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ON IDEMPOTENTS AND GREEN RELATIONS IN THE ALGEBRAS OF MANY-PLACED FUNCTIONS

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Let I be a non-void set of natural numbers. The union $A_I=\bigcup_{i\in I}A_i$ of mutually disjoint non-void sets A_i is called Menger system [6] if for every $a_1, \ldots, a_m\in A_n$, $b\in A_m$, $n, m\in I$ the element $a_1\ldots a_m b\in A_n$ is uniquely defined and the superassociativity law

$$a_1 \ldots a_m(b_1 \ldots b_k c) = (a_1 \ldots a_m b_1) \ldots (a_1 \ldots a_m b_k) c \tag{1}$$

holds for all $a_1, \ldots, a_m\in A_n$, $b_1, \ldots, b_k\in A_m$, $c\in A_k$, $n, m, k\in I$.

Denote by $\varphi_n(M)$ the set of all n-place functions on the set M. Define for every $f_1, \ldots, f_m\in\varphi_n(M)$, $g\in\varphi_m(M)$ the function $f_1\ldots f_m g\in\varphi_n(M)$ by

$$x_1 \ldots x_n(f_1 \ldots f_m g) = (x_1 \ldots x_n f_1) \ldots (x_1 \ldots x_n f_m) g, \quad x_1, \ldots, x_n\in M. \tag{2}$$

For an operation so defined the identity (1) holds, therefore the set $\varphi_I(M)=\bigcup_{i\in I}\varphi_i(M)$ is a Menger system, called a *full function system* (on the set M). Every subsystem of $\varphi_I(M)$ is called a *function system* (on M). Every Menger system is isomorphic to some function system [6].

Particular cases of function systems are clone algebras by Cohn [2]. Clone algebras are the function systems A_I which for every $n\in I$ contain projectors $\pi_i^n\in\varphi_n(M)$, $i=1, \ldots, n$ (defined by $x_1\ldots x_n\pi_i^n=x_i$ for every $x_1, \ldots, x_n\in M$). Semigroups and full transformation semigroups are also particular cases ($I=\{1\}$) correspondingly of Menger systems and full function systems.

The aim of the previous paper is to investigate in Menger systems concepts corresponding to idempotents and Green relations in semigroups. Two analogues of regular elements are termed and their properties are found to be similar to the properties of regular elements in semigroups. An analogue of maximal subgroup containing a given idempotent is also discussed.

Call element e of a Menger system an *idempotent*, if $e\ldots ee=e$. By the above definition and (2) function $f\in\varphi_n(M)$ is an idempotent if and only if

$$x \ldots xf = x \tag{3}$$

for every element x from the range $M^n f$ of the function f.

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Call element $a \in A_n$ of the Menger system A_i *weakly regular*, if $a = a \dots \dots a(z_1 \dots z_n a)$ for some $z_1, \dots, z_n \in A_m$, $m \in I$. Call *a regular*, if $a = a \dots a(z \dots za)$ for some $z \in A_m$, $m \in I$. Call a Menger system A_I *(weakly) regular*, if every element of A_I is (weakly) regular.

Green equivalences $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$ and $\mathscr{H}$ were defined for Menger systems in the following way [3]:

$(a, b) \in \mathscr{L}$ $(a \in A_n, b \in A_m, n, m \in I)$ if and only if $a = b$ or there exist $s_1, \dots, s_n \in A_m$, $t_1, \dots, t_m \in A_n$ such that $a = t_1 \dots t_m b$, $b = s_1 \dots s_n a$;

$(a, b) \in \mathscr{R}$ if and only if $a, b \in A_n$ for some $n \in I$ and $a = b$ or there exist $s \in A_m$, $t \in A_k$, $m, k \in I$ such that $a = b \dots bt$, $b = a \dots as$;

$\mathscr{D} = \mathscr{L} \cup \mathscr{R} (= \mathscr{L} \cdot \mathscr{R}$ c.f. [3]), $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$.

Denote by $L(a)$ $\big(R(a), D(a), H(a)\big)$ the $\mathscr{L}(\mathscr{R}, \mathscr{D}, \mathscr{H})$-class of the element a.

Define for arbitrary $a \in A_n$, $b \in A_m$, $n, m \in I$ binary relations $\leqq_L$, $\leqq_R$, $\leqq_H$ by

$a \leqq_L b$ if and only if $a = b$ or $a = t_1 \dots t_m b$ for some $t_1, \dots, t_m \in A_n$;

$a \leqq_R b$ if and only if $a = b$ or $a = b \dots bt$ for some $t \in A_k$, $k \in I$ (clearly then $n = m$);

$a \leqq_H b$ if and only if $a \leqq_L b$ and $a \leqq_R b$.

All these relations are preorderings [2] and by definitions $(a, b) \in \mathscr{X}$ if and only if $a \leqq_X b$ and $b \leqq_X a$, where $\mathscr{X} = \mathscr{L}, \mathscr{R}$ or $\mathscr{H}$.

Lemma 1. *Let e be an idempotent element of a Menger system A_I. The element e is a right identity for an element a* (i.e. $a = a \dots ae$) *if and only if $a \leqq_L e$, a left identity for a* (i.e. $a = e \dots ea$) *if and only if $a \leqq_R e$ and a two-sided identity for a if and only if $a \leqq_H e$.*

Proof. If $a \leqq_L e$, $e \in A_n$, $a \in A_m$, then $a = s_1 \dots s_n e$ for some $s_1, \dots, s_n \in A_m$ (the case $a = e$ is trivial) and $a \dots ae = (s_1 \dots s_n e) \dots (s_1 \dots s_n e)e = s_1 \dots s_n (e \dots ee) = s_1 \dots \dots s_n e = a$ by (1). On the other hand, from $a \dots ae = a$ follows $a \leqq_L e$ by the definition. The other assertions follow quite similarly.

Corollary 1. *If e is an idempotent, then e is a right identity for $L(e)$, a left identity for $R(e)$ and a two-sided identity for $H(e)$.*

Corollary 2. *If e, f are idempotents, then $e \leqq_L f$ if and only if $e \dots ef = e$, $e \leqq_R f$ if and only if $f \dots fe = e$ and $e \leqq_H f$ if and only if $e \dots ef = f \dots fe = e$.*

Theorem 1. *An element of a Menger system is*

1) *weakly regular, if and only if there is an idempotent in its $\mathscr{L}$-class*
2) *regular if and only if there is an idempotent in its $\mathscr{R}$-class.*

Proof. Let $a \in A_n$ be a weakly regular element of a Menger system A_I, i.e. $a = a \dots a(z_1 \dots z_n a)$ for some $z_1, \dots, z_n \in A_m$, $m \in I$. From here $z_1 \dots z_n a \in L(a)$. Denote $z_1 \dots z_n a = e$. By (1) $e \dots ee = (z_1 \dots z_n a) \dots (z_1 \dots z_n a)(z_1 \dots z_n a) = z_1 \dots \dots z_n \big(a \dots a(z_1 \dots z_n a)\big) = z_1 \dots z_n a = e$, i.e. e is idempotent.

If a is regular, i.e. $a = a \dots a(z \dots za)$, $z \in A_m$, $m \in I$ then for $f = a \dots az$ we have

$f\dots fa=(a\dots az)\dots(a\dots az)a=a\dots a(z\dots za)=a$, $f\dots ff=f\dots f(a\dots az)=(f\dots fa)\dots$ $\dots(f\dots fa)z=a\dots az=f$. Thus $f\in R(a)$ is an idempotent.

On the other hand, if there exists an idempotent $e\in L(a)$ or an idempotent $f\in R(a)$, then $e=z_1\dots z_n a, f=a\dots az$ for some $z_1, \dots, z_n, z$. By Corollary 1 $a=a\dots$ $\dots ae=a\dots a(z_1\dots z_n a)$ or $a=e\dots ea=(a\dots az)\dots(a\dots az)a=a\dots a(z\dots za)$, thus a is weakly regular or regular.

Corollary 1. *An element of a Menger system is weakly regular if and only if its $\mathcal{D}$-class contains an idempotent. Hence if an element of some $\mathcal{D}$-class D is weakly regular, then all elements from D are weakly regular and elements from containing idempotents $\mathcal{R}$-classes are regular.*

Proof. Suppose $a\in D$ is weakly regular. Then there exists an idempotent $e\in L(a)\subset D$. All elements from $R(e)$ are regular (therefore also weakly regular). Since $R(e)$ meets every $\mathcal{L}$-class from D [3], every $\mathcal{L}$-class from D contains weakly regular elements, hence by Theorem 1 consists of weakly regular elements and so does also D as a union of its $\mathcal{L}$-classes. In the other direction the assertion is obvious, since every idempotent is regular.

A subsystem B_J of a Menger system A_I ($J\subset I$, $B_n\subset A_n$ for every $n\in I$) is called [3]
a *left ideal*, if $I=J$ and $x_1\dots x_m y\in B_n$ for every $x_1, \dots, x_m\in A_n$, $y\in B_m$, $n, m\in I$,
a *right ideal*, if $x_1\dots x_m y\in B_n$ for every $x_1, \dots, x_m\in B_n$, $y\in A_m$, $m\in I$ and
a *two-sided ideal*, if B_J is a left and a right ideal.

Now Theorem 1 can be reworded in a manner always known for semigroups [1]:

Corollary 2. *An element a of a Menger system is weakly regular if and only if the left ideal generated by the element a has an idempotent generator. If an element a is regular, then the right ideal generated by the element a has an idempotent generator. Conversely, if the right ideal generated by an element a has an idempotent generator and has a right identity, then the element a is regular.*

Proof. The left ideal generated by an element a $L_a=\{z|z\leqq_L a\}$. By the definition $(a, b)\in\mathcal{L}$ if and only if $L_a=L_b$. Hence the assertion about weakly regular elements follows directly from the last theorem.

If a has a right identity, i.e. $a=a\dots ax$ for some x (in particular, if a is weakly regular), then the right ideal generated by the element a $R_a=\{z|z\leqq_R a\}$ and the assertion follows quite similarly. Notice that if a right identity for a does not exist, the above expression for R_a need not hold.

For arbitrary $f\in\varphi_n(M)$, $g\in\varphi_m(M)$ we have [3]

$(f, g)\in\mathcal{L}$ if and only if f and g have the same range, i.e. $M^n f=M^m g$;

$(f, g)\in\mathcal{R}$ if and only if $n=m$ and f and g have the same partition (partition $\tau(h)$ corresponding to the function $h\in\varphi_k(M)$ is an equivalence on M^k defined by $(\bar{x}, \bar{y})\in\tau(h)$ if and only if $\bar{x}h=\bar{y}h$, $\bar{x}, \bar{y}\in M^k$);

$(f, g)\in\mathcal{D}$ if and only if the ranges of f and g have the same cardinality and

$(f, g)\in\mathcal{H}$ if and only if f and g have the same range and the same partition.

Call equivalence $\tau \subset M^n$ *diagonal* if every τ-class contains at least one element of the form $(x \dots x)$, $x \in M$. From (3) it is obvious that a function $f \in \varphi_n(M)$ is idempotent if and only if the partition $\tau(f)$ is diagonal and for every $x \in M^n$ $\bar{x}f = y$, where $\bar{x}$ and $(y \dots y)$ belong to the same $\tau(f)$-class. Since $(f, g) \in \mathcal{R}$ if and only if $\tau(f) = \tau(g)$, from the above theorem follows

Corollary 3. *A function f is regular if and only if the partition $\tau(f)$ is diagonal.*

Theorem 2. *The full function system $\varphi_n(M)$ is weakly regular.*

Proof. Let $g \in \varphi_n(M)$. Let $f \in \varphi_n(M)$ be such that
1) $M^n f = M^n g$
2) $(x \dots x) f = x$ for every $x \in M^n g$
Because of 1) $(f, g) \in \mathcal{L}$, because of 2) f is an idempotent.

Corollary. *Every Menger system can be embedded into a weakly regular Menger system generated by idempotents.*

Proof. For $I = \{1\}$, i.e. for semigroups the assertion was established in [7]. If $I \neq \{1\}$, Menger system A_I can be embedded into a weakly regular Menger system $\varphi_I(M)$ [6] and $\varphi_I(M)$ is generated by idempotents [5].

Call elements $b_1, \dots, b_n \in A_m$ *weakly inverses* of an element $a \in A_n$, $n, m \in I$, if $a \dots a(b_1 \dots b_n a) = a$, $b_1 \dots b_n(a \dots ab_i) = b_i$, $i = 1, \dots, n$. Call elements $a \in A_n$, $b \in A_m$ *inverses* (of each other), if $a \dots a(b \dots ba) = a$, $b \dots b(a \dots ab) = b$.

Theorem 3. *An element of a Menger system has weakly inverses if and only if it is weakly regular and inverses if and only if it is regular.*

Proof. If an element has (weakly) inverses then by the above definition it is weakly regular. On the other hand, let element $a \in A_n$ be weakly regular, i.e. $a = a \dots a(z_1 \dots z_n a)$ for some $z_1, \dots, z_n \in A_m$, $m \in I$. Let $b_i = z_1 \dots z_n(a \dots az_i)$, $i = 1, \dots, n$. Then $a \dots a(b_1 \dots b_n a) = a \dots a((z_1 \dots z_n(a \dots az_1)) \dots (z_1 \dots z_n(a \dots az_n))a) = a \dots a((a \dots az_1) \dots (a \dots az_n)a) = a \dots a(z_1 \dots z_n(a \dots a(z_1 \dots z_n a))) = a \dots a(z_1 \dots z_n a) = a$, $b_1 \dots b_n(a \dots ab_i) = (z_1 \dots z_n(a \dots az_1)) \dots (z_1 \dots z_n(a \dots az_n))(a \dots a(z_1 \dots z_n(a \dots az_i))) = z_1 \dots z_n((a \dots az_1) \dots (a \dots az_n)((a \dots a(z_1 \dots z_n a)) \dots (a \dots a(z_1 \dots z_n a)) z_i)) = z_1 \dots z_n(a \dots a(z_1 \dots z_n(a \dots az_i))) = z_1 \dots z_n((a \dots a(z_1 \dots z_n a)) \dots (a \dots a(z_1 \dots z_n)) z_i = z_1 \dots z_n(a \dots az_i) = b_i$, i.e. $b_1, \dots, b_n$ are weakly inverses to a. The proof for the case when element a is regular is quite similar.

Corollary. *An element of a Menger system has weakly inverses if and only if there is an idempotent in its $\mathcal{D}$-class and inverse if and only if there exists an idempotent in its $\mathcal{R}$-class.*

Theorem 4. *In a weakly regular Menger system each element has regular weakly inverses.*

Proof. Let $b_1, \ldots, b_n \in A_m$ be weakly inverses of an element $a \in A_n$ of a weakly regular Menger system A_I, $n, m \in I$. Then $e = b_1 \ldots b_n a$ is idempotent and $(e, a) \in \mathscr{L}$. Furthermore, $e \ldots eb_i = (b_1 \ldots b_n a) \ldots (b_1 \ldots b_n a) b_i = b_1 \ldots b_n (a \ldots ab_i) = b_i$, $i = 1, \ldots, n$. Since A_I is weakly regular for every b_i there are $z_1, \ldots, z_m \in A_k$, $k \in I$ such that $b_i = b_i \ldots b_i (z_1 \ldots z_m b_i)$. Let $y = z_1 \ldots z_m e$. Then $b_i \ldots b_i (y \ldots yb_i) = b_i \ldots b_i ((z_1 \ldots z_m e) \ldots$ $\ldots (z_1 \ldots z_m e) b_i) = b_i \ldots b_i (z_1 \ldots z_m (e \ldots eb_i)) = b_i \ldots b_i (z_1 \ldots z_m b_i) = b_i$, i.e. b_i is regular.

Theorem 5. *Let e, f be idempotents from the same $\mathscr{D}$-class of a Menger system A_I. For every $a \in L(f) \cap R(e)$ there exists an element $b \in R(f) \cap L(e)$ such that a, b are inverses of each other and $a \ldots ab = e$, $b \ldots ba = f$.*

Conversely, if elements a, b are inverses of each other, then the elements $e = a \ldots ab$, $f = b \ldots ba$ are idempotents and $e \in R(a) \cap L(b)$, $f \in L(a) \cap R(b)$.

Proof. Let $a \in L(f) \cap R(e)$, where $e \in A_n$, $f \in A_m$ are idempotents of a Menger system A_I. Then $e = a \ldots as$, $f = s_1 \ldots s_n a$ for some $s_1, \ldots, s_n \in A_m$, $s \in A_k$, $m, k \in I$ and $e \ldots ea = a \ldots af = a$ by Lemma 1. Denote $b = f \ldots f(s \ldots se)$. Now $a \ldots ab = a \ldots a(f \ldots$ $\ldots f(s \ldots se)) = (a \ldots af) \ldots (a \ldots af)(s \ldots se) = a \ldots a(s \ldots se) = (a \ldots as) \ldots (a \ldots as)e =$ $= e \ldots ee = e$, $b \ldots ba = (f \ldots f(s \ldots se)) \ldots (f \ldots f(s \ldots se))a = f \ldots f((s \ldots se) \ldots (s \ldots se))a =$ $= f \ldots f(s \ldots s(e \ldots ea)) = f \ldots f(s \ldots sa) = (s_1 \ldots s_n a) \ldots (s_1 \ldots s_n a)(s \ldots sa) = s_1 \ldots$ $\ldots s_n((a \ldots as) \ldots (a \ldots as)a) = s_1 \ldots s_n (e \ldots ea) = s_1 \ldots s_n a = f$. Hence $a \ldots a(b \ldots ba) = a \ldots$ $\ldots af = a$, $b \ldots b(a \ldots ab) = b \ldots be = (f \ldots f(s \ldots se)) \ldots (f \ldots f(s \ldots se))e = f \ldots$ $\ldots f(s \ldots s(e \ldots ee)) = f \ldots f(s \ldots se) = b$, i.e. a, b are mutually inverses. From $b \ldots be =$ $= b$, $a \ldots ab = e$ follows $(e, b) \in \mathscr{L}$, from $b \ldots ba = f$, $f \ldots fb = b$ follows $(b, f) \in \mathscr{R}$.

On the other hand, let a, b be inverses, $a \ldots a(b \ldots ba) = a$, $b \ldots b(a \ldots ab) = b$. Then elements $e = a \ldots ab$, $f = b \ldots ba$ are idempotens such that $a \ldots af = a$, $b \ldots$ $\ldots be = b$. Hence $(a, f) \in \mathscr{L}$, $(b, e) \in \mathscr{R}$. Furthermore, $a = (a \ldots ab) \ldots (a \ldots ab)a = e \ldots$ $\ldots ea$, $b = (b \ldots ba) \ldots (b \ldots ba)b = f \ldots fb$, therefore $(a, e) \in \mathscr{R}$, $(b, f) \in \mathscr{L}$. Consequently, $e \in R(a) \cap L(b)$, $f \in L(a) \cap R(b)$.

Theorem 6. *Let a, b be elements of a subsystem B_J of a Menger system A_I, $I \subset J$. If the element b is weakly regular in B_J, then $a \leqq_L b$ in B_J if and only if $a \leqq_L b$ in A_I.*

Proof. It is obvious that from $a \leqq_L b$ in B_J follows $a \leqq_L b$ in A_I, so we have to prove the contrary.

Let $a \in B_n$, $b \in B_m$ and $a \leqq_L b$ in A_I, i.e. $a = t_1 \ldots t_m b$ for some $t_1, \ldots, t_m \in A_n$. Since b is weakly regular in B_J, there exists an idempotent $e \in B_k$, $k \in J$ such that $(e, b) \in \mathscr{L}$. From here $b \leqq_L e$ in B_J and since $a \leqq_L b$, $a \leqq_L e$ in A_I. From here $a \ldots ae = a$ by Lemma 1.

Since $(e, b) \in \mathscr{L}$, $e = z_1 \ldots z_m b$ for some $z_1, \ldots, z_m \in B_k$. Therefore $(a \ldots az_1) \ldots$ $\ldots (a \ldots az_m)b = a \ldots a(z_1 \ldots z_m b) = a \ldots ae = a$, i.e. $a \leqq_L b$ in B_J.

By the definition $(a, b) \in \mathscr{L}$ if and only if $a \leqq_L b$ and $b \leqq_L a$. Consequently, from the above theorem follows

Corollary. *If an element b is weakly regular in a subsystem B_J of a Menger system A_I, then $L_B(b)=L_A(b)\cap(B_J\times B_J)$, where $L_A(b)$, $L_B(b)$ are the $\mathcal{L}$-classes of the element b in systems A_I, B_J respectively. Consequently, if a subsystem B_J of a Menger system A_I is weakly regular, $\mathcal{L}_B=\mathcal{L}_A\cap(B_J\times B_J)$, where $\mathcal{L}_A$, $\mathcal{L}_B$ are $\mathcal{L}$-equivalences in systems A_I, B_J respectively. In particular, $\mathcal{L}_{\varphi_n(M)}=\mathcal{L}_{\varphi_I(M)}\cap(\varphi_n(M)\times\varphi_n(M))$ for every natural number n.*

Theorem 7. *Let a, b be elements of a Menger system A_I, $a\leqq_L b$. If a is weakly regular, for every idempotent $e\in L(b)$ there exists an idempotent $f\in L(a)$ such that $f\leqq_H e$.*

Proof. Let $a\in A_n$, $b\in A_m$, $a\leqq_L b$ and let $e\in A_I$ be an idempotent from $L(b)$. Since $a\leqq_L b$, $a=t_1\ldots t_m b$ for some $t_1,\ldots,t_m\in A_n$. Since $e\in L(b)$, $e=z_1\ldots z_m b$ for some $z_1,\ldots,z_m\in A_I$ and $b\ldots be=b$. From $a\leqq_L b$, $b\leqq_L e$ follows $a\leqq_L e$, hence $a\ldots ae=a$ by Lemma 1.

Since a is weakly regular, $a=a\ldots a(x_1\ldots x_n a)$ for some $x_1,\ldots,x_n\in A_k$ $k\in I$ and the element $e'=x_1\ldots x_n a$ is idempotent, $e'\leqq_L a$. From $a\leqq_L e$ follows $e'\leqq_L e$, hence $e'\ldots e'e=e'$. Denote $e\ldots ee'=f$. Then $f\ldots ff=(e\ldots ee')\ldots(e\ldots e')(e\ldots ee')=e\ldots$ $\ldots e((e'\ldots e'e)\ldots(e'\ldots e'e)e')=e\ldots e(e'\ldots e'e')=e\ldots ee'=f$, i.e. f is idempotent. By the definition $f\leqq_R e$. Since $f\leqq_L e'$, also $f\leqq_L e$. Hence $f\leqq_H e$. From $e'\ldots e'f=$ $=e'\ldots e'(e\ldots ee')=(e'\ldots e'e)\ldots(e'\ldots e'e)e'=e'\ldots e'e'=e'$ follows that $e'\leqq_L f$. Now from $f\in\leqq_L e'$ it follows that $(f,e')\in\mathcal{L}$ and since $e'\in L(a)$, also that $f\in L(a)$.

A Menger system A_I is called *group-like* [4] if the only left ideal of A_I is A_I itself and the only right ideals of A_I are $A_J=\bigcup_{n\in J}A_n$, $J\subset I$.

For every $m\in I$, $a\in A_m$ the set $\{x\ldots xa|x\in A_n, n\in I\}$ is a left ideal of the Menger system A_I. Similarly, for every $n,m\in I$, $a\in A_n$ the set $\{a\ldots ax|x\in A_m\}$ is a right ideal of the Menger system A_I. Since every ideal contains ideals of that kind, the above definition is equivalent to the following: a Menger system A_I is group-like, if

(4) for every $a\in A_n$, $b\in A_m$, $n,m\in I$ there exists $x\in A_m$ such that $x\ldots xa=b$

and

(5) for every $a,b\in A_n$, $n\in I$ there exists $y\in A_n$ such that $a\ldots ay=b$.

From (4) it follows that $b\leqq_L a$ for every $a,b\in A_I$ and because of (5) $b\leqq_R a$ for every $a,b\in A_n$, $n\in I$. Consequently, all elements in group-like Menger system A_I are $\mathcal{L}$-equivalent and all elements from every A_n are $\mathcal{R}$-equivalent (therefore also $\mathcal{H}$-equivalent).

Theorem 8. *A Menger system A_I is group-like if and only if every A_n, $n\in I$ is weakly regular and contains exactly one idempotent.*

Proof. From (1) it follows that in every Menger system A_I the defined by

$$a\cdot b=a\ldots ab,\quad a\in A_n,\quad b\in A_m,\quad n,m\in I$$

binary operation $\cdot$ is associative. If A_I is group-like, by (4), (5) every $\{A_n,\cdot\}$ is group, hence contains exactly one idempotent — the identity element e^n. From (4), (5) it also follows that every A_n is weakly regular (in fact, even regular).

On the other hand, let every subsystem A_n, $n \in I$ of a Menger system A_I be weakly regular and contain idempotent e^n. Consequently for every $a \in A_n$, $n \in I$ $a \dots ae^n = a$ and $e^n = x_1 \dots x_n a$ for some $x_1, \dots, x_n \in A_n$. Take $b \in A_m$, $m \in I$ arbitrary. Since $e^m \dots e^m a \in A_m$, $e^m = z_1 \dots z_m (e^m \dots e^m a)$ for some $z_1, \dots, z_m \in A_m$. Therefore, $b = b \dots be^m = b \dots b(z_1 \dots z_m(e^m \dots e^m a)) = (b \dots b(z_1 \dots z_m e^m)) \dots (b \dots b(z_1 \dots z_m e^m))a$ and for $x = b \dots b(z_1 \dots z_m e^m)$ (4) holds.

Let $a \in A_n$, $n \in I$. Take $x \in A_n$ such that $x \dots xa = e^n$. Then $a = a \dots ae^n = a \dots \dots a(x \dots xa)$ and $(a \dots ax) \dots (a \dots ax)(a \dots ax) = a \dots a(x \dots x(a \dots ax)) = a \dots \dots a((x \dots xa) \dots (x \dots xa)x) = (a \dots a(x \dots xa)) \dots (a \dots a(x \dots xa)x = a \dots ax$, i.e. the element $a \dots ax$ is an idempotent. Hence $a \dots ax = e^n$. For arbitrary $b \in A_n$ $a \dots a(x \dots xb) = (a \dots ax) \dots (a \dots ax)b = e^n \dots e^n b = b$, i.e. for $y = x \dots xb$ (5) holds also.

For every idempotent e of a semigroup S there is uniquely defined a maximal subgroup of the semigroup S, containing e as identity. This will be the set $H(e)$ [1]. In a Menger system A_I too with a set $\{e^k, k \in J \subset I\}$ of idempotents, all from one $\mathscr{L}$-class and each from different A_k is connected some group-like subsystem of A_I. For instance, such will be the set $\{e^k, k \in J\}$ itself, since from $(e^k, e^l) \in \mathscr{L}$ follows $e^k \dots e^k e^l = e^k$ for every $k, l \in J$. However, there need not exist maximal group-like subsystems, even for only one idempotent.

For example, let the set $M = \{1, 2, 3\}$. Define functions $e, a, b, c \in \varphi_2(M)$ by

x	y	$(xy)e$	$(xy)a$	$(xy)b$	$(xy)c$
1	1	1	1	2	2
2	1	1	1	2	2
2	2	2	3	1	3
1	2	2	3	1	3
2	3	2	3	1	3
3	1	2	3	1	3
3	3	3	2	3	1
3	2	3	2	3	1

Functions e, a, b, c have all the same range, equal to M. Hence they are all $\mathscr{L}$-equivalent. They also have the same partition τ (classes of τ are $\{(11, (21)\}$, $\{(22), (12), (23), (31)\}$, $\{(33), (32)\}$). Thus they all are $\mathscr{R}$-equivalent, and therefore $\mathscr{H}$-equivalent also. The sets $\{e, a\}$ and $\{e, b\}$ are group-like subsystems of the system $\varphi_2(M)$, containing an idempotent e. However, e, a, b cannot belong together to any group-like subsystem, since $(xy)(abe) = 2$ for every $x, y \in M$, i.e. the range of the function abe is $\{2\}$ and $(e, abe) \notin \mathscr{L}$. Moreover, the element $c \in H(e)$ cannot belong to any group-like system either, since $(xy)(ece) = 2$ for every $x, y \in M$.

For further investigation of group-like subsystems we quote some results from [3].

Let a, b A_n be elements from $\mathscr{H}$-class H of a Menger system A_I. Let $s_1, \dots, s_n \in A_n$, $t \in A_m$ be such that $s_1 \dots s_n a = b$, $a \dots at = b$ (because of $(a, b) \in \mathscr{H}$ such elements exist). Then all mappings $x \to s_1 \dots s_n x$, $x \in H$ (for every $(a, b) \in \mathscr{H}$) form a simple transitive group $\Lambda(H)$ of permutations of the set H. Similarly all mappings $y \to y \dots yt$, $y \in H$ also form a simple transitive group $\Gamma(H)$ of permutations of H. The groups $\Lambda(H)$, $\Gamma(H)$ are anti-isomorphic. If H' is another $\mathscr{H}$-class from the containing H $\mathscr{D}$-class D, the groups $\Gamma(H)$ and $\Gamma(H')$ are isomorphic. Therefore the groups $\Gamma(H)$, $H \subset D$ are all isomorphic to some group G_D. Call G_D the Schützenberger group of D.

If $\mathscr{H}$-class H contains an idempotent, $\{H, \cdot\}$ is a group isomorphic to the group G_D. The corresponding isomorphism π_H: $H \to G_D$ may be considered as the right regular representation of H.

Let H' be another containing idempotent f' $\mathscr{H}$-class from D. The mapping $\tau_{HH'}$: $x \to (x)\tau_{HH'} = f' \dots f' x$, $x \in H$ is an isomorphism from the group $\{H, \cdot\}$ to the group $\{H', \cdot\}$. Furthermore,

$$\tau_{HH'}\pi_{H'} = \pi_H \tag{6}$$

and $\tau_{HH'}\tau_{H'H}$ is the identity mapping on the set H.

Let H be containing idempotent f $\mathscr{H}$-class from $\mathscr{L}$-class L. Define for every idempotent $e \in L$, $e \in A_n$ a partial $(n-1)$-place function Φ_e: $H^{n-1} \to H$ by

$$\Phi_e(a_1 \dots a_{n-1}) = \begin{cases} fa_1 \dots a_{n-1}e, & \text{if } fa_1 \dots a_{n-1}e \in H \\ \text{undefined otherwise.} \end{cases}$$

Here $a_1, \dots, a_{n-1} \in H$.

Lemma 2. *Let H, H' be $\mathscr{H}$-classes from $\mathscr{L}$-class L. Let $e, f, f' \in L$ be idempotents, $f \in H$, $f' \in H'$. Let $a_i \in H$, $b_i = (a_i)\tau_{HH'} \in H'$, $i = 1, \dots, n-1$. Then $fa_1 \dots a_{n-1}e \in H$ if and only if $f'b_1 \dots b_{n-1}e \in H'$ and if $fa_1 \dots a_{n-1} \in H$ then $(fa_1 \dots a_{n-1})\tau_{HH'} = f'b_1 \dots b_{n-1}e$.*

Proof. Let $fa_1 \dots a_{n-1}e \in H$ for some $a_1, \dots, a_{n-1} \in H$. Then $f'b_1 \dots b_{n-1}e = f'((a_1)\tau_{HH'}) \dots ((a_{n-1})\tau_{HH'})e = (f' \dots f'f)(f' \dots f'a_1) \dots (f' \dots f'a_{n-1})e = f' \dots f'(fa_1 \dots a_{n-1}e) = (fa_1 \dots a_{n-1}e)\tau_{HH'} \in H'$ (here $f' \dots f'f = f'$ because of $(f, f') \in \mathscr{L}$). So from $fa_1 \dots a_{n-1}e \in H$ follows $f'b_1 \dots b_{n-1}e \in H'$.

On the other hand, let $f'b_1 \dots b_{n-1}e \in H'$. Since $\tau_{HH'}\tau_{H'H}$ is the identity mapping on H, $(b_i)\tau_{H'H} = a_i$ for every $i = 1, \dots, n-1$. Now $fa_1 \dots a_{n-1}e = f((b_1)\tau_{H'H}) \dots ((b_{n-1})\tau_{H'H})e = (f \dots ff')(f \dots fb_1) \dots (f \dots fb_{n-1})e = f \dots f(f'b_1 \dots b_{n-1}e) = (f'b_1 \dots b_{n-1}e)\tau_{H'H} \in H$.

By the last lemma the function Φ_e does not depend essentially on the choice of the $\mathscr{H}$-class H from its $\mathscr{L}$-class L and can be considered as (depending on L) a function Ψ_e^L on the group G_D, defined by

$$\Psi_e^L(a_1 \dots a_{n-1}) = \big(\Phi_e((a_1)\pi_H^{-1} \dots (a_{n-1})\pi_H^{-1})\big)\pi_H \tag{7}$$

where $a_1, \ldots, a_{n-1} \in G_D$ and H is an arbitrary $\mathscr{H}$-class from $L''=L(e)$), containing an idempotent.

Theorem 8. *Let $B_J=\bigcup_{n\in J} B_n$, $B_n\subset A_n$ be a subset of a Menger system A_J, $J\subset I$. The set B_J is group-like subsystem of A_I if and only if* 1)—3) *hold:*

1) *B_J belongs to one $\mathscr{L}$-class L of A_I (thus also to one $\mathscr{D}$-class D)*

2) *for every B_n there exists containing an idempotent e^n $\mathscr{H}$-class H_n of A_I such that $B_n\subset H_n$*

3) *there exists subgroup G of the Schützenberger group G_D such that if we denote $\pi_{H_n}=\pi_n$, $\Psi^L_{e^n}=\Psi_n$, then for every $n\in J$*

3.1 $(B_n)\pi_n=G$

3.2 *the function Ψ_n is everywhere defined on G and G is closed under Ψ_n.*

Proof. Let $B_j=\bigcup_{n\in J} B_n$ be a group-like subsystem of a Menger system A_I, $B_n\subset A_n$, $J\subset I$. As mentioned above, $\mathscr{L}_B$ consists of one $\mathscr{L}_B$-class, every B_n forms one $\mathscr{H}_B$-class and every B_n contains an idempotent e^n. Hence B_J belongs also to one $\mathscr{L}_A$-class L (and thus to one $\mathscr{D}_A$-class D), every B_n is contained in some $\mathscr{H}_A$-class H_n and idempotent e^n belongs to H_n.

Since B_n is subsystem of A_I, $e^n a_1 \ldots a_{n-1} e^n \in B_n$ for every $a_1, \ldots, a_{n-1}\in B_n$. Thus $\Phi_n(a_1\ldots a_{n-1})=e^n a_1\ldots a_{n-1} e^n$ is always defined on B_n. By (7) Ψ_n is then also always defined on the set $(B_n)\pi_n$ and the last set is closed for Ψ_n. By (4), (5) $\{B_n, \cdot\}$ is a group. Since this group is a subgroup of the group $\{H_n, \cdot\}$, $(B_n)\pi_n$ is a subgroup of the group G_D. By (6) for every $a\in B_n$ $(a)\pi_n=(a)\tau_{nm}\pi_m=((a)\tau_{nm})\pi_m\subset(B_m)\pi_m$, i.e. $(B_n)\pi_n\subset(B_m)\pi_m$. Similarly $(B_m)\pi_m\subset(B_n)\pi_n$. Thus $(B_n)\pi_n=(B_m)\pi_m$ for every $n, m\in J$.

On the other hand, suppose 1)—3) hold for the subset $B_J=\bigcup_{n\in J} B_n$, $B_n\subset A_n$, $J\subset I$.

Since $(B_n)\pi_n=G$ is a group and π_n is an isomorphism, $\{B_n, \cdot\}$ is also a group. Let e^n be its identity. Clearly e^n is an idempotent, and from $B_n\subset H_n$ follows that e^n is the idempotent from H_n and the functions τ_{nm}, π_n are defined for every $n, m\in J$.

Since e^n is the identity of the group B_n, $e^n\cdot x=x\cdot e^n=x$ for every $x\in B_n$. By 1) $(e^n, e^m)\in\mathscr{L}$ for every $n, m\in J$, thus $e^n\cdot e^m=e^n\ldots e^n e^m=e^n$.

Denote for arbitrary $a\in B_n$, $n, m\in J$ $a(m)=((a)\pi_n)\pi_m^{-1}$, $a^{-1}(m)=(((a)\pi_n)^{-1})\pi_m^{-1}$. Clearly $a(n)=a$, denote similarly $a^{-1}(n)=a^{-1}$.

By (6) $\tau_{mn}\pi_n=\pi_m$ for every $n, m\in J$. Hence $(\pi_m)^{-1}=(\tau_{mn}\pi_n)^{-1}=\pi_n^{-1}\tau_{nm}$ and for every $y\in B_m$ $y=((y)\pi_m)\pi_m^{-1}=((y)\pi_m)\pi_n^{-1}\tau_{nm}=e^m\ldots e^m(((y)\pi_m)\pi_n^{-1})=e^m\cdot y(n)$. Thus for arbitrary $x_1, \ldots, x_m\in A_n$ $x_1\ldots x_m y=(x_1\cdot e^n)(x_1\cdot(x_1^{-1}\cdot x_2))\ldots(x_1\cdot(x_1^{-1}\cdot x_m))\cdot$ $\cdot(e^m\cdot y(n))=x_1\cdot(e^n(x_1^{-1}\cdot x_2)\ldots(x_1^{-1}\cdot x_m)e^m)\cdot y(n)$. Denote the last expression by w, $(x_1^{-1}\cdot x_{i+1})\pi_n=a_i$, $i=1, \ldots, n-1$. Then by (7) $e^n(x_1^{-1}\cdot x_2)\ldots(x_1^{-1}\cdot x_m)e^m=$ $=\Phi_m((x_1^{-1}\cdot x_2)\ldots(x_1^{-1}\cdot x_m))=(\Psi_m(a_1\ldots a_{n-1}))\pi_n^{-1}$ and by 3.2 $\Psi_m(a_1\ldots a_{n-1})\in G$ is defined. Since π_n is isomorphism, $w=(w)\pi_n\pi_n^{-1}=((x_1\cdot(\Psi_m(a_1\ldots a_{n-1}))\pi_n^{-1}y(n))\pi_n\pi_n^{-1}=$ $=((x_1)\pi_n\cdot(\Psi_m(a_1\ldots a_{n-1}))\pi_n^{-1}\pi_n\cdot(y(n))\pi_n)\pi_n^{-1}=((x_1)\pi_n\cdot\Psi_m(a_1\ldots a_{n-1})\cdot(y(n))\pi_n)\pi_n^{-1}$. Now $(x_1)\pi_n$, $\Psi_m(a_1\ldots a_{n-1})$, $(y(n))\pi_n\in G$ and G is subgroup of the group G_D. Consequently, $(x_1)\pi_n \cdot \Psi_m(a_1 \dots a_{n-1}) \cdot (y(n)\pi_n) \in G$ also. So $x_1 \dots x_m y = ((x_1)\pi_n \cdot \Psi_m(a_1 \dots a_{n-1}) \cdot$ $\cdot (y(n)\pi_n))\pi_n^{-1} \in B_n$. Thus B_J is a subsystem of the Menger system A_I.

For every $a \in B_n$, $b \in B_m$ $(b \cdot a^{-1}) \dots (b \cdot a^{-1})a = b \cdot (a^{-1} \dots a^{-1}a) = b \cdot (a^{-1} \cdot a) =$ $= b \cdot e^n = b$, i.e. (4) holds for B_J. Similarly, for every $a, b \in B_n$ $a \dots a(a^{-1} \cdot b) =$ $= a \cdot (a^{-1} \cdot b)$ $a^{-1} \cdot a \cdot b = e^n \cdot b = b$, i.e. (5) holds also. Thus the subsystem B_J is group-like.

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