ON THE EXISTENCE OF AUTOMORPHIC QUASIMEROMORPHIC MAPPINGS IN $\mathbb{R}^n$

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1. Introduction

Let $G$ be a Möbius group in $\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$, $n \geq 2$, and $D$ a domain in $\mathbb{R}^n$. A mapping $f: D \to \mathbb{R}^n$ is said to be automorphic with respect to $G$ if $f$ is continuous, open, discrete, sense-preserving, and $f \circ g = f$ for all $g \in G$. Note that if $G$ has automorphic mappings $f: D \to \mathbb{R}^n$, then $G$ is discrete and $D$ is invariant under $G$. Discrete Möbius groups which have invariant domains are called function groups.

A mapping $f: D \to \mathbb{R}^n$ is called quasiregular, abbreviated $qr$, if $f$ is continuous, ACL$^n$, and

(1.1) $|f'(x)|^n \equiv KJ(x, f)$

a.e. in $D$ for some $K \in [1, \infty)$. If $f: D \to \mathbb{R}^n$, then the ACL$^n$ condition and (1.1) can be checked at $f^{-1}(\infty)$ by means of auxiliary Möbius transformations. If these conditions hold the mapping is then said to be quasimeromorphic, abbreviated $qm$. If $n=2$ and (1.1) holds with $K=1$, then $f$ is meromorphic.

The purpose of this note is to prove

1.2. Theorem. Let $G$ be a discrete Möbius group acting on $B^n$, $n \geq 2$, with $V(B^n/G) < \infty$. Then $G$ has $qm$ automorphic mappings $f: B^n \to \mathbb{R}^n$.

In the above theorem $V(B^n/G)$ denotes the hyperbolic volume of the orbit space $B^n/G$, see [4].

The proof is constructive. It is based on a modification of the method of Alexander [1], on basic properties of Möbius groups, see Chapter 3, and on the properties of radial strechtings, see Chapter 2. We shall not estimate the dilatations of $f$ in terms of $G$. For the sake of simplicity we shall restrict ourselves to the case $n=3$. The same method applies to $n=3$ and to $n=2$.

It is known that every function group in $\mathbb{R}^2$ has meromorphic automorphic mappings. We do not know whether function groups in $\mathbb{R}^n$, $n>2$, have $qm$ automorphic mappings, nor we know whether the condition $V(B^n/G) < \infty$ in Theorem 1.2 is essential. We have examples, see [4, 4.2], of $qm$ automorphic mappings $f: B^n \to \mathbb{R}^n$ for infinite groups with $V(B^n/G) = \infty$.

doi:10.5186/aasfm.1977.0317
The notation and terminology will be as in [4]. In particular we denote 
\( x = (x_1, \ldots, x_n) = \sum x_i e_i \) for \( x \in \mathbb{R}^n \), \( B^n(a, r) = \{ x \in \mathbb{R}^n : |x-a| < r \} \), \( B^n(0, r) = B^n(1, r) \), \( S^{n-1}(a, r) = \partial B^n(a, r) \), \( S^{n-1}(0, r) = S^{n-1}(1, r) \), \( H^n(h) = \{ x \in \mathbb{R}^n : x_1 > h \} \), and \( H^n = H^n(0) \). For Möbius groups \( G \) acting on \( B^n \) we let 
\[ \text{Fix} \ G = \{ x \in B^n : g(x) = x \ \text{for some} \ g \in G \setminus \{ \text{id} \} \} \).

2. Radial stretchings

2.1. In this chapter we consider a special class of bi-lipschitzian mappings. A mapping \( f : A \to \mathbb{R}^n, A \subset \mathbb{R}^n \), is called bi-lipschitzian if

\[ |x-y|/L \equiv |f(x)-f(y)| \equiv L|x-y| \]
for all \( x \) and \( y \) in \( A \) and for some \( L \equiv 1 \). The smallest \( L \) for which (2.2) holds will be denoted by \( L(f) \).

2.3. A bounded domain \( D \subset \mathbb{R}^n \) is said to be strictly star shaped if each ray \( L \) from 0 meets \( \partial D \) at exactly one point. It follows that \( 0 \in D \) and that the mapping \( \phi^* : \partial D \to S^{n-1} \) which sends \( L \cap \partial D \) to \( L \cap S^{n-1} \) is a homeomorphism. We let \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) denote the radial linear extension of \( \phi^* \), i.e. \( \phi(x) = x\phi^*(x)/|x^*|, x \neq 0 \), and \( \phi(0) = 0 \) where \( \{ x^* \} = \partial D \cap \{ t x : t > 0 \} \). This mapping \( \phi \) which is an automorphism of \( \mathbb{R}^n \) and maps \( D \) onto \( B^n \) will be called the radial linear stretching defined by \( D \).

2.4. Lemma. Suppose that \( D \) is strictly star shaped and \( \phi^* : \partial D \to S^{n-1} \) is bi-lipschitzian. Then \( \phi \) is bi-lipschitzian.

**Proof.** Let \( M = \sup \{|x| : x \in \partial D \} \) and \( m = \inf \{|x| : x \in \partial D \} \). Let \( x, y \in \mathbb{R}^n \). We may assume that \( x \neq 0 \) and that \( \phi(x) = x \) i.e. \( \phi^*(x^*) = x^* \) since otherwise we consider the mapping \( \phi \circ F_x \) where \( F_x(z) = |x^*|z \). Then \( F_x \) is bi-lipschitzian with \( L(F_x) \equiv \max \{ M, 1/m \} \), \( \phi \circ F_x(x) = x \), and \( \phi \) is bi-lipschitzian if and only if \( \phi \circ F_x \) is. Let \( \alpha \in [0, \pi] \) denote the angle between the vectors \( x \) and \( y \). If \( y = 0 \) we set \( \alpha = 0 \). We claim that

\[ |\phi(y) - y| \equiv |y|(L(\phi^*) + 1)\alpha/m. \]

If \( y = 0 \) then (2.5) is trivial. Suppose \( y \neq 0 \). Now

\[ |\phi(y) - y| = |y||\phi^*(y^*) - y^*|/|y^*| \]
\[ \equiv |y||\phi^*(y^*) - x^*| + |y^* - x^*| + |x^* - x^*|/m \]
\[ \equiv |y||\phi^*(y^*) - \phi^*(x^*) + L(\phi^*)\phi^*(y^*) - \phi^*(x^*)| + m \]
\[ \equiv |y|(\alpha + L(\phi^*)\alpha)/m \]
and (2.5) follows.
On the existence of automorphic quasimeromorphic mappings in $\mathbb{R}^n$

To prove that $|\varphi(x) - \varphi(y)| \equiv K|x - y|$ suppose first that $\alpha = \pi/2$. Then $|x - y| \equiv |y|$ and (2.5) yields

$$|\varphi(x) - \varphi(y)| / |x - y| \equiv 1 + \pi(L(\varphi^* + 1)/m.$$  

If $\alpha = \pi/2$, then $|x - y| \equiv |y| \sin \alpha$ and (2.5) implies

$$|\varphi(x) - \varphi(y)| / |x - y| \equiv 1 + \alpha(L(\varphi^* + 1)/(m \sin \alpha))$$

$$< 1 + \pi(L(\varphi^* + 1)/m.$$  

To prove the opposite inequality let $x, y \in \mathbb{R}^n \setminus \{0\}$. Then

$$|\varphi(x) - \varphi(y)| = |x|\varphi^*(x^*)/|x^*| - |y|\varphi^*(y^*)/|y^*| \equiv |x - y|/M$$

and since this inequality is trivial when $x = 0$ or $y = 0$, the lemma follows.

2.6. Let $D$ be a bounded domain in $\mathbb{R}^n$ with $0 \in D$ and let $\beta \in (0, \pi/4]$. We say that $D$ satisfies the $\beta$-cone condition if the open cone

$$C(x, \beta) = \{z \in \mathbb{R}^n : |z - x| < |x|, (x - z) \cdot \alpha > |x - z| |x| \cos \beta\}$$

with vertex $x$ and central angle $\beta$ lies in $D$ whenever $x \in \partial D$. Note that if $D$ satisfies the $\beta$-cone condition, then $D$ is strictly star shaped.

2.7. Lemma. Suppose that $D$ satisfies the $\beta$-cone condition for some $\beta > 0$. Then $\varphi^*$ is bi-lipschitzian.

Proof. Let $x, y \in \partial D$. Then

$$|\varphi^*(x) - \varphi^*(y)| \equiv |x - y|/m$$

where $m = \inf \{|z| : z \in \partial D\}$.

To prove the other inequality we may assume that $\varphi^*(x) = x$ and $|y| \equiv |x|$. Let $\alpha \in (0, \pi]$ denote the angle between $x$ and $y$. Suppose first that $\alpha = \pi/2$. Then

$$|\varphi^*(x) - \varphi^*(y)| \equiv \sqrt{2} \equiv |x - y|/\sqrt{2}.$$  

Suppose now that $\alpha < \pi/2$. Since $y$ is outside the cone $C(x, \beta)$, elementary trigonometry yields

$$|x - y| \equiv |y| \sin \frac{\alpha}{\sin \beta} \equiv |\varphi^*(x) - \varphi^*(y)|/\sin \beta,$$

and the lemma follows.

Since every bi-lipschitzian mapping of $\mathbb{R}^n$ is quasiconformal, see [3], the above lemmas imply:

2.8. Corollary. Suppose that $D$ satisfies the $\beta$-cone condition for some $\beta > 0$. Then the radial linear stretching $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $D$ is quasiconformal.

2.9. Remarks. (a) We shall mainly use Corollary 2.8 to show that the homeomorphism $\varphi|D$ onto $B^n$ is quasiconformal.
(b) It is easy to see that if $D$ is strictly star shaped, then $\phi$ is bi-lipschitzian if and only if $D$ satisfies the $\beta$-cone condition for some $\beta > 0$. Moreover, for $n=3$ the cone condition is equivalent to the boundary condition of [3, 5.3].

(c) We shall later use the following elementary property of linear stretchings: Let $D_1$ and $D_2$ be strictly star shaped domains and $\varphi_1$, $\varphi_2$ the corresponding linear stretchings. Define $\varphi = \varphi_2^{-1} \circ \varphi_1$. Suppose that $x, y \in \partial D_1$ and $E(x) = y$ for some $E \in O(n)$. If $|\varphi(x)| = |\varphi(y)|$, then $E \circ \varphi(x) = \varphi \circ E(x)$.

3. Fundamental polyhedra for Möbius groups

3.1. Normal fundamental polyhedra. Let $G$ be a discrete Möbius group acting on $B^n$. Then $G$ is countable and thus $B^n \setminus \text{Fix } G \neq \emptyset$. The normal fundamental polyhedron $P$ centered at a point $x_0 \in B^n \setminus \text{Fix } G$ is defined by

$$P = \{x \in B^n : d(x, x_0) < d(x, g(x_0)) \text{ for all } g \in G \setminus \{\text{id}\}\}.$$  

Here $d$ denotes the hyperbolic distance in $B^n$. $P$ is a convex polyhedron in the hyperbolic sense, possibly with infinite number of faces. Each $(n-1)$-face, considering only $\partial P \cap B^n$, lies in a hyperbolic $(n-1)$-plane

$$H(A, x_0) = \{x \in B^n : d(x, x_0) = d(x, A(x_0))\}$$

for some $A \in G \setminus \{\text{id}\}$. Since $AH(A^{-1}, x_0) = H(A, x_0)$, the $(n-1)$-faces of $P$ are pairwise $G$-equivalent. Note also that $H(A^{-1}, 0)$ is contained in the isometric sphere $I(A) = \{x \in \mathbb{R}^n : |A'(x)| = 1\}$ of $A$ and $A = E \circ I$ where $I$ is the reflection in $H(A^{-1}, 0)$ and $E$ is an orthogonal transformation in $\mathbb{R}^n$. Indeed, $(I \circ A^{-1})B^n = B^n$ and $(I \circ A^{-1})(0) = 0$ and so $I \circ A^{-1} \in O(n)$. Therefore $A = E \circ I$ for some $E \in O(n)$, and $|A'(x)| = 1$ for all $x \in H(A^{-1}, 0)$. Finally, recall that if $G$ is discrete and the hyperbolic measure $V(B^n/G)$ is finite, then, see [2], [5], or [6], every normal fundamental polyhedron $P$ has finitely many faces and $P \cap S^{n-1}$ is either empty, which happens only when $B^n/G$ is compact, or consists of finitely many points, called boundary vertices. We summarize the above facts:

3.2. Lemma. Let $G$ be a discrete Möbius group acting on $B^n$ with $V(B^n/G) < \infty$. Suppose that $0 \in \text{Fix } G$ and let $P$ be a normal fundamental polyhedron centered at $0$. Then $P$ is of the form

$$P = B^n \setminus \bigcup_{i=1}^{2k} \overline{B^n(x_i, r_i)}$$

where each $S_i = S^{n-1}(x_i, r_i)$, $i=1, \ldots, 2k$, is orthogonal to $S_i$, $r_i = r_{i+k}$, and $T_i S_i = S_{i+k}$ for some $T_1, \ldots, T_k \in G$. Furthermore, each $T_i$, $i=1, \ldots, k$, is of the form $T_i = E_i \circ I_i$ where $I_i$ denotes the reflection in $S_i$ and $E_i \in O(n)$. 

3.3. Simple fundamental polyhedron. Let $G$ be a discrete Möbius group acting on $B^n$ with $V(B^n/G)<\infty$. A normal fundamental polyhedron $P$ for $G$ is said to be simple if no two boundary vertices of $P$ are $G$-equivalent. In other words, $P$ is simple if and only if for each boundary vertex $p \in \partial P \cap S^{n-1}$ all the $(n-1)$-faces of $P$ which meet at $p$ are pairwise $G$-equivalent. By [4, Lemma 3.5], $G$ has always simple fundamental polyhedra. To understand the action of $G$ near a boundary vertex $p$ of a simple fundamental polyhedron $P$ centered at $x_0 \in B$, choose a Möbius transformation $A$ with $AB^n=H^n$, $A(p)=\infty$, and $A(x_0)=e_n$. Then $P_1=AP$ is a simple fundamental polyhedron centered at $e_n$ for the group $G_1=AGA^{-1}$ with a boundary vertex at $\infty$. The $(n-1)$-faces of $P_1$ which meet at $\infty$ are pairwise equivalent via elements of $G_1$ that generate the stabilizer $G_m=\{g \in G_1: g(\infty)=\infty\}$. Each $g$ in $G_m \setminus \{id\}$ is a similarity in $R^n$ with a unique fixed point at $\infty$ and acts on each $(n-1)$-plane $\partial H^n(h)$, $h>0$, in the same manner, see [4]. The normal fundamental polyhedron $P_2$ for $G_m$ centered at $e_n$ is of the form $P_2=Q \times (0, \infty)$ where $Q$ is a finite bounded convex euclidean $(n-1)$-dimensional polyhedron. There exists $h_0>0$ such that $P_1 \cap H^n(h)=P_2 \cap H^n(h)$ for all $h \geq h_0$.

For the sake of notational simplicity we shall from now on restrict our considerations to the case $n=3$. The extension to the general case $n>3$ and $n=2$ is quite straightforward.

3.4. Lemma. Let $G$ be a discrete Möbius group acting on $B^3$ with $V(B^3/G)<\infty$. Suppose that $G$ has a simple fundamental polyhedron $P$ centered at 0. Then there exist a finite convex euclidean 3-dimensional polyhedron $Q \subset B^3$ with all its vertices in $S^2$ and a homeomorphism $h: \overline{P} \to \overline{Q}$ such that

(i) $h|P$ is quasiconformal,

(ii) $h \circ T_i(x)=E_i \circ h(x)$ for all $x \in S_i \cap \partial P$, $i=1, \ldots, k$, where $S_i, T_i,$ and $E_i$ are as in Lemma 3.2.

Proof. Case 1: $B^3/G$ is compact. Let $z_1, \ldots, z_m$ be the vertices of $P$ and let $\varphi_1$ and $\varphi_2$ be the radial linear stretchings defined by $P$ and the euclidean polyhedron $Q$ which is spanned by $\varphi_1(z_1), \ldots, \varphi_1(z_m)$, respectively. Then $h=\varphi_2^{-1} \circ \varphi_1$ is the required mapping. Indeed, $h$ maps $\overline{P}$ homeomorphically onto $\overline{Q}$, and since $P$ and $Q$ satisfy the $\beta$-cone condition for some $\beta>0$, $\varphi_1$ and $\varphi_2$ are quasiconformal by Corollary 2.8 and consequently so is $h|P$. For (ii) let $x \in S_i \cap \partial P$. Then by Lemma 3.2

$h \circ T_i(x)=h \circ E_i \circ I_i(x)=h \circ E_i(x)$.

Since $|x_i|=|x_{i+k}|$ and $r_i=r_{i+k}$ it follows by the nature of $h|\partial P$ and $E_i$, see 2.9 (c), that $h \circ E_i(x)=E_i \circ h(x)$ and so (ii) follows.

Case 2: $B^3/G$ is non-compact. Since now $P$ does not satisfy the $\beta$-cone condition for any $\beta>0$, we first map $P$ quasiconformally onto a domain $R \subset B^3$ which satisfies the $\beta$-cone condition and then proceed as in Case 1.

Let $p_1, \ldots, p_q$ be the boundary vertices of $P$. For each $j=1, \ldots, q$ choose a Möbius transformation $A_j$ with $A_jB^3=H^3$, $A_j(p_j)=\infty$, and $A_j(0)=e_3$. Pick
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\( h > 0 \) such that \( A_j^{-1}(H^3(h)) \cap A_i^{-1}(H^3(h)) = \emptyset \) for \( j \neq i \) and so that \( \bigcup A_j^{-1}(H^3(h)) \) does not contain any vertex of \( P \). By 3.3 each set \( C_j = A_j(P) \cap H^3(h) \) is of the form \( Q \times (h, \infty) \) where \( Q \) is a convex, bounded, and finite 2-dimensional euclidean polyhedron containing 0. Let \( \psi_j^*: \mathbb{R}^3 \to \mathbb{R}^3 \) be the radial linear stretching defined by \( Q \) and let \( \psi_j: \mathbb{R}^3 \times [h, \infty) \to H^3 \) be the mapping \( \psi_j(x, s) = (\psi_j^*(x), s - h) \). Then \( \psi_j \) is bi-lipschitzian and hence quasiconformal and maps \( C_j \) onto the semi-infinite cylinder \( Z = B^2 \times (0, \infty) \). Next we construct a quasiconformal mapping of \( Z \) onto a finite cylinder.

Let \((r, \varphi, x_3)\) and \((t, \varphi, \Theta)\) denote the cylinder and spherical coordinates of \( \mathbb{R}^3 \), respectively. The polar angle is measured from the positive half of the \( x_3 \)-axis. The mapping \( f_1: Z \to B^3 \cap H^3 \) which is defined by \( t = e^{-x_3}, \varphi = \varphi, \) and \( \Theta = \pi r / 2 \) is quasiconformal, see [3, p. 47], surjective, and maps the discs \( B^2 \times \{x_3\}, x_3 > 0 \), onto the hemispheres \( S^2(e^{-x_3}) \cap H^3 \). Let \( f_2 \) denote the reflection in the sphere \( S^2(e^3/2, 2/3) \). Then \( f_2 \) is anti-conformal and maps \( B^3 \cap H^3 \) onto \( H^3 \setminus \overline{B}(e_2/2, 1/2), \ B^3(1/3) \cap H^3 \) onto \( \{x \in H^3: x_3 > -1/3\} \) and the 2-planes through the \( x_3 \)-axis onto the spheres centered on the line \( L = \{x \in \partial H^3: x_3 = -1/3\} \) and passing through the points \( e_2/3 \) and \( -e_2 \). Using cylinder coordinates \((r, \varphi, x_3)\) with \( L \) as the axis of symmetry and \( \varphi \) measured from the direction of the vector \( -e_2 \) to the direction of \( e_3 \), we define a mapping \( f_3: H^3 \to H^3 \) by

\[
 f_3(r, \varphi, x_3) = (r, \pi/2 + \varphi/4, x_3), \quad 0 \equiv \varphi < 2\pi/3, \\
= (r, \varphi, x_3), \quad 2\pi/3 \equiv \varphi < \pi.
\]

Then \( f_3 \) is quasiconformal, maps \( H^3 \setminus \overline{B}(e_2/2, 1/2) \) onto \( \{x \in H^3: x_3 > -1/3\} \) \( \setminus \overline{B}(e_2/2, 1/2) \), maps each sphere which is centered on \( L \) and which passes through the points \( e_2/3 \) and \( -e_2 \) into itself, and maps half-planes through \( L \) rigidly into half-planes through \( L \).

Finally, let \( \psi = f_1^{-1} \circ f_2^{-1} \circ f_3 \circ f_2 \circ f_1 \). Then \( \psi \) maps the semi-infinite cylinder \( Z \) quasiconformally onto the finite cylinder \( Z' = B^2 \times (0, \log 3) \) and has the following properties:

(a) \( \psi \) has a homeomorphic extension to \( \overline{Z} \) denoted again by \( \psi \),

(b) \( \psi(x) = x \) for \( x \in \partial Z \cap \partial H^3 \), and

(c) each plane through the \( x_3 \)-axis is mapped into itself in the same manner.

To map \( P \) onto a domain \( R \) in \( B^2 \) which satisfies the \( \beta \)-cone condition for some \( \beta > 0 \), define \( \varphi: P \to B^2 \) by \( \varphi(x) = A_j^{-1} \circ \psi_j^{-1} \circ \psi \circ A_j(x) \) for \( x \in P \cap A_j^{-1}(H^3(h)) \), \( j = 1, \ldots, q \), and \( \varphi(x) = x \) otherwise. Then \( \varphi \) is quasiconformal with a homeomorphic extension, denoted by \( \varphi \), to \( \overline{P} \). The image domain \( R = \varphi P \) is obtained from \( P \) by cutting away balls tangent to \( S^2 \) at the boundary vertices and hence \( R \) satisfies the \( \beta \)-cone condition for some \( \beta > 0 \). Because of (c),

\[
(3.5) \quad \varphi \circ E_i(x) = E_i \circ \varphi(x)
\]

for every \( x \in \partial P \cap S_i \).
Let \( \{z_1, \ldots, z_m\} \) be the set of all vertices of \( P \) and let \( \varphi_1 \) and \( \varphi_2 \) be the radial linear stretchings defined by \( R \) and by the euclidean polyhedron \( Q \) which is spanned by \( \{\varphi_1(z_1), \ldots, \varphi_1(z_m), p_1, \ldots, p_q\} \), respectively. Finally, let \( h = \varphi_2^{-1} \circ \varphi_1 \circ \varphi \). Then \( h \) maps \( P \) quasiconformally onto \( Q \) with a homeomorphic extension to \( \bar{P} \).

To prove (ii) note that, by Lemma 3.2 and by the symmetry of \( Q \), see 2.9 (c),

\[
h \circ T_i(x) = h \circ E_i \circ I_i(x) = h \circ E_i(x) = \varphi_2^{-1} \circ \varphi_1 \circ E_i(x) = E_i \circ \varphi_2^{-1} \circ \varphi_1(x) = E_i \circ h(x)
\]
for \( x \in \partial P \cap \bigcup A_j^{-1}(H^3(h)) \). Suppose now that \( x \in \partial P \cap A_j^{-1}(H^3(h)) \cap S_i \) for some \( j \) and \( i \). Then \( h(x) = \varphi_2^{-1} \circ \varphi_1 \circ \varphi(x) \) and hence by (3.5) and Lemma 3.2, \( h \circ T_i(x) = \varphi_2^{-1} \circ \varphi_1 \circ E_i \circ \varphi(x) \). By the same reason as above we can now interchange \( E_i \) and \( \varphi_2^{-1} \circ \varphi_1 \). This yields (ii) and the proof is complete.

4. Construction of a \( QM \) automorphic mapping in \( B^3 \)

4.1. Simplices in \( R^3 \). Given a simplex \( \sigma = (x_0, x_1, x_2, x_3) \) in \( R^3 \) we let \( |\sigma| \) denote the closed tetrahedron in \( R^3 \) which is spanned by the vertices \( x_0, x_1, x_2, x_3 \) and let \( \operatorname{C}(|\sigma|) = \overline{R^3 \setminus \text{int} |\sigma|} \). All 3-simplices in \( R^3 \) will be oriented by the sign of the standard determinant function associated with the basis \( e_1, e_2, e_3 \).

4.2. Lemma. Given any two simplices \( \sigma = (x_0, x_1, x_2, x_3) \) and \( \tau = (y_0, y_1, y_2, y_3) \) in \( R^3 \) there exists a sense-preserving homeomorphism \( h = h_{\sigma \tau} \) from \( |\sigma| \) into \( R^3 \) such that

(i) \( h(|\sigma|) = |\tau| \) if \( \sigma \) and \( \tau \) have the same orientation and \( h(|\sigma|) = \operatorname{C}(|\tau|) \) otherwise,

(ii) \( h(x_i) = y_i, \ i = 0, 1, 2, 3, \)

(iii) \( h|\text{int} |\sigma| \) is a piecewise linear homeomorphism,

(iv) \( h|\text{int} |\sigma| \) is quasiconformal.

Proof. If \( \sigma \) and \( \tau \) have the same orientations, then the piecewise linear map of \( |\sigma| \) onto \( |\tau| \) which is defined by (ii) satisfies (i), (iii), and (iv).

Suppose now that \( \sigma \) and \( \tau \) have different orientations. We may assume that \( 0 \in \text{int} |\tau| \). The radial linear stretching \( \varphi \) defined by \( \text{int} |\tau| \) is quasiconformal in \( R^3 \) and has a quasiconformal extension to \( \overline{R^3} \) with \( \varphi(\infty) = \infty \). Let \( I \) denote the reflection in \( S^2 \) and \( h_1 : |\sigma| \to |\tau| \) the sense-reversing piecewise linear map which satisfies \( h_1(x_i) = y_i, \ i = 0, 1, 2, 3 \). Then it is easy to check that \( h = \varphi^{-1} \circ I \circ \varphi \circ h_1 \) is the required map.

4.3. Proof of Theorem 1.2 for \( n = 3 \). Let \( P \) be a simple fundamental polyhedron for \( G \) centered at \( x_0 \in B^3 \). We may assume that \( x_0 = 0 \), otherwise consider first \( AGA^{-1} \) for some Möbius transformation \( A \) with \( AB^3 = B^3 \) and \( A(x_0) = 0 \).

Let \( T_i \) and \( E_i, \ i = 1, \ldots, k, \) be as in Lemma 3.2 and \( Q \) and \( h : \overline{P} \to \overline{Q} \) as in Lemma 3.4. Using planes through 0 triangulate \( Q \) so that any two \( E_i \)-equivalent faces in \( \partial Q \) have \( E_i \)-equivalent sub-triangulations. In this triangulation, call it \( K \), identify \( E_i \)-equivalent faces in \( \partial Q \) and thus denote vertices of \( K \) which are \( E_i \)-equiv-
alent, \(i=1, \ldots, k\), by the same symbols. Now \(K\) can be made fine enough so that all four vertices of any 3-simplex are distinct. Let \(S=\{\sigma_1, \ldots, \sigma_4\}\) be the set of all 3-simplices in \(K\) and \(\{0, x^2, \ldots, x^N\}\) the set of all vertices in \(K\). Then each \(\sigma_i\) is of the form \((0, x^{i_1}, x^{i_2}, x^{i_3})\). Choose \(N\) points \(0, y^2, \ldots, y^N\) in general position in \(R^3\), i.e. no four points are coplanar, and associate with each 3-simplex \(\sigma_i=(0, x^{i_1}, x^{i_2}, x^{i_3})\) in \(S\) the simplex \(\tau_i=(0, y^{i_1}, y^{i_2}, y^{i_3})\). Now define \(g: Q \rightarrow R^3\) by \(g[|\sigma_i|=h_{\sigma_i\tau_i}\) for \(i=1, \ldots, v\). Here \(h_{\sigma_i\tau_i}\) is as in Lemma 4.2.

Finally let \(f: B^3 \rightarrow R^3\) be defined for \(x\in T^{-1}(P) \cap B^3, T\in G\), by \(f(x)=g \circ h \circ T(x)\). Then \(f\) is continuous by Lemma 3.4 (ii) and by the construction of \(K\) and \(g, f\) is automorphic with respect to \(G\), and since \(g \circ h\) is \(q m\) in \(P\) by Lemma 3.4 and Lemma 4.2, it follows that \(s o\) is \(f\) in \(B^3\).

4.4. Remark. The automorphic mapping \(f\) constructed above has the property \(N(A,f) < \infty\) where \(A\) is any fundamental set for \(G\) and \(N(A,f)=\sup \text{card} (f^{-1}(y) \cap A)\) over all \(y\in R^3\).

References


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Received 10 March 1976