A similar example in higher dimensions will be given in 4.11.

(5) Fattening the Fox—Artin arc we obtain a LIP 3-ball B in R^3 which is not even locally TOP flat. See Gehring [12, Theorem 3, p. 316]. Thus the LIP version of Newman's theorem [25, 3.13] is false. The boundary of B is a LIP 2-sphere in R^3 which is not locally TOP flat.

(6) The set $\{(x, y, z)\in R^3 | 0\leqq z\leqq 1,\ x^2+y^2\leqq z^4\}$ is a locally TOP flat 3-ball in R^3, but it is not locally LIP flat, because it contains no LIP arc through the origin.

3.11. Theorem. *Every connected* LIP 1-*manifold M is lipeomorphic to exactly one of the following* LIP 1-*manifolds:* $(0, 1)$, $[0, 1)$, $[0, 1]$, S^1.

Proof. It is well known that there is a homeomorphism $f\colon M_0\to M$ where M_0 is one of the manifolds listed in the theorem. Using a locally finite cover of M by LIP arcs such that no three of them intersect, we may choose a locally finite family of points $(x_j)_{j\in J}$ in M_0 with the following properties: (1) The indexing set J is $\mathbf{Z}$ for $M_0=(0, 1)$, $\mathbf{N}$ for $M_0=[0, 1)$, and $\{1, \dots, r\}$ for a compact M_0. (2) The points x_j are in the positive order. (3) If A_j is the arc from x_j to x_{j+1} (A_r is from x_r to x_1 if $M_0=S^1$), then $fA_j=B_j$ is a LIP arc in M. (4) The arcs B_j cover M. Choose a homeomorphism $g\colon M_0\to M$ such that $g(x_j)=f(x_j)$ for all $j\in J$, and g maps A_j lipeomorphically onto B_j. Since M_0 and M are locally quasiconvex, it follows from 2.36 that g is a lipeomorphism. □

3.12. Corollary. *Let M and N be* LIP 1-*manifolds. If M and N are homeomorphic, they are lipeomorphic.* □

3.13. Theorem. *Let M be a* LIP *manifold. Then there is a* LIP *manifold DM, called the double of M, which has the following properties: DM contains* LIP *submanifolds M_1, M_2 such that $DM=M_1\cup M_2$, $M_1\cap M_2=\partial M_1=\partial M_2$, and there are lipeomorphisms $f_i\colon M\to M_i$ such that $f_1|\partial M=f_2|\partial M$. The triple (DM, M_1, M_2) is unique up to a lipeomorphism. Moreover, $\partial DM=\emptyset$, and the submanifolds M_1, M_2, ∂M_1 are locally* LIP *flat in DM.*

Proof. The uniqueness of (DM, M_1, M_2) is clear. If $\partial M=\emptyset$, we may put $DM=M\times\{0, 1\}$. If $\partial M\neq\emptyset$, DM can be constructed for example as follows: Define $f_1, f_2\colon M\to M\times R^1$ by $f_1(x)=(x, 0)$ and $f_2(x)=(x, d(x, \partial M))$. Then each f_i is a LIP embedding of M onto M_i with $M_1\cap M_2=\partial M_1=\partial M_2=\partial M\times 0$. Set $DM=M_1\cup M_2$. It remains to prove that each point $(a, 0)\in M_1\cap M_2$ has a neighborhood V in DM such that $(V, V\cap M_1)$ is lipeomorphic to (R^n, R^n_+). Choose a lipeomorphism $h\colon U\to R^n_+$ of an open neighborhood U of a in M. Then $V=f_1U\cup \cup f_2U$ is an open neighborhood of $(a, 0)$ in DM. Define $p\colon R^n\to R^n$ by $p(x)= =(x_1, \dots, x_{n-1}, -x_n)$ and $h^*\colon V\to R^n$ by $h^*|f_1U=hf_1^{-1}|f_1U$ and $h^*|f_2U= =phf_2^{-1}|f_2U$. Then h^* is a homeomorphism and defines lipeomorphisms $f_1U\to R^n_+$ and $f_2U\to pR^n_+$. To prove that h^* is a lipeomorphism it suffices to show that $f_1U\cup \cup f_2U$ is a LIP proper union. Let $(b, 0)\in f_1U\cap f_2U=(U\cap\partial M)\times 0$. Choose $r>0$ with $B(b, 2r)\subset U$. For every $x\in B(b, r)\cap \operatorname{int} M$, we have $d(x, U\cap\partial M)=d(x, \partial M)$. Using the metric d_1 of 2.39 in $M\times R^1$ we obtain

$$\frac{d_1(f_2(x), f_1U\cap f_2U)}{d_1(f_2(x), V\setminus f_2U)}=\frac{d(x, U\cap\partial M)+d(x, \partial M)}{d(x, \partial M)}=2.$$

By 2.24, the union $f_1U\cup f_2U$ is LIP proper. □

4. Embedding

4.1. By Whitney's embedding theorem, every DIFF n-manifold can be DIFF embedded into R^{2n+1}. Similar results hold for PL and TOP manifolds. It is therefore natural to ask whether every LIP n-manifold can be LIP embedded into R^{2n+1}. We have not been able to solve this problem. However, we shall prove in 4.5 and 4.9 that a LIP n-manifold can be LIP embedded into $R^{n(n+1)}$ and that there is a locally LIP flat LIP embedding into $R^{n(n+2)}$. Before that, we prove a weaker result, which gives a new proof for the equivalence of the two definitions 3.1 and 3.3 of a LIP manifold.

4.2. Theorem. *Let M be a* LIP *n-manifold in the atlas sense* 3.3. *If M has a countable basis, then M is lipeomorphic to a closed subset of $R^{(n+1)^2}$.*

Proof. For every $x\in M$, choose a chart (U_x, h_x) at x such that $\overline{U}_x$ is compact and h_xU_x is open in R^n_+. By [2?, 2.7], the cover $(U_x)_{x\in M}$ has a locally finite countable open refinement $\mathscr{B}$ such that $\mathscr{B}=\mathscr{B}_0\cup\dots\cup\mathscr{B}_n$ where the members of each $\mathscr{B}_i$ are pairwise disjoint. Let $\mathscr{B}_i=\{U_{i1}, U_{i2}, \dots\}$, and choose a LIP embedding h_i of $V_i=\cup\mathscr{B}_i$ into R^n_+ such that for every j, h_iU_{ij} is an open subset of $R^n_+\cap B^n(3je_1, 1)$. Since M is normal, it has an open cover $\{W_i | 0\leq i\leq n\}$ such that $\overline{W}_i\subset V_i$. From 2.5 it easily follows that there is a LIP map $\varphi_i\colon M\to I$ such that $\varphi_i|\overline{W}_i=1$ and $\operatorname{spt}\varphi_i\subset V_i$. Then the product φ_ih_i, extended by zero to M, is LIP. Let $s=(n+1)^2$ and define $f\colon M\to R^s$ by $f=(\varphi_0, \varphi_0h_0, \dots, \varphi_n, \varphi_nh_n)$. Then f is LIP. We show that f is the desired LIP embedding.

Since $\varphi_i|\overline{W}_i=1$, we have $|f(h_i^{-1}(x))-f(h_i^{-1}(y))|\geq|x-y|$ for all $x, y\in h_iW_i$. Thus $fh_i^{-1}\colon h_iW_i\to fW_i$ is a lipeomorphism, which implies that f defines a lipeomorphism of W_i onto fW_i. Hence f is a LIP immersion in the obvious sense. To prove that f is injective, assume $f(x)=f(y)$. If $x\in W_i$, then $\varphi_i(y)=\varphi_i(x)=1$. Thus $y\in V_i$, and $h_i(x)=\varphi_i(x)h_i(x)=\varphi_i(y)h_i(y)=h_i(y)$, whence $y=x$.

We complete the proof by showing that f is proper. Let A be a compact set in R^s. Since M is metrizable, it suffices to show that an arbitrary sequence (x_ν) in $f^{-1}A$ has a convergent subsequence. Choose a positive integer k such that $A\subset B^s(3k+2)$. We may assume that for some i, $x_\nu\in W_i$ for all ν. Since $\varphi_i|W_i=1$, we have $|h_i(x_\nu)|\leqq|f(x_\nu)|<3k+2$ for all ν. Since $h_iU_{ij}\subset B^n(3je_1, 1)$, this implies $x_\nu\in\bigcup\{U_{ij}|1\leqq j\leqq k\}$ for all ν. Since each $\overline{U}_{ij}$ is compact, (x_ν) has a convergent subsequence. □

4.3. Remarks. (1) It follows from the proof of 4.2 that the map $(\varphi_0 h_0, \dots, \varphi_n h_n)\colon M\to R^{n(n+1)}$ is a proper LIP immersion.

(2) One can show that the embedding constructed above is locally LIP flat. We omit the proof, since a better result will be given in 4.9.

(3) Theorem 4.2 holds, with essentially the same proof, for all locally compact separable metric spaces which can be locally LIP embedded into R^n.

4.4. Lemma. *Let M be a* LIP *n-manifold (in the sense of* 3.1*), let $s>2n$, let $f\colon M\to R^s$ be a* LIP *immersion and let $\varepsilon\colon M\to(0,\infty)$ be continuous. Then there is an injective* LIP *immersion $g\colon M\to R^s$ such that $|f(x)-g(x)|<\varepsilon(x)$ for all $x\in M$. If f is injective in a neighborhood U of a closed set A, we may choose $g|A=f|A$.*

Proof. We shall give a LIP version of Milnor's proof [21, 1.29] for the corresponding DIFF result. Choose an open locally finite refinement $(U_i)_{i\in Z}$ of the cover $(M\setminus A, U)$ of M such that $\overline{U}_i$ is compact and $f|U_i$ is a LIP embedding for all i. The indexing is chosen so that $\{i|i\leqq 0$ and $U_i\neq\emptyset\}=\{i|\emptyset\neq U_i\subset U\}$. Next choose an open cover $(V_i)_{i\in Z}$ of M with $\overline{V}_i\subset U_i$ for all i. By 2.5 there are LIP maps $\varphi_i\colon M\to I$, $i>0$, such that $\varphi_i|\overline{V}_i=1$ and $\operatorname{spt}\varphi_i\subset U_i$.

We shall inductively construct LIP immersions $g_j\colon M\to R^s$, $j\geqq 0$, such that $g_0=f$, $g_j=g_{j-1}+\varphi_j b_j$, where $b_j\in R^s$ is yet to be chosen. The first requirement is that $|b_j|<2^{-j}\min\varepsilon\overline{U}_j$ (or $b_j=0$ if $U_j=\emptyset$). Then $|g_j(x)-g_{j-1}(x)|<2^{-j}\varepsilon(x)$ for all $x\in M$. Choose an open cover $\{B_1, \dots, B_r\}$ of $\overline{U}_j$ and a positive number l such that $|g_{j-1}(x)-g_{j-1}(y)|\geqq l d(x, y)$ for all x, y in B_k, $1\leqq k\leqq r$. The second requirement is that $|b_j|<l/2\operatorname{lip}(\varphi_j|\overline{U}_j)$. Then $|g_j(x)-g_j(y)|\geqq|g_{j-1}(x)-g_{j-1}(y)|-|\varphi_j(x)-\varphi_j(y)||b_j|\geqq 2^{-1}l d(x, y)$ for all x, y in $U_j\cap B_k$, $1\leqq k\leqq r$. Since $g_j(x)=g_{j-1}(x)$ outside $\operatorname{spt}\varphi_j$, it follows that g_j is a LIP immersion. Finally, let N be the open set in $M\times M$ consisting of pairs (x, y) such that $\varphi_j(x)\neq\varphi_j(y)$. Define a LIP map $\psi\colon N\to R^s$ by $\psi(x, y)=(g_{j-1}(y)-g_{j-1}(x))/(\varphi_j(x)-\varphi_j(y))$. Since $M\times M$ is a LIP $2n$-manifold and $s>2n$, the set ψN is of Hausdorff s-measure zero (see 6.2). Hence we may require $b_j\notin\psi N$. Then $g_j(x)=g_j(y)$ if and only if $\varphi_j(x)=\varphi_j(y)$ and $g_{j-1}(x)=g_{j-1}(y)$.

Define $g\colon M\to R^s$ by $g(x)=\lim_{j\to\infty} g_j(x)$. For each $x_0\in M$, there is a neighborhood V of x_0 and an integer j such that $g(x)=g_j(x)$ for $x\in V$. Hence g is a

LIP immersion. Furthermore, $|f(x)-g(x)|<\varepsilon(x)$ for all $x\in M$, and $g|A=f|A$. It remains to prove that g is injective. Suppose that $g(x)=g(y)$ with $x\neq y$. Then $\varphi_j(x)=\varphi_j(y)$ and $g_{j-1}(x)=g_{j-1}(y)$ for all $j>0$. For $j=1$ this yields $f(x)=f(y)$. Hence x and y cannot be in the same set U_i. If $x\in V_i$ for some $i>0$, then $\varphi_i(y)=$ $=\varphi_i(x)=1$, and thus $y\in U_i$. Hence x and y are in U. This is impossible, since $f|U$ is injective. □

4.5. Theorem. *Let M be a* LIP *n-manifold. Then there is a closed* LIP *embedding $f\colon M\to R^{n(n+1)}$.*

Proof. For $n=1$ the theorem follows from 3.11. Assume $n\geqq 2$. By 4.3 (1), there is a proper LIP immersion $F\colon M\to R^{n(n+1)}$. Since $n(n+1)>2n$, it follows from 4.4 that there is an injective LIP immersion $f\colon M\to R^{n(n+1)}$ such that $|f(x)-F(x)|<1$ for all $x\in M$. Then f is a proper map and hence a closed LIP embedding. □

4.6. Remark. Lemma 4.4 and Theorem 4.5 can be generalized for locally compact separable metric spaces which can be locally LIP embedded into R^n, $n\geqq 2$, see Remark 4.3 (3).

4.7. Theorem. *Let D be a* LIP *k-ball in R^n. Then there is a strong lipeomorphism φ of $R^{n+k}=R^n\times R^k$ onto itself such that $\varphi D=I^k$.*

Proof. This is the LIP version of a theorem of Klee [26, Theorem 2.5.1, p. 74]. Using 5.6 instead of Tietze's theorem, all maps occurring in the proof can be made Lipschitz. □

4.8. Corollary. *If M is a* LIP *k-submanifold of R^n, M is locally* LIP *flat in $R^{n+k}=R^n\times R^k$.* □

4.9. Theorem. *If M is a* LIP *n-manifold, there is a closed locally* LIP *flat* LIP *embedding $f\colon M\to R^{n(n+2)}$.*

Proof. This follows from 4.5 and 4.8. □

4.10. Remark. Theorem 4.9 is one dimension better than the result announced in 4.3 (2).

4.11. *Projections.* Suppose that $f\colon M\to R^s$ is a LIP embedding of a LIP n-manifold M, $s>2n+1$. One might think that the dimension of the target space could be lowered by choosing a suitable projection of R^s onto an $(s-1)$-dimensional linear subspace. In the DIFF category this is possible, see [7, Théorème 5, p. 12]. Set $N=fM$, and let $\sigma(N)=\{(x-y)/|x-y|\,|x,y\in N,\ x\neq y\}\subset S^{s-1}$ be the set of directions of all secants of N. It is easy to see that the Hausdorff $(s-1)$-measure of $\sigma(M)$ is zero. Hence there is $a\in S^{s-1}\setminus\sigma(N)$. Let p be a projection of R^s in the direction a onto an $(s-1)$-dimensional subspace V. Then $p|N$ is injective and Lipschitz. Continuing similarly we obtain a LIP injection of N into R^{2n+1}. However, $p|N$ need not be a LIP embedding. In fact, we shall next construct a strongly LIP flat arc $A\subset R^s$ such that $\overline{\sigma(U)}=S^{s-1}$ for every neighborhood U of a point in A. It is clear that no projection of R^s onto a proper subspace can define a LIP immersion of A.

For $s=1$ take $A=I^1$. For $s=2$ see 3.10 (4). Let $s\geqq 3$, and let (r, φ, z) be the cylindrical coordinates in R^s. Thus $x_1=r\cos\varphi$, $x_2=r\sin\varphi$, and $(x_3, \dots, x_s)=(z_1, \dots, z_{s-2})$. Define $g\colon R^s\to R^s$ by $g(r, \varphi, z)=(r, \varphi+\log|x|, z)$. It is easy to verify that g is an L-lipeomorphism for some L. For example, computing the derivative $g'(x)$ shows that it is bounded for $x\neq 0$. Moreover, $g^{-1}(r, \varphi, z)=(r, \varphi-\log|x|, z)$. Choose a countable dense set $\{a_j|j\in\mathrm{N}\}$ in $\{x\in S^{s-1}|x_1=0\}$. For each $j\in\mathrm{N}$ choose a rotation h_j of R^s such that $h_j(e_1)=e_1$, $h_j(e_2)=a_j$. Set $f_j=h_jgh_j^{-1}$ and observe that $f_j|S^{s-1}(e^{2\pi k})=\mathrm{id}$ for all integers k. Hence we can define a homeomorphism $f\colon R^s\to R^s$ by $f(x)=f_j(x)$ for $e^{-2j\pi}\leqq|x|\leqq e^{-2(j-1)\pi}$, $j\in\mathrm{N}$, and $f(x)=x$ for $x=0$ and $|x|\geqq 1$. By 2.35, $f|R^s\setminus 0$ is an L-lipeomorphism. Hence f is an L-lipeomorphism. Let J be the segment $\{te_2|-1\leqq t\leqq 1\}$. Then $A=fJ$ is a strongly LIP flat (in the obvious sense) arc in R^s consisting of two "twisted" logarithmic spirals and the origin. Clearly $\mathrm{cl}\,\sigma(A\cap B^s(\varepsilon))=S^{s-1}$ for all $\varepsilon>0$.

4.12. *Non-manifolds.* Every finite-dimensional separable metric space can be topologically embedded into a euclidean space, see [14, Theorem V3, p. 60]. We shall next show that the corresponding result is not true for LIP embeddings by constructing a countable metric space which cannot be LIP embedded into any euclidean space.

Let X be the set consisting of all positive integers and the point ∞. Let $a(1)>a(2)>\dots$ be a sequence such that $a(i)\to 0$ and the series $\sum_i a(i)^n$ diverges for every $n\in\mathrm{N}$. For example, we may choose $a(i)=1/\log(i+1)$. Define a metric d in X by $d(i,j)=a(\min(i,j))$ if $i\neq j$ and by $d(i,i)=0$. Then X has its usual topology and is compact. Suppose that $f\colon X\to R^n$ is a LIP embedding. Then there is $q>0$ such that $|f(i)-f(j)|\geqq qd(i,j)$ for all i, j in X. Since $d(i,j)\geqq a(i)$ whenever $i,j\in X\setminus\infty$ and $i\neq j$, it follows that the balls $B_i=B^n(f(i), qa(i)/2)$ are disjoint for $i\in X\setminus\infty$. Since the series $\sum_i a(i)^n$ diverges, the set $E=\cup\{B_i|i\in X\setminus\infty\}$ has an infinite measure. On the other hand, since fX is compact, E is bounded, and we obtain a contradiction.

Observe that we did not make use of the fact that f is LIP. Hence there is no homeomorphism $f\colon X\to Y\subset R^n$ such that f^{-1} is LIP.

5. Extension and approximation

5.1. The main results of this section deal with maps of a metric space X into a LIP manifold M. In 5.12 we show that if A is closed in X, every LIP map $f\colon A\to M$ has a LIP extension to a neighborhood of A in X. This result will be used in 5.18 to prove that every map $f\colon X\to M$ can be approximated by a LIP map.

5.2. Definition. A LIP *partition of unity* subordinated to an open cover $(U_j)_{j\in J}$ of a metric space X is family $(\varphi_j)_{j\in J}$ of LIP maps $\varphi_j\colon X\to I$ such that the supports $\operatorname{spt}\varphi_j=\operatorname{cl}\varphi_j^{-1}(0,1]$ form a locally finite family, $\operatorname{spt}\varphi_j\subset U_j$ for all $j\in J$, and $\sum_{j\in J}\varphi_j(x)=1$ for all $x\in X$.

5.3. Theorem. *Let $(U_j)_{j\in J}$ be an open cover of a metric space X. Then there is a* LIP *partition of unity subordinated to this cover.*

Proof. We may assume that $U_j\neq X$ for all j. Choose a locally finite open refinement $(V_j)_{j\in J}$ of $(U_j)_{j\in J}$ such that $\overline{V}_j\subset U_j$ [8, p. 162]. Set $\psi_j(x)=d(x, \complement V_j)$ and $\varphi_j=\psi_j/\sum_{j\in J}\psi_j$. Then $\operatorname{spt}\varphi_j\subset\overline{V}_j$, and the theorem follows. □

5.4. Theorem. *Let X be a metric space and let f_0, f_1 be real-valued functions on X such that f_0 is upper semicontinuous, f_1 is lower semicontinuous, and $f_0(x)<f_1(x)$ for all $x\in X$. Then there is a* LIP *map $g\colon X\to R^1$ such that $f_0(x)<g(x)<f_1(x)$ for all $x\in X$.*

Proof. The proof given in [8, 4.3, p. 171] for the paracompact case yields a LIP map g if a LIP partition of unity is used. □

5.5. A metric space X is an *absolute* LIP *extensor* (ALE) if for every closed set B in every metric space Y, every LIP map $f\colon B\to X$ has a LIP extension to Y. If every such f has a LIP extension to a neighborhood of B, X is an *absolute* LIP *neighborhood extensor* (ALNE). Our next goal is to show that every LIP manifold is an ALNE. The proof is based on LIP versions of Tietze's theorem, due to McShane, and Hanner's theorem, which states that being an ALNE is a local property.

5.6. Lemma. *Let A be a subset of a metric space X and let $f\colon A\to R^n$ be Lipschitz. Then f has a Lipschitz extension $g\colon X\to R^n$ with* $\operatorname{lip} g\leqq n^{1/2}\operatorname{lip} f$.

Proof. Apply McShane [19, Theorem 1] to each coordinate map of f. □

5.7. Theorem. *R^n and R^n_+ are* ALE*'s.*

Proof. Since R^n_+ is a LIP retract of R^n, it suffices to show that R^n is an ALE. Let B be closed in Y, and let $f\colon B\to R^n$ be LIP. For each $b\in B$ choose an open neighborhood U_b in Y such that $f|U_b\cap B$ is Lipschitz. By 5.6, $f|U_b\cap B$ has a LIP extension $f_b\colon Y\to R^n$. Let $(\varphi_i)_{i\in I}$ be a LIP partition of unity subordinated to the cover $\{Y\setminus B\}\cup\{U_b|b\in B\}$ of Y. Set $J=\{j\in I|B\cap\operatorname{spt}\varphi_j\neq\emptyset\}$. For each $j\in J$ choose $b\in B$ with $\operatorname{spt}\varphi_j\subset U_b$ and set $g_j=f_b$. Then $g(x)=\sum_{j\in J}\varphi_j(x)g_j(x)$ defines a LIP map $g\colon X\to R^n$. If $x\in B$ and $\varphi_i(x)>0$, then $i\in J$ and $g_i(x)=f(x)$, whence $g(x)=f(x)$. □

5.8. Lemma. *Let $\mathcal{U}$ be an open cover of a metric space X satisfying the following conditions:*

(1) *If $U\in\mathcal{U}$ and V is open in U, then $V\in\mathcal{U}$.*

(2) *If U and V are in $\mathcal{U}$, then $U\cup V\in\mathcal{U}$.*

(3) *If* $\{U_i|i\in I\}$ *is a pairwise disjoint subfamily of* $\mathcal{U}$, *then* $\bigcup\{U_i|i\in I\}\in\mathcal{U}$. *Then* $\mathcal{U}$ *contains all open sets of* X.

Proof. This follows directly from Michael [20, 2.1 (e) and 3.3]. □

5.9. We shall give a general version of Hanner's theorem and obtain the LIP result as a special case. Let F be a class of maps between metric spaces. We say that a metric space X is an ANE_F if for every closed set B in every metric space Y, every F-map $f\colon B\to X$ has an F-extension $g\colon U\to X$ to an open neighborhood U of B. Thus ALNE means ANE_F for $F=\mathrm{LIP}$.

5.10. Theorem. *Let F be a class of maps between metric spaces such that the following conditions are satisfied:*

(1) *If $f\colon Y\to X$ is in F and if B is open or closed in Y, then $f|B\colon B\to X$ is in F.*
(2) *Let A be open in X, let $j\colon A\to X$ be the inclusion, and let $f\colon Y\to A$. Then $f\in F$ if and only if $jf\in F$.*
(3) *If $f\colon Y\to X$ is a map such that every point in Y has a neighborhood U such that $f|U\in F$, then $f\in F$.*

Suppose that X is a metric space such that every point in X has a neighborhood which is an ANE_F. *Then X is an* ANE_F.

Proof. The proof of Hanner [13, 3.1] shows that every open subset of an ANE_F is an ANE_F. Hence every point of X has an open neighborhood which is an ANE_F. Let $\mathcal{U}$ be the family of all open subsets of X which are ANE_F's. Then $\mathcal{U}$ is an open cover of X. It suffices to show that $\mathcal{U}$ satisfies the conditions of 5.8. The condition (1) follows from the first remark of the present proof. The condition (3) is proved exactly as in Michael [20, 4.1 (c)]. To prove (2), we can follow Hanner's proof [13, 3.3, a)] with a slight modification. Indeed, the maps g and F constructed in the proof need not be in F. To arrange this, we choose the sets Y_1, Y_2 so that $\overline{Y}_1\cap\overline{Y}_2=\emptyset$. Then choose open neighborhoods V_i of $\overline{Y}_i$ such that $\overline{V}_1\cap\overline{V}_2=\emptyset$. Let $V_0=Y\setminus(V_1\cup V_2)$. Then $g|(U_0\cap V_0)\cup B$ is in F, since it is locally in F. Replacing Y_i by V_i we may thus assume that g is in F. To show that the map F is in the family F, we observe that the sets $W_1=U_1\setminus\overline{Y}_2$ and $W_2=U_2\setminus\overline{Y}_1$ form an open cover of U and $F|W_i=g_i|W_i$, $i=1, 2$. □

5.11. Corollary. *Let X be a metric space such that every point has a neighborhood which is an* ALNE. *Then X is an* ALNE. □

5.12. Theorem. *Every* LIP *manifold is an* ALNE.

Proof. Let M be a LIP manifold. Every point in M has a neighborhood U lipeomorphic to R^n or R^n_+. Since the property ALE is obviously a LIP invariant, it follows from 5.7 that U is an ALE and hence an ALNE. By 5.11, M is an ALNE. □

5.13. Theorem. *Let M be a* LIP *submanifold of R^n. Then M is a* LIP *neighborhood retract of R^n.*

Proof. Since M is locally compact, it has an open neighborhood U such that M is closed in U. By 5.12, the identity map $M \to M$ has a LIP extension to an open neighborhood V of M in U. $\square$

5.14. *Approximation.* Suppose that $f\colon X \to Y$ is a map between metric spaces X, Y. It is natural to ask whether f can be approximated in some sense by LIP maps. If Y is too general, the answer is negative, as is seen from the following counterexample: Let $X=[-1, 1]$, let $f(x)=(x, x \sin(1/x))$, $f(0)=(0, 0)$, and let $Y=\operatorname{im} f \subset R^2$. Then Y is an arc which is not locally rect fiable at 0. If $g\colon X \to Y$ is LIP, then $\operatorname{im} g$ lies entirely either in the left half plane or in the right half plane. Thus f cannot be approximated by LIP maps.

We shall show that the approximation is always possible if Y is a LIP manifold. Moreover, the approximating map can be obtained from f by a small homotopy. A relative version will also be given.

5.15. *Function spaces.* Given two metric spaces X, Y we let $T(X, Y)$ denote the set of all maps $f\colon X \to Y$. We shall use the *majorant topology* in $T(X, Y)$. A basis for this topology consists of sets $U(f, \varepsilon)=\{g \mid d'(f(x), g(x))<\varepsilon(x) \text{ for all } x \in X\}$. Here d' is the metric of Y, $f \in T(X, Y)$, and $\varepsilon\colon X \to (0, \infty)$ is an arbitrary map. This topology is equal to the *graph topology* (Whitehead [31, (5.2), p. 172]), whose basis consists of sets $W_U=\{f \mid \Gamma(f) \subset U\}$. Here $\Gamma(f)$ is the graph of $f\colon X \to Y$ and U is a arbitrary open set in $X \times Y$. Hence the topology of $T(X, Y)$ is independent of d'.

The elements of $U(f, \varepsilon)$ are called *ε-approximations* of f. A homotopy $h\colon X \times I \to Y$ is called an *ε-homotopy* if h_t is an ε-approximation of h_0 for every $t \in I$.

Let X, Y, Z be metric spaces, and let $\varphi\colon Y \to Z$ be a map. Then φ induces a function $\varphi_*\colon T(X, Y) \to T(X, Z)$ by $\varphi_*(f)=\varphi f$. Using the graph topology, it is easy to give a straightforward proof for the following result:

5.16. Lemma. *$\varphi_*\colon T(X, Y) \to T(X, Z)$ is continuous.* $\square$

5.17. Theorem. *Let M be a* LIP *manifold, X a metric space, $f\colon X \to M$ continuous, $\varepsilon\colon X \to (0, \infty)$ continuous, and $A \subset X$ closed. Then there is a continuous $\delta\colon A \to (0, \infty)$ such that if $g\colon A \to M$ is* LIP *and a δ-approximation to $f|A$, then g has a* LIP *extension $u\colon X \to M$ which is ε-homotopic to f. Moreover, if $f|A$ is* LIP *and $g=f|A$, the homotopy can be chosen to be fixed on A.*

Proof. Special case: M is an open subset W of R^n. We may assume that $\varepsilon(x)<d(f(x), R^n \setminus W)$ for all $x \in X$. We show that $\delta=\varepsilon|A$ satisfies the conditions of the theorem.

So let $g\colon A \to W$ be a LIP map, which is an $\varepsilon|A$-approximation to $f|A$. For every $x \in X$ choose an open neighborhood U_x as follows: If $x \in X \setminus A$, then $U_x \subset X \setminus A$ and $d(fU_x)<\inf \varepsilon U_x$. If $x \in A$, then $g|U_x \cap A$ is Lipschitz. By 5.6, we can choose a LIP extension $g_x\colon X\to R^n$ of $g|U_x\cap A$. We may assume, replacing U_x by a smaller neighborhood, that $|g_x(y)-f(y)|<\varepsilon(y)$ for all $y\in U_x$ and for all $x\in A$. Let $(\varphi_x)_{x\in X}$ be a LIP partition of unity subordinated to the cover $(U_x)_{x\in X}$. For each $x\in X$ define $u_x\colon X\to R^n$ by $u_x=g_x$ if $x\in A$ and by $u_x(y)=f(x)$ if $x\in X\setminus A$ and $y\in X$. Set $u=\sum_{x\in X}\varphi_x u_x$. Then u is clearly LIP. If $a\in A$ and $\varphi_x(a)>0$, then $u_x(a)=g_x(a)=g(a)$, whence $u(a)=g(a)$. If $y\in X$ and $\varphi_x(y)>0$, then $|u_x(y)-f(y)|<\varepsilon(y)$,w hence $|u(y)-f(y)|<\varepsilon(y)$. Setting $h(x,t)=(1-t)f(x)+$ $+tu(x)$ we obtain an ε-homotopy from f to u. Since $\varepsilon(x)<d(f(x), R^n\setminus W)$, $\operatorname{im} h\subset W$. If $g=f|A$, h is fixed on A. The special case is proved.

General case. By 4.2, there is a LIP embedding of M into a euclidean space R^n for some n. From 5.16 it follows that we may assume $M\subset R^n$. By 5.13, there is a LIP retraction $r\colon W\to M$ of an open neighborhood W of M. By 5.16, there is a continuous $\delta\colon X\to(0,\infty)$ such that if $v\colon X\to W$ is a δ-approximation to f, then $rv\colon X\to M$ is an ε-approximation to $rf=f$. We may assume $\delta(x)<d(f(x), R^n\setminus W)$ for all $x\in X$. Let $g\colon A\to M$ be LIP and a $\delta|A$-approximation to $f|A$. By the special case, g has a LIP extension $v\colon X\to W$ which is δ-homotopic to f in W. Then $u=rv\colon X\to M$ is a LIP extension of g. If $h\colon X\times I\to W$ is a δ-homotopy from f to v, then rh is an ε-homotopy from f to u in M. If $g=f|A$ and if h is fixed on A, then also rh is fixed on A. □

5.18. Corollary. *Let M be a* LIP *manifold, X a metric space, $f\colon X\to M$ continuous, and $\varepsilon\colon X\to(0,\infty)$ continuous. Then there is a* LIP *map $g\colon X\to M$ which is ε-homotopic to f. Moreover, if $f|A$ is* LIP *for a closed set $A\subset X$, the homotopy can be chosen to be fixed on A.* □

6. General position

6.1. General position is an important tool in PL topology. A typical example is the following result: Let P and Q be compact polyhedra of dimensions p and q in the interior of a PL n-manifold M. If $p+q\leqq n-1$, there is for every $\varepsilon>0$ a PL homeomorphism $h\colon M\to M$ such that $d(h,\mathrm{id})<\varepsilon$ and $P\cap hQ=\emptyset$. In this section we try to find LIP analogues of results like this. It turns out that on a LIP manifold, polyhedral conditions can often be replaced by assumptions concerning rectifiability and Hausdorff measure. For example, we shall prove a LIP version of the above result, assuming that P is p-rectifiable and Q is of Hausdorff $(q+1)$-measure zero.

6.2. *Hausdorff measure and rectifiability.* An excellent reference on these topics is Federer [10], and we shall use his terminology and notation. Thus we let $\mathscr{H}^q(X)$, $q\geqq 0$, denote the q-dimensional Hausdorff measure [10, p. 171] of a separable metric space X. Recall that $\mathscr{H}^0$ is the counting measure. If f is an L-Lipschitz map of X, then $\mathscr{H}^q(fX)\leqq L^q\mathscr{H}^q(X)$. Hence the following properties are LIP invariants of a separable metric space X: (1) $\mathscr{H}^q(X)=0$, (2) $\mathscr{H}^q(X)$ is σ-finite, (3) the Hausdorff dimension of X is q. For a compact space X, the property $\mathscr{H}^q(X)<\infty$ is also a LIP invariant.

A metric space X is *p-rectifiable* if there is a bounded set $F\subset R^p$ and a Lipschitz map of F onto X. If X is a countable union of p-rectifiable sets, it is *countably p-rectifiable*. Equivalently, X is countably p-rectifiable if there is a set $F\subset R^p$ and a LIP map of F onto X. Thus countable p-rectifiability is a LIP invariant property.

We shall use the phrase "μ almost all" in its usual sense, meaning all except for a set of μ measure zero, and we may omit μ if it is the ordinary Lebesgue measure ($\mathscr{H}^n$ in R^n). Following Federer [10], we do not make a distinction between measure and outer measure.

6.3. Lemma. *Suppose that $A\subset R^p$ with $\mathscr{H}^p(A)<\infty$, α: $A\to R^n$ is Lipschitz, $E\subset R^n$, and g: $E\times A\to R^n$ is a map such that $\varphi(x,y)=(g(x,y),y)$ defines a Lipschitz embedding φ: $E\times A\to R^n\times A$. Suppose also that $Q\subset R^n$ with $\mathscr{H}^q(Q)<\infty$ and that $0\leqq k\leqq p+q$. Then $\mathscr{H}^k(Q\cap(\alpha+g_x)A)<\infty$ for $\mathscr{H}^{p+q-k}$ almost all $x\in E$, where $g_x(y)=g(x,y)$. In particular, if $p+q\leqq n$, then $Q\cap(\alpha+g_x)A$ is finite for almost all $x\in E$.*

Proof. Setting $\psi(x,y)=\varphi(x,y)+(\alpha(y),0)$ we obtain another Lipschitz embedding ψ: $E\times A\to R^n\times A$. Indeed, if φ_1: $\operatorname{im}\varphi\to E\times A$ is the inverse of φ, then $\psi_1(x,y)=\varphi_1(x-\alpha(y),y)$ defines a Lipschitz inverse ψ_1: $\operatorname{im}\psi\to E\times A$ of ψ. By [10, 2.10.45], $\mathscr{H}^{p+q}(Q\times A)<\infty$. Hence the set $Q_1=\psi^{-1}[Q\times A]$ is of finite $\mathscr{H}^{p+q}$ measure. For $x\in R^n$ set $D(x)=\{y\in R^p|(x,y)\in Q_1\}$. Applying [10, 2.10.27] with the substitution $Y\mapsto R^n$, $Z\mapsto A$, $A\mapsto Q_1$, $m\mapsto p+q-k$ yields $\mathscr{H}^k(D(x))<\infty$ for $\mathscr{H}^{p+q-k}$ almost all $x\in R^n$. If $\alpha(y)+g(x,y)=z\in Q$, then $(z,y)=\psi(x,y)$ and $y\in D(x)$. Hence $Q\cap(\alpha+g_x)A\subset p_1\psi[x\times D(x)]$ where p_1: $R^n\times A\to R^n$ is the projection. The lemma follows. □

6.4. Lemma. *Suppose that $A\subset R^p$, α: $A\to R^n$ is* LIP, *$E\subset R^n$, and g: $E\times A\to R^n$ is a map such that $\varphi(x,y)=(g(x,y),y)$ defines a* LIP *embedding φ: $E\times A\to R^n\times A$. Suppose also that $Q\subset R^n$ with $\mathscr{H}^q(Q)=0$ and that $0\leqq k\leqq p+q$. Then $\mathscr{H}^k(Q\cap(\alpha+g_x)A)=0$ for $\mathscr{H}^{p+q-k}$ almost all $x\in E$. In particular, if $p+q\leqq n$, then $Q\cap(\alpha+g_x)A=\emptyset$ for almost all $x\in E$.*

Proof. A slight modification of the proof of 6.3. □

6.5. Theorem. *Suppose that $P\subset R^n$ is countably p-rectifiable and $Q\subset R^n$ with $\mathscr{H}^q(Q)=0$. If $p+q\leqq n$, then $Q\cap(P+x)=\emptyset$ for almost all $x\in R^n$.*

Proof. Apply 6.4 with $E=R^n$ and $g(x,y)=x$. □

6.6. *Notation.* We let $\|x\|=\max(|x_1|,\dots,|x_n|)$ denote the Banach norm of a vector $x\in R^n$. For $z\in\operatorname{int}I^n$, let w_z: $I^n\to I^n$ be the PL homeomorphism defined by $w_z(x)=x+(1-\|x\|)z$. Thus w_z is the z-cone extension of $\operatorname{id}|\partial I^n$ to $I^n=0(\partial I^n)$.

6.7. Lemma. *Let* $P \subset \operatorname{int} I^n$ *be countably* p*-rectifiable and let* $Q \subset \operatorname{int} I^n$ *with* $\mathscr{H}^q(Q)=0$. *If* $p+q \geqq n$, *then* $\mathscr{H}^{p+q-n}(Q \cap w_z P)=0$ *for almost all* $z \in \operatorname{int} I^n$. *If* $p+q \leqq n$, *then* $Q \cap w_z P=\emptyset$ *for almost all* $z \in \operatorname{int} I^n$.

Proof. Choose a LIP map α of a set $A \subset R^p$ onto P. Set $E=\operatorname{int} I^n$ and define $g: E \times A \to E$ by $g(z, y)=(1-\|\alpha(y)\|)z$. Then $\varphi(z, y)=(g(z, y), y)$ defines a LIP embedding $\varphi: E \times A \to R^n \times A$, since it has a LIP inverse φ_1: $\operatorname{im} \varphi \to E \times A$, defined by $\varphi_1(z, y)=(z/(1-\|\alpha(y)\|), y)$. Since $\alpha+g_z=w_z\alpha$, the lemma follows from 6.4. □

6.8. Definition. A LIP *isotopy* of a metric space X is a level preserving lipeomorphism $F: X \times I \to X \times I$ such that $F_0=\mathrm{id}$. Here we use the customary notation $F(x, t)=(F_t(x), t)$. If $d(F_t(x), x)<\varepsilon$ for all $x \in X$ and $t \in I$, F is said to be an ε*-isotopy*. If $F|(X \setminus U) \times I=\mathrm{id}$, F is *supported* by U.

6.9. Theorem. *Let M be a* LIP *n-manifold, let $P \subset \operatorname{int} M$ be compact and countably p-rectifiable, let $Q \subset M$ with $\mathscr{H}^q(Q)=0$, let U be a neighborhood of $P \cap \bar{Q}$, and let $\varepsilon>0$. Then there is a* LIP *ε-isotopy F of M supported by U such that $\mathscr{H}^{p+q-n}(Q \cap F_1 P)=0$. For $p+q \leqq n$ this means $Q \cap F_1 P=\emptyset$. If M is a* PL *manifold, F can be chosen to be a* PL *isotopy.*

Proof. Choose LIP n-balls $B_1, \ldots, B_k$ in U so that $P \cap \bar{Q} \subset \bigcup\{\operatorname{int} B_i | 1 \leqq i \leqq k\}$. Choose lipeomorphisms $\varphi_i: B_i \to I^n$ and set $P_1=\varphi_1[P \cap \operatorname{int} B_1]$, $Q_1=\varphi_1[Q \cap \operatorname{int} B_1]$. By 6.7, there is $z_1 \in \operatorname{int} I^n$ such that $\mathscr{H}^{p+q-n}(Q_1 \cap w_{z_1} P_1)=0$. The Alexander trick [25, p. 37] gives a PL isotopy of I^n fixed on ∂I^n and finishing with w_{z_1}. With the aid of φ_1, we can transfer this isotopy to a LIP isotopy of B_1. Extending this isotopy by the identity, we obtain a LIP isotopy F^1 of M supported by B_1 such that $\mathscr{H}^{p+q-n}(Q \cap F_1^1 P \cap \operatorname{int} B_1)=0$. Setting $P_2=\varphi_2[F_1^1 P \cap \operatorname{int} B_2]$, $Q_2=\varphi_2[Q \cap \operatorname{int} B_2]$ we similarly choose z_2 with $\mathscr{H}^{p+q-n}(Q_2 \cap w_{z_2} P_2)=0$ and obtain a LIP isotopy F^2 of M supported by B_2 with $\mathscr{H}^{p+q-n}(Q \cap F_1^2 F_1^1 P \cap (\operatorname{int} B_1 \cup \operatorname{int} B_2))=0$. After k steps we have a LIP isotopy $F=F^k \ldots F^1$ of M supported by U such that $\mathscr{H}^{p+q-n}(Q \cap F_1 P)=0$. Since the points z_i can be chosen to be arbitrarily close to the origin, F can be chosen to be an ε-isotopy. If M is PL, all maps can be chosen to be PL. □

6.10. Theorem. *The condition* $\mathscr{H}^{p+q-n}(Q \cap F_1 P)=0$ *of* 6.9 *can be replaced by* $\mathscr{H}^{p+q-n}(P \cap F_1 Q)=0$.

Proof. Replace the isotopy F by its inverse F^{-1}: $M \times I \to M \times I$. □

6.11. Remark. There are several obvious modifications of 6.9. We may use 6.3 instead of 6.4 and obtain a finiteness condition on $\mathscr{H}^{p+q-n}(Q \cap F_1 P)$ instead of the zero condition. For example, if P^p and Q^q are compact LIP submanifolds of M^n with $p+q \leqq n$, then there is a small LIP isotopy of M which carries P onto P' with $Q \cap P'$ finite.

6.12. *Applications.* We shall give three applications of LIP general position. First, we show that removing a set of sufficiently small Hausdorff dimension does not change the lowdimensional homotopy groups of a LIP manifold. Next, we show that for a compact set $X \subset R^n$, $\mathcal{H}^{q+1}(X)=0$ implies $\dim X \leqq q$ where dem means demension in the sense of Štanko. Since dim$\leqq$dem, this is a stronger result than the classical $\dim X \leqq q$ [14, p. 104]. The third application deals with PL engulfing. The usual engulfing theorems [26, Chapter 4] are concerned with engulfing a polyhedron of dimension r. Using 6.9 we can show that the polyhedron can often be replaced by an arbitrary compact set of $\mathcal{H}^{r+1}$ measure zero.

6.13. Theorem. *Let M be a* LIP *n-manifold without boundary, let E be closed in M with $\mathcal{H}^q(E)=0$, and let $x_0 \in M \setminus E$. Then the homomorphisms $\pi_i(M\setminus E, x_0) \to \pi_i(M, x_0)$ induced by the inclusion are injective for $0 \leqq i \leqq n-q-1$ and surjective for $0 \leqq i \leqq n-q$.*

Proof. Suppose first $0 \leqq i \leqq n-q$. Let $\alpha \in \pi_i(M, x_0)$. By 5.18, α has a LIP representative $f\colon (I^i, \partial I^i) \to (M, x_0)$. By 6.9, there is a LIP isotopy of M which carries $\operatorname{im} f$ off E and keeps x_0 fixed. Hence f is homotopic rel ∂I^i to a map into $M\setminus E$.

Next assume $0 \leqq i \leqq n-q-1$. Let $f\colon (I^i, \partial I^i) \to (M\setminus E, x_0)$ be a map homotopic to the constant map c rel ∂I^i in M. We must show that $f \simeq c$ rel ∂I^i in $M\setminus E$. By 5.18, we may assume that f is LIP. Choose a homotopy $H\colon (I^i \times I, \partial I^i \times I) \to (M, x_0)$ from f to c. By 5.18, we may again assume that H is LIP. Now use 6.9 to isotope $\operatorname{im} H$ off E keeping $x_0 \cup \operatorname{im} f$ fixed, and the theorem follows. □

6.14. Remarks. The above result is true, of course, for PL and DIFF manifolds. For $i=0$ it means that no component of M is contained in E if $\mathcal{H}^n(E)=0$ and that each component of M contains exactly one component of $M\setminus E$ if $\mathcal{H}^{n-1}(E)=0$. These are classical results, see [14, Theorem VII 3, p. 104 and Corollary 1, p. 48]. The theorem is also true for manifolds with boundary. A special case of 6.13 was proved in [17, 3.3].

6.15. Theorem. *Let $X \subset R^n$ be compact with $\mathcal{H}^{q+1}(X)=0$. Then* dem $X \leqq q$.

Proof. We shall use the dual demension Dem of Štanko [28], see Edwards [9, Proposition 1.2 (2′)]. Let P be a closed polyhedron in R^n with $\dim P = p \leqq n-q-1$, let U be a neighborhood of $X \cap P$, and let $\varepsilon > 0$. Then P is countably p-rectifiable. By 6.10, there is an ε-isotopy of R^n supported by U which carries X off P. Hence dem $X \leqq q$. □

6.16. Theorem (Engulfing). *Suppose that M is an r-connected* PL *n-manifold without boundary with $r \leqq n-3$. If $X \subset M$ is compact and $\mathcal{H}^{r+1}(X)=0$, then X is contained in a* PL *n-ball.*

Proof. Choose a compact polyhedral neighborhood Y of X and a triangulation K of Y such that no simplex of K meets both X and ∂Y. Let J_1 be the $(n-r-1)$-

skeleton of K and let $J_2=\{A\in K|A\cap X=\emptyset\}$. By 6.10, there is a PL homeomorphism $h\colon M\to M$ such that $|J_1|\cap hX=\emptyset$ and $h|(M\setminus Y)\cup|J_2|=\mathrm{id}$. Let L be the subcomplex of the barycentric subdivision of K consisting of those simplexes which do not meet $|J_1|\cup|J_2|$. Then $\dim L=r$. By the engulfing theorem of Stallings [26, p. 150], $|L|$ is contained in the interior of a PL ball $B\subset M$. Using the map described in 2.43 (not in 2.42!), we find a PL homeomorphism $g\colon M\to M$ which keeps $|L|\cup|J_1|\cup|J_2|\cup(M\setminus Y)$ fixed and maps $\operatorname{int} B$ onto a set containing hX. Then X is contained in the PL ball $h^{-1}gB$. □

6.17. The above results deal with general position of *sets*. We have been less successful with the general position of *maps*. For example, we would like to show that a LIP map $M^n\to N^{2n+1}$ can be LIP approximated by a LIP embedding, but we have not even been able to prove this in the euclidean case. However, we have established the following weaker result:

6.18. Theorem. *Let M be a compact* LIP *manifold with $\partial M=\emptyset$, and let $g\colon M\to R^{2n+1}$ be a* LIP *map. Then for every $\varepsilon>0$ there is an injective* LIP *map $g_1\colon M\to R^{2n+1}$ such that $d(g_1,g)<\varepsilon$ and* $\operatorname{lip}(g_1-g)<\varepsilon$.

Proof. Since the result is rather unsatisfactory, we only give a sketch. Let E be the space of all LIP maps $M\to R^{2n+1}$. Then E is a Banach space with the norm $\|f\|=\sup_{x\in M}|f(x)|+\operatorname{lip} f$. For $\varepsilon>0$, the set $G_\varepsilon=\{f\in E|d(f^{-1}(y))<\varepsilon$ for all $y\in R^{2n+1}\}$ is open in E. By Baire's theorem, it suffices to show that G_ε is dense. Let $f\in E$. Cover M with interiors $U_1,\dots,U_k$ of LIP balls such that $d(U_i)<\varepsilon$ for all i. Using 6.4 we can find a map f_1 such that $f_1(x)=f(x)$ for $x\in M\setminus U_1$, $f[M\setminus U_1]\cap f_1U_1=\emptyset$, and $\|f_1-f\|$ is small. Then modify similarly f_1 in U_2. After k steps we obtain a map f_k such that each fiber of f_k is contained in some U_i. Thus $f_k\in G_\varepsilon$, and the theorem follows. □

7. Collaring and the Schönflies problem

7.1. In this section we first prove a LIP version of an important result of Brown [5], which states that a locally collared set is collared. In particular, the boundary of a LIP manifold M has a collar in M. Next we prove a Schönflies theorem in the LIP category. In particular, a PL $(n-1)$-sphere in R^n always bounds a LIP ball.

7.2. Definitions. We let I' denote the interval $[0,1)$. Let Y be a subset of a metric space X. A LIP *collar* of Y in X is a LIP embedding $c\colon Y\times I'\to X$ such that $c(x,0)=x$ for all $x\in Y$ and $\operatorname{im} c$ is an open neighborhood of Y in X. A *local* LIP *collar* is a family $(U_j,c_j)_{j\in J}$ such that $(U_j)_{j\in J}$ is a cover of Y, U_j is open in Y, and c_j is a LIP collar of U_j in X.

7.3. Lemma. *Let X be a metric space, let $Y\subset X$, let $c\colon Y\times I'\to X$ be a map such that $c(x,0)=x$ for all $x\in Y$, and let $\varepsilon\colon Y\to(0,\infty)$ be continuous. Then there* *is a* LIP *map* $\delta\colon Y\to(0,1]$ *such that* $d(x,c(x,t))<\varepsilon(x)$ *whenever* $x\in Y$ *and* $0\le t<<\delta(x)$.

Proof. For $x\in Y$ set $g(x)=\sup\{t|c[x\times[0,t]]\subset B(x,\varepsilon(x))\}$. It is easy to see that g is lower semicontinuous. By 5.4, there is a LIP map $\delta\colon Y\to R^1$ such that $0<\delta(x)<g(x)$ for all x. □

7.4. Theorem. *Let Y be a subset of a metric space X. If Y has a local* LIP *collar in X, then Y has a* LIP *collar in X.*

Proof. Let $\mathscr{U}$ be the family of all sets U open in Y such that U has a LIP collar in X. Then $\mathscr{U}$ is a cover of Y. It suffices to show that $\mathscr{U}$ satisfies the conditions of 5.8. The condition (1) is clear. Next it is easy to verify that Brown's proof for the TOP case of (3) [26, Lemma 1.7.1, p. 35] yields a LIP collar if all given collars are LIP. We shall prove (2) using an idea of Connelly [6]. However, since we do not assume that Y is closed in X, an additional argument is needed.

Assume that $U_1, U_2\in\mathscr{U}$, and set $U=U_1\cup U_2$. Let $c_i\colon U_i\times I'\to X$ be a LIP collar of U_i, $i=1,2$. Applying 7.3 we find LIP maps $\delta_i\colon U_i\to(0,1]$ such that $d(x,c_i(x,\delta_i(x)t))<d(x,U\setminus U_i)$ for all $(x,t)\in U_i\times I'$, $i=1,2$. Thus $d(c_i(x,\delta_i(x)t),U\setminus U_i)>0$ for all $(x,t)\in U_i\times I'$. Replacing c_i by the LIP collar $(x,t)\mapsto c_i(x,\delta_i(x)t)$ we may therefore assume that U is closed in $N=\operatorname{im} c_1\cup\operatorname{im} c_2$. Moreover, we may assume that $c_i^{-1}U=U_i\times 0$. From now on, all closures will be taken in N. Choose a LIP partition of unity (φ_1,φ_2) on U such that $\operatorname{spt}\varphi_i==A_i\subset U_i$. Then choose an open neighborhood V_i of A_i in N such that $\overline{V}_i\subset\operatorname{im} c_i$. Setting $U_i'=V_i\cap Y$ we have $U=U_1'\cup U_2'$. By 7.3, there are LIP maps $\delta_i\colon U_i'\to(0,1]$ such that $d(x,c_i(x,\delta_i(x)t))<d(x,N\setminus V_i)$ for all $(x,t)\in U_i'\times I'$. Setting $B_i==\{c_i(x,\delta_i(x)t)|x\in A_i,\ 0\le t\le 1/2\}$ we thus have $B_i\subset V_i$. Since B_i is closed in $\operatorname{im} c_i$ and $\overline{B}_i\subset\overline{V}_i\subset\operatorname{im} c_i$, B_i is closed in N. Replacing U_i by U_i', c_i by $c_i'(x,t)=(x,\delta_i(x)t)$, and N by $\operatorname{im} c_1'\cup\operatorname{im} c_2'$, we may therefore assume that $c_i[A_i\times[0,1/2]]$ is closed in N for $i=1,2$.

Set $M=N\cup U\times[-1,0]$, where $(x,0)$ is identified with x. We shall construct a lipeomorphism $g\colon N\to M$ such that $g(x)=(x,-1)$ for $x\in U$. Then $(x,t)\mapsto\mapsto g^{-1}(x,t-1)$ will give a LIP collar of U in N, and hence in X.

Define $h_i\colon U_i\times[-1,1)\to M$ by $h_i(x)=c_i(x)$ for $x\in U_i\times I'$ and by $h_i(x)=x$ otherwise. It is easy to see, for example by 2.22, that h_i is a LIP embedding. Let $f_1\colon U_1\times I'\to U_1\times[-1,1)$ be defined by $f_1(x,t)=(x,r_x(t))$ where $r_x(t)=t$ for $t\in[1/2,1)$ and r_x maps $[0,1/2]$ affinely onto $[-\varphi_1(x),1/2]$. From 2.40 it follows that f_1 is a LIP embedding. Next define $g_1\colon N\to M$ by $g_1(x)=h_1(f_1(h_1^{-1}(x)))$ for $x\in\operatorname{im} c_1$ and by $g_1(x)=x$ otherwise. Since $c_1[A_1\times[0,1/2]]$ is closed in N, g_1 is a LIP embedding. Moreover, $h_2^{-1}g_1N=\{(x,t)|x\in U_2,\ -\varphi_1(x)\le t<1\}$. Let $f_2\colon h_2^{-1}g_1N\to U_2\times[-1,1)$ be defined by $f_2(x,t)=(x,s_x(t))$ where $s_x(t)=t$ for $t\in[1/2,1)$ and s_x maps $[-\varphi_1(x),1/2]$ affinely onto $[-1,1/2]$. By 2.40, f_2 is a lipeomorphism. Define $g_2\colon g_1N\to M$ by $g_2(x)=h_2(f_2(h_2^{-1}(x)))$ for $x\in\operatorname{im} h_2\cap g_1N$

and by $g_2(x)=x$ otherwise. Since $h_2[A_2\times[-1, 1/2]]$ is closed in M, g_2 is a lipeomorphism. Then $g=g_2g_1$: $N\to M$ is the sought-for lipeomorphism. □

7.5. Corollary. *If M is a* LIP *manifold, then ∂M has a* LIP *collar in M.* □

7.6. *Schönflies problem.* Let f: $S^{n-1}\to S^n$ be a topological embedding. If $n=2$, then f can be extended to a homeomorphism $S^n\to S^n$, according to the classical Schönflies theorem. Because of the wild embeddings, the result is not true for $n\geqq 3$. However, if f can be extended to an embedding g of an annulus $\bar{B}^n\setminus B^n(a)$, $0<a<1$, then f can be extended to a homeomorphism of $\bar{B}^n$ onto $\bar{D}$ where D is the component of $S^n\setminus fS^{n-1}$ containing $g[B^n\setminus B^n(a)]$. This result is due to Brown [3] and also to Mazur [18] and Morse [22]. Similarly, we may consider the CAT Schönflies problem, where CAT is one of the categories DIFF, PL, LIP, TOP: Suppose that B is a CAT n-ball, S a CAT n-sphere, f a CAT embedding of a neighborhood of ∂B in B into S. Does $f|\partial B$ have an extension to a CAT embedding of B into S? Note that in this case, $f\partial B$ has a CAT collar in $\bar{D}$ where D is as above. In other words, $\bar{D}$ is a CAT manifold with boundary $f\partial B$. Stated in this form, the answer is known to be negative for CAT=DIFF, since there are diffeomorphisms $S^{n-1}\to S^{n-1}$ which cannot be extended to a diffeomorphism of $\bar{B}^n$. In the PL case, the answer is positive for $n\neq 4$ and unknown for $n=4$. We shall show that the answer is positive for CAT=LIP. Without any collaring condition, a LIP $(n-1)$-sphere in S^n need not even bound a topological ball. A counterexample can be constructed with the aid of a fattened Fox—Artin arc, see 3.10.

A *quasiconformal* Schönflies theorem was proved by Gehring [11], who used an explicit version of the Mazur—Morse method. His proof needs only slight changes to yield the LIP theorem. To avoid repetition, we refer to Gehring's proof as given in [29, Section 41] and give only the modifications needed in the LIP case.

7.7. Theorem. *Suppose that $0<a<1$ and that f is a* LIP *embedding of the annulus $E=\bar{B}^n\setminus B^n(a)$ into S^n. Then $f|S^{n-1}$ can be extended to a* LIP *embedding f^*: $\bar{B}^n\to S^n$.*

Proof. We may assume $n\geqq 2$. We work in the compactified space $\bar{R}^n=R^n\cup\infty$. With the spherical metric q [29, p. 37], it is a metric space lipeomorphic to S^n. Moreover, the identity map is a lipeomorphism of (R^n, q) onto R^n with the euclidean metric. All complements and closures are taken in $\bar{R}^n$.

Step 1. (See [29, 41.1].) Suppose that (1) D_1, D_2 are domains such that $\bar{D}_1\cap\bar{D}_2=\emptyset$ and $\bar{D}_1\cup\bar{D}_2\subset B^n$, (2) B_1 and B_2 are open round balls such that $\bar{B}_1\cap\bar{B}_2=\emptyset$ and $\bar{B}_1\cup\bar{B}_2\subset B^n$, (3) f is a lipeomorphism of $\complement(D_1\cup D_2)$ onto $\complement(B_1\cup B_2)$ such that $f\partial D_i=\partial B_i$, (4) $f(x)=x$ in a neighborhood of $\complement B^n$. Then there exists a lipeomorphism f^*: $\complement D_2\to\complement B_2$ such that $f^*|\partial D_2=f|\partial D_2$. All LIP properties are taken in the spherical metric.

The construction of f^* is exactly as in [29]. Only the LIP property of f^* at ∞ needs a separate argument.

The set $A=\mathbf{C}B^n\setminus\infty$ is $(\pi/2)$-quasiconvex in the euclidean metric. Moreover, A is a locally finite union of closed sets A_i such that each $f^*|A_i$ is a composite map of translations, f, f^{-1}, and a single affine map, common to all i. By 2.35, $f^*|A$ is L-Lipschitz for some L in the euclidean metric. Since $|f^*(x)-x|<5$ for all finite x, $|f^*(x)|\geqq|x|/2$ for $|x|\geqq 10$. Therefore, for all x, y in $\mathbf{C}B^n(10)\setminus\infty$, we obtain

$$\begin{aligned} q(f^*(x), f^*(y)) &= |f^*(x)-f^*(y)|(1+|f^*(x)|^2)^{-1/2}(1+|f^*(y)|^2)^{-1/2} \\ &\leqq 4Lq(x, y). \end{aligned}$$

Hence f^* is LIP at ∞. A similar argument shows that $(f^*)^{-1}$ is LIP at ∞, and Step 1 is proved.

Step 2. (See [29, 41.2].) In addition to (1), (2), and (3) of Step 1, suppose that (4′) $0\in D_2$, (5′) $\mathbf{C}f\mathbf{C}\bar{B}^n\subset B^n$. Then there exists a lipeomorphism $f^*\colon \mathbf{C}D_2\to\mathbf{C}B_2$ such that $f^*|\partial D_2=f|\partial D_2$.

The proof of Step 2 in [29] is also valid in our case except that we must replace the map g of [29] by the following map $g\colon \bar{R}^n\to\bar{R}^n$ (which could be used in the qc case as well): $g(0)=0$, $g(\infty)=\infty$, and $g(x)=\varphi(|x|)|x|^{-1}x$ for $x\neq 0, \infty$, where $\varphi\colon R^1_+\to R^1_+$ is the PL homeomorphism which maps $[0, a]$ linearly onto $[0, b]$, $[a, 1]$ affinely onto $[b, 1]$ and is the identity on $[1, \infty)$. Then g is a lipeomorphism.

Step 3. The proof of [29, 41.3] can be directly translated to the LIP case. Since Möbius transformations are diffeomorphisms of S^n, they are lipeomorphisms in the spherical metric of $\bar{R}^n$. This completes the proof of 7.7. □

7.8. Theorem. *Let S be a* LIP *n-sphere, and let S_1 be a locally* LIP *flat* LIP *$(n-1)$-sphere in S. Then (S, S_1) is lipeomorphic to the standard pair (S^n, S^{n-1}).*

Proof. Choose a lipeomorphism $h\colon S^{n-1}\to S_1$. Let D_1, D_2 be the components of $S\setminus S_1$. Since S_1 is locally LIP flat, it has a local LIP collar in $\bar{D}_1$. By 7.4, it has a LIP collar $c\colon S_1\times I'\to\bar{D}_1$ in $\bar{D}_1$. Then h can be extended to a LIP embedding $f\colon \bar{B}^n\setminus 0\to\bar{D}_1$ by $f(x)=c(h(x/|x|), 1-|x|)$. By 7.7, h can be extended to a lipeomorphism $g_1\colon \bar{B}^n\to\bar{D}_1$. Similarly we find an extension of h to a lipeomorphism $g_2\colon \bar{R}^n\setminus B^n\to\bar{D}_2$. Then $g_1\cup g_2$ is a lipeomorphism of $(\bar{R}^n, S^{n-1})$ onto (S, S_1). Here $\bar{R}^n=R^n\cup\infty$ with the spherical metric. □

7.9. Theorem. *Let M be a compact* LIP *n-manifold which is the union of two open* LIP *n-balls. Then M is a* LIP *n-sphere.*

Proof. See the TOP case [26, Theorem 1.8.4, p. 49]. □

7.10. Theorem. *Let S be a* PL *n-sphere and let $S_1\subset S$ be a* PL *$(n-1)$-sphere. Then (S, S_1) is lipeomorphic to the standard pair (S^n, S^{n-1}). In particular, the closure of each component of $S\setminus S_1$ is a* LIP *n-ball.*

Proof. The theorem is obvious for $n=1$. Proceeding inductively, we assume that the theorem is true for $n \leqq p$, and suppose $n=p+1$. By 7.8, it suffices to show that S_1 is locally LIP flat in S. Let $x \in S_1$, and choose a triangulation (K, L) of (S, S_1) such that x is a vertex. By induction, the PL sphere pair $(|\mathrm{lk}\,(x, K)|, |\mathrm{lk}\,(x, L)|)$ is lipeomorphic to (S^p, S^{p-1}). The cone construction yields a lipeomorphism of $(|\mathrm{st}\,(x, K)|, |\mathrm{st}\,(x, L)|)$ onto $(\bar{B}^{p+1}, \bar{B}^p)$. Hence S_1 is locally LIP flat in S. □

8. LIP structures of R^n, S^n, and I^n

8.1. The aim of this section is to prove the Lipvermutung (LIP Hauptvermutung) for the manifolds R^n and S^n for $n \neq 4$, and for I^n for $n \neq 4$, 5. By this we mean that if a LIP manifold is homeomorphic to one of these manifolds, it is lipeomorphic to it. A LIP version of the Poincaré conjecture follows then directly from the corresponding TOP result, proved by Newman [24]. In fact, this section is a slightly enlarged LIP version of Newman's paper. We begin by the LIP version of Newman's engulfing theorem [24, Theorem 5]. A subset X of a LIP n-manifold M is called LIP *p-dominated* if for every $x \in X$ there is a neighborhood N of x and a lipeomorphism $f\colon N \to I^n$ such that $f[N \cap X]$ is contained in a polyhedron of dimension at most p.

8.2. Theorem (Engulfing). *Let M be a* LIP *n-manifold without boundary, and let $X \subset M$ be closed and* LIP *p-dominated with $p \leqq n-3$. Let V be an open set in M such that (M, V) is p-connected and $X \setminus V$ is compact. Then there is a lipeomorphism $h\colon M \to M$ such that $X \subset hV$ and $h=\mathrm{id}$ outside a compact set.*

Proof. The theorem is proved by rewriting Newman's proof in the LIP category. It is only necessary to be sure that all maps occurring in the proof can be made LIP. The previous sections of our paper give all the tools needed for this. Since the proof is long, we must leave the details to the reader. However, we shall give some hints for this translation work together with some remarks on Newman's proof [24, Sections 1—15, pp. 555—569] concerning omission of certain unneeded hypotheses and correction of some slight inaccuracies.

Lemma 1. To obtain a LIP map φ, apply 5.4.

Lemma 2. Assume that X is a LIP manifold and F compact. In the proof, set $f_t=(1-t)f_0+tf_1$ and replace f_{1+} by f_b and φ' by f_a for suitable $0<a<1<b$. A computation shows that $h|\pi(f_0, f_b)$ is LIP. By 2.36, h is LIP. Similarly h^{-1} is LIP. In the definition of engulfing, the isotopy condition (2) can be omitted.

Theorem 1. Assume that Y and Z are LIP manifolds, F is compact, and g_1 is LIP. In case 1 of the proof, assume that g_0 is also LIP. The set V is chosen to be $A \times \mathrm{int}_Z F$ where A is a relatively compact neighborhood of o. Defining $\varphi(y, z)= =(d(y, o)g_0(z)+d(y, \partial A)g_1(z))/(d(y, o)+d(y, \partial A))$ yields a LIP engulfing h. The theorem is only needed with (4.3) replaced by (4.3′).

Theorem 3. Assume that L, K are compact and f is LIP. Since f' is constructed by a coning process, it follows from our § 2 that f' is LIP.

Theorem 4. Assume that L, K are compact and f is LIP. Newman's map φ seems to be a standard mistake of the second kind (see 2.43), but this is easily corrected.

Theorem 6. Assume that H, L, Γ are compact, M a LIP manifold, X LIP p-dominated, and f LIP. Replace the condition $\dim \Gamma \leqq p$ by the following one: $\dim \Gamma \leqq n$ and $\dim H \leqq p$. Indeed, the theorem will be applied on p. 569 with the substitution $\Gamma \mapsto G^*$ where $\dim G^*$ may be $n-2$. Claim that g is LIP and h is a lipeomorphism.

Section 11. Replace the additional condition of $B(q)$ by the following one: $H \setminus f^{-1}V$ has a neighborhood in H of dimension $\leqq q$. This helps in the proof of $C(q, 1) \Rightarrow C(q, m)$ and allows us to change (11.2) into the form: $\operatorname{int} \sigma^q$ is a neighborhood of $H \setminus f^{-1}V$ in H. This in turn will imply in Section 12 that σ^q is a principal simplex of H.

Lemma 6. In the proof, choose Γ_0 so that $\dim \overline{H \setminus P} \leqq q$.

Section 12. By 5.18, the extended map f can be chosen to be LIP.

Lemma 7. Assume that G, L, K are compact, $\dim G \leqq n$, $\dim K \leqq p$, and f is LIP. In the proof, we could not see why the map g_{i+1} satisfies (13.3): $g_{i+1}|L \cup G^\alpha = f'|L \cup G^\alpha$, although obviously $g_{i+1}(x) = f'(x)$ for $x \in L \cup (G^\alpha \cap D)$. We obtained g_{i+1} by applying 5.17 with the substitution $X \mapsto G$, $Y \mapsto L \cup G^\alpha \cup D$, $f \mapsto f'$, $g \mapsto (f'|L \cup G^\alpha) \cup (g'|L \cup D)$. Moreover, when defining the polyhedron G^*, one should identify points in $L \cup (D \cap G^\alpha)$ with the same g_i-image, not only in $D \cap G^\alpha$, since otherwise $f^*|L^*$ is not necessarily an embedding. When applying $B(q-1)$ on p. 569, the new condition of $B(q-1)$ is satisfied, since $W = D^* \setminus p[K \cup |_i Q_1^q|]$ is a neighborhood of $D^* \setminus f^{*-1} h_i V$ in D^* and $W \subset pP$.

The rest of the proof is essentially unchanged. □

8.3. Theorem. *Let X be a metric space which is the union of open sets $V_1 \subset V_2 \subset \ldots$ such that every V_i is lipeomorphic to R^n. Then X is lipeomorphic to R^n.*

Proof. The corresponding TOP result is due to Brown [4]. It is easy to check that all maps in Brown's paper can be made LIP. □

8.4. Theorem. *Let M be a* LIP *manifold homeomorphic to R^n, $n \neq 4$. Then M is lipeomorphic to R^n.*

Proof. The case $n=1$ follows from 3.11. For $n \geqq 2$, Theorem 8.3 implies that it is sufficient to show that each compact set $A \subset M$ is contained in an open LIP ball. We divide the rest of the proof into two cases: $2 \leqq n \leqq 3$ and $n \geqq 5$.

Let $n=2$ or 3, and let $A \subset M$ be compact. Choose a homeomorphism $h\colon M \to R^n$

and a closed n-cube Q containing hA. Divide Q into congruent closed cubes $Q_1, \dots, Q_k$ so that for each i, $P_i = Q_1 \cup \dots \cup Q_i$ is a PL n-ball, $P_i \cap Q_{i+1}$ is a PL $(n-1)$-ball in ∂Q_{i+1}, and $Q_i' = h^{-1}Q_i$ is contained in an open LIP n-ball $B_i \subset M$. It suffices to show that $P_i' = h^{-1}P_i$ is contained in an open LIP ball for all $i=1, \dots, k$. This is clearly true for $i=1$. Proceeding inductively, assume that P_i' is contained in an open LIP ball B.

We can obviously choose an open neighborhood U of Q_{i+1}' in B_{i+1} and a homeomorphism f of U onto an open set $V \subset R^n$ such that $fQ_{i+1}' = I^{n-1} \times I$ and $f[U \cap P_i'] \subset \subset \{x \in R^n | x_n \leqq 0\}$. Choose $\varepsilon \in (0, 1)$ such that $(1+\varepsilon)(I^{n-1} \times I) \subset V$ and $(1+\varepsilon)I^{n-1} \times \times [0, \varepsilon] \subset f[U \cap B]$. Let S be the PL $(n-1)$-sphere $\partial((1+\varepsilon)I^{n-1} \times [\varepsilon, 1+\varepsilon])$, and let S_1 be the PL $(n-2)$-sphere $\partial((1+\varepsilon)I^{n-1} \times \varepsilon)$. Choose a lipeomorphism g of U onto an open set $W \subset R^n$. Then $\varphi = gf^{-1}$: $V \to W$ is a homeomorphism. For every $\delta > 0$, there is a homeomorphism u of φS onto a PL $(n-1)$-sphere E such that $d(u, \mathrm{id}) < \delta$. For $n=2$ this is elementary, and for $n=3$ this follows from the approximation theorem of Bing [1, Theorem 1]. The same argument in the dimension $n-1$ yields a homeomorphism v of $u\varphi S_1$ onto a PL $(n-2)$-sphere $F \subset E$ such that $d(v, \mathrm{id}) < \delta$. Let D_1, D_2 be the components of $E \setminus F$, and let D be the bounded component of $R^n \setminus E$. Choosing δ small enough, we may assume that $g[Q_{i+1}' \setminus B] \subset D$, $\bar{D} \subset W$, $\bar{D}_1 \subset g[B \cap U]$, and $E \cap gQ_{i+1}' \subset D_1$.

By 7.10 or by the PL Schönflies theorem, $\bar{D}$ is a LIP ball. Choose a lipeomorphism q: $\bar{D} \to I^n$. Also by 7.10, (E, F) is lipeomorphic to $(\partial I^n, \partial I^{n-1})$. Using the cone construction, we may therefore assume that $qD_2 = \partial I^n \cap R^n_+$. It is easy to construct LIP maps α, β: $I^{n-1} \to (0, 1)$ such that $\alpha | \partial I^{n-1} = \beta | \partial I^{n-1} = 0$, $\{(x, t) | -1 \leqq t \leqq \alpha(x)\} \subset \subset q[\bar{D} \cap g[B \cap U]]$, and $q[\bar{D} \cap gQ_{i+1}'] \subset \{(x, t) | -1 \leqq t < \beta(x)\}$. Applying 2.40 we find a lipeomorphism r: $I^n \to I^n$ such that $r | \partial I^n = \mathrm{id}$ and $r(x, \alpha(x)) = (x, \beta(x))$ for all $x \in I^{n-1}$. Define w: $M \to M$ by $w | g^{-1}D = g^{-1}q^{-1}rqg | g^{-1}D$ and by $w | M \setminus g^{-1}D = \mathrm{id}$. Then w is a lipeomorphism, and wB is an open LIP ball containing P_{i+1}'. The case $2 \leqq n \leqq 3$ is proved.

Suppose that $n \geqq 5$. Assume that $A \subset M$ is compact. Using the proof of Newman [24, Theorem 7, p. 570], attributed by Newman to Connell, we can first show that A can be covered with two open LIP balls. The proof makes use of 2.42 and 8.2. Then a slight modification of this proof shows that A can be covered with a single open LIP ball. We shall give the latter proof in detail.

We may assume that A is a locally flat TOP n-ball. Choose LIP balls B_1, B_2 with $A \subset \mathrm{int}\, B_1 \cup \mathrm{int}\, B_2$. Using lipeomorphisms $B_i \to I^n$ we introduce PL structures on B_i, $i=1, 2$. Choose smaller concentric "cubes" $C_i \subset \mathrm{int}\, B_i$, $D_i \subset \mathrm{int}\, C_i$, $i=1, 2$, such that $A \subset \mathrm{int}\, D_1 \cup \mathrm{int}\, D_2$. Choose a triangulation K of C_2 such that a subcomplex triangulates D_2 and no simplex meets both ∂D_1 and ∂C_1. Set $M_1 = = M \setminus C_1$, $V_1 = \mathrm{int}\, B_1 \setminus C_1$, $X_1 = |K^2| \setminus C_1$. Since M_1 and V_1 are 2-connected, it follows from 8.2 that there is a lipeomorphism h_1: $M_1 \to M_1$ such that $h_1 V_1 \supset X_1$ and $h_1 = \mathrm{id}$ near ∂M_1. We extend h_1 by identity to a lipeomorphism h_1: $M \to M$. Then $U_1 = h_1\, \mathrm{int}\, B_1$ is an open LIP ball and $|K^2| \cup C_1 \subset U_1$.

Next choose a derived subdivision of K, and let L be the dual skeleton of K^2. Choose a TOP ball A_1 such that $A \subset A_1 \subset D_1 \cup D_2$ and $A_1 \setminus \operatorname{int} A$ is homeomorphic to $S^{n-1} \times I$. Set $M_2 = \operatorname{int} A_1$, $V_2 = \operatorname{int} A_1 \setminus A$, $X_2 = |L| \cap \operatorname{int} A_1$. Since M_2 and V_2 are $(n-3)$-connected, it follows from 8.2 that there is a lipeomorphism $h_2 \colon M_2 \to M_2$ such that $h_2 V_2 \supset X_2$ and $h_2 = \mathrm{id}$ near ∂M_2. We extend h_2 by identity to a lipeomorphism $h_2 \colon M \to M$. Then $|L| \cup (M \setminus A_1) \subset h_2[M \setminus A] = U_2$.

We have now the situation described at the end of 2.42. Hence there is a stretching lipeomorphism $h \colon C_2 \to C_2$ such that $D_2 \subset h[U_1 \cap C_2] \cup (U_2 \cap C_2)$ and $h | \partial C_2 = \mathrm{id}$. We extend h by identity to a lipeomorphism $h \colon M \to M$. Since h maps simplexes of K onto themselves, $D_1 \subset hC_1 \subset hU_1$. Thus $M = hU_1 \cup U_2$, and A is contained in the open LIP ball $h_2^{-1} hU_1$. □

8.5. Theorem. *Let M be a* LIP *manifold homeomorphic to S^n, $n \neq 4$. Then M is lipeomorphic to S^n.*

Proof. Fix a point $x \in M$. Choose a LIP ball neighborhood B_1 of x. By 8.4, $S^n \setminus x$ is lipeomorphic to R^n. Hence there is a LIP ball $B_2 \subset S^n \setminus x$ with $\partial B_1 \subset \operatorname{int} B_2$. The theorem follows from 7.9. □

8.6. Theorem. *Let M be a* LIP *manifold homeomorphic to I^n, $n \neq 4, 5$. Then M is lipeomorphic to I^n.*

Proof. By 3.13, the double DM of M is a LIP manifold containing M as a locally LIP flat submanifold. By 8.5, DM is a LIP n-sphere and ∂M is a LIP $(n-1)$-sphere. By 7.8, M is a LIP n-ball. □

9. Open problems

9.1. *Elementary problems.* (1) Can every LIP n-manifold be LIP embedded into R^{2n+1}?

(2) The LIP annulus conjecture for $n=2$: Let S_1 and S_2 be disjoint locally LIP flat LIP 1-spheres in a LIP 2-sphere S, and let D be the domain whose boundary is $S_1 \cup S_2$. Is $\bar{D}$ lipeomorphic to $S^1 \times I$?

(3) Is a LIP arc in R^2 always locally LIP flat?

(4) More generally, is a locally TOP flat LIP arc in R^n always locally LIP flat?

(5) Does a locally quasiconvex metric space have a basis consisting of quasiconvex sets?

9.2. *Advanced problems.* (1) The LIP annulus conjecture for $n \geqq 3$.

(2) Does every TOP manifold have a LIP structure?

(3) Lipvermutung: If two LIP manifolds are homeomorphic, are they lipeomorphic?

(4) If two PL manifolds are lipeomorphic, are they PL homeomorphic?

9.3. Remarks. By Siebenmann [27, 2.1, p. 137], there exists a LIP manifold which has no PL structure. The answers to the questions (3) and (4) of 9.2 cannot be both positive, since they would yield the PL Hauptvermutung.

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