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A REAL-ANALYTIC QUASICONFORMAL EXTENSION OF A QUASISYMMETRIC FUNCTION

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An increasing homeomorphism $\varphi\colon \mathbf{R}\to\mathbf{R}$ is ϱ-quasisymmetric if

$$\varrho^{-1} \leqq (\varphi(x+t)-\varphi(x))/(\varphi(x)-\varphi(x-t)) \leqq \varrho \tag{1}$$

for all real x and t, $t\neq 0$. It is well-known that every ϱ-quasisymmetric function can be extended to a K-quasiconformal mapping $f\colon H\to H$, where H denotes the upper half-plane. This was first shown by Beurling and Ahlfors [2] who gave an explicit construction for an extension f. This extension is in the class $C^1(H)$. In the present note we shall introduce an analytic kernel into the integerals which define f and thus obtain a real-analytic solution of the boundary value problem. The construction follows closely the one in [2]. Special attention, however, is paid to the estimation of the maximal dilatation K. The elementary but laborious computations in [2] leading to the estimate $K<\varrho^2$ for $\varrho>1$ seem to contain some slips. In fact, the same computations yield a somewhat sharper bound.

Remark. Recently Lehto [4] has demonstrated the existence of a real-analytic solution of the boundary value problem for quasiconformal mappings by a different, less explicit method. The maximal dilatation of his solution seems to be essentially larger than that of ours.

We state our result as

Theorem. *Every ϱ-quasisymmetric function has a K-quasiconformal real-analytic extension to the upper half-plane. There is a number $\varrho_0(=1.925\,057\ldots)$ such that $K<\varrho^{3/2}$ if $1<\varrho<\varrho_0$ and $K<3\varrho^2/4$ if $\varrho\geqq\varrho_0$.*

Proof. Let $\varrho>1$ and φ be a fixed ϱ-quasisymmetric function. (Since a 1-quasisymmetric function is linear, there is nothing to prove in the case $\varrho=1$.) Our construction is based on the functions $u_k\colon H\to\mathbf{R}$ defined by

$$u_k(x+iy) = \int_{-\infty}^{\infty} \varkappa_k(t)\,\varphi(x+yt)\,dt, \quad k = 1, 2, \ldots, \tag{2}$$

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where

$$\varkappa_k(z) = c_k \exp\left(-(2z)^{4^k}\right),$$

and the constant c_k is so chosen that the integral of $\varkappa_k$ over the real line equals 1/2. The convergence of the integral in (2) for any $x+iy \in H$ is guaranteed by the inequalities [3, p. 245]

$$\psi(t) \leqq 2t^B \quad \text{for} \quad 0 \leqq t \leqq 1, \tag{3}$$

$$\psi(t) \leqq (2t)^A \quad \text{for} \quad 1 \leqq t, \tag{4}$$

with $A = \log_2(1+\varrho)$, $B = \log_2(1+\varrho^{-1})$, holding for all ϱ-quasisymmetric ψ normalized by $\psi(0)=0$, $\psi(1)=1$. — For future use we note that (3) applied to $t \mapsto 1-\psi(1-t)$ implies

$$\psi(1-2^{-k}) - \psi(2^{-k}) \geqq 1 - 4\varrho^k/(\varrho+1)^k, \quad k = 1, 2, \ldots. \tag{5}$$

Lemma 1. *The functions* u_k *are real-analytic in* H *and*

$$\lim_{z \to x_0} u_k(z) = \varphi(x_0)/2 \tag{6}$$

for every real x_0.

Proof. Fix $x_0 + iy_0$ in H. Inequality (4) implies the uniform and absolute convergence of the integrals

$$g_k(z, w) = \int_{-\infty}^{\infty} \varkappa_k((t-z)/w)\varphi(t)\,dt,$$

$$\int_{-\infty}^{\infty} \varkappa_k'((t-z)/w)\varphi(t)\,dt, \quad \int_{-\infty}^{\infty} t\varkappa_k'((t-z)/w)\varphi(t)\,dt$$

in a sufficiently small polydisc $U \subset \mathbf{C} \times \mathbf{C}$, with center (x_0, y_0). It follows by a standard argument that g_k has complex derivatives with respect to z and w in U. Hence g_k is holomorphic at (x_0, y_0); since $u_k(x+iy) = yg_k(x, y)$ for $x+iy \in H$, the function u_k is real-analytic at x_0+iy_0. To prove (6), let x_n+iy_n with $y_n > 0$ converge to x_0 and apply Lebesgue's convergence theorem to the integrals

$$\int_{-\infty}^{\infty} \varkappa_k(t)(\varphi(x_n+y_n t) - \varphi(x_0))\,dt.$$

By (4) they are majorized by an integrable function and thus converge to zero as n tends to infinity.

The following two lemmas contain the necessary estimates of the partial derivatives of u_k.

Lemma 2.

$$\limsup_{k \to \infty} \frac{|(u_k)_y(z)|}{(u_k)_x(z)} \leqq \frac{\varrho-1}{2(\varrho+1)},$$

uniformly in H.

Proof. Given $z=x+iy\in H$, set

$$\psi(t)=\varphi(x+yt),\quad t_k=(2\cdot 4^{k4^{-k}})^{-1},\quad \text{and}\quad a_k=\psi(t_k)-\psi(0).$$

Observe that $t\mapsto t\varkappa_k(t)$ takes its largest value at t_k. Differentiation of (2) and a subsequent integration by parts shows that

$$y(u_k)_x(z)=\int_{-\infty}^{\infty}\varkappa_k(t)\,d\psi(t),\quad y(u_k)_y(z)=\int_{-\infty}^{\infty}t\varkappa_k(t)\,d\psi(t).$$

Because ψ is increasing and $\varkappa_k$ is positive, and because of (1),

$$\int_{-\infty}^{\infty}\varkappa_k(t)\,d\psi(t)>\int_{-t_k}^{t_k}\varkappa_k(t)\,d\psi(t)>(1+\varrho^{-1})a_k\varkappa_k(t_k).$$

On the other hand,

$$\int_{-\infty}^{t_k}t\varkappa_k(t)\,d\psi(t)<\int_{-\infty}^{\infty}t\varkappa_k(t)\,d\psi(t)<\int_{-t_k}^{\infty}t\varkappa_k(t)\,d\psi(t). \tag{7}$$

The integral on the right hand side of (7) is smaller than

$$\varkappa_k(t_k)\int_{-t_k}^{0}t\,d\psi(t)+\varkappa_k(0)\int_{0}^{t_k}t\,d\psi(t)+\varkappa_k(t_k)(\psi(1-t_k)-\psi(t_k)) \tag{8}$$
$$+\varkappa_k(1-t_k)\varrho a_k+a_kt_k\sum_{n=2}^{\infty}n\varkappa_k(nt_k)\varrho^n.$$

By a lemma of Beurling and Ahlfors [1, p. 67]

$$\int_{-t_k}^{0}t\,d\psi(t)=t_k\psi(-t_k)-\int_{-t_k}^{0}\psi(t)\,dt$$
$$\leqq -\frac{t_k}{\varrho+1}(\psi(0)-\psi(-t_k))\leqq\frac{-t_ka_k}{\varrho(\varrho+1)},$$

and similarly

$$\int_{0}^{t_k}t\,d\psi(t)\leqq\frac{t_ka_k\varrho}{\varrho+1}.$$

By (3),

$$\psi(1-t_k)-\psi(t_k)<2(\psi(2t_k)-\psi(t_k))(1-2t_k)^B\leqq 2a_k\varrho(1-2t_k)^B.$$

The left hand side of (7) can be estimated from below in the same way. Combining the estimates, dividing by a_k, and observing that t_k, $\varkappa_k(0)$ and $\varkappa_k(t_k)$ tend to 1/2 while $\varkappa_k(1-t_k)$ and the infinite sum in (8) tend to zero as $k\to\infty$, we obtain the desired inequality.

Lemma 3. *If* $\lambda=1/2$ *or* $-1/2$,

$$\limsup_{k\to\infty}\frac{(u_k)_x(x+\lambda y+iy)}{(u_k)_x(x-\lambda y+iy)}\leqq\varrho,$$

uniformly for $x+iy\in H$.

Proof. We may suppose $\lambda=1/2$. Then, with ψ as in the previous lemma and by (5) and (1),

$$\begin{aligned} y(u_k)_x(x-y/2+iy) &> \varkappa_k(2^{-1}-2^{-k})(\psi(-2^{-k})-\psi(-1+2^{-k}))\\ &\geqq \varrho^{-1}\varkappa_k(2^{-1}-2^{-k})(1-4\varrho^k/(\varrho+1)^k)(\psi(1)-\psi(0)). \end{aligned}$$

In the other direction we have

$$\begin{aligned} y(u_k)_x(x+y/2+iy) &< \varkappa_k(0)(\psi(1)-\psi(0))\\ &+\varkappa_k(2^{-1})(\psi(1+2^{-k})-\psi(1)+\psi(0)-\psi(-2^{-k}))\\ &+\varkappa_k(2^{-1}+2^{-k})(\psi(2)-\psi(1)+\psi(0)-\psi(-1))\\ &+2(\psi(1)-\psi(0))\sum_{n=1}^{\infty}\varkappa_k(2^{-1}+n)\,\varrho^{n+1}. \end{aligned}\tag{9}$$

By (3) and (1)

$$\psi(1+2^{-k})-\psi(1)\leqq 2\cdot 2^{-kB}\varrho(\psi(1)-\psi(0))$$

and

$$\psi(0)-\psi(-2^{-k})\leqq\varrho(\psi(2^{-k})-\psi(0))\leqq 2\cdot 2^{-kB}\varrho(\psi(1)-\psi(0)).$$

To obtain the asserted inequality it now suffices to combine the estimates, divide by $\psi(1)-\psi(0)$ and observe that $\varkappa_k(2^{-1}+2^{-k})$ and the infinite sum in (9) tend to zero as $k\to\infty$.

We now proceed to the definition of the quasiconformal extension. To this end, set $\alpha_k(z)=u_k(z+y/2)$, $\beta_k(z)=u_k(z-y/2)$ $(z=x+iy)$ and

$$f_k=\alpha_k+\beta_k+ir(\alpha_k-\beta_k),$$

where r is a positive parameter. By the chain rule, the partial derivatives of α_k and β_k are

$$\begin{aligned} (\alpha_k)_x(z) &= (u_k)_x(z+y/2),\quad (\beta_k)_x(z)=(u_k)_x(z-y/2),\\ (\alpha_k)_y(z) &= (u_k)_x(z+y/2)/2+(u_k)_y(z+y/2),\\ (\beta_k)_y(z) &= -(u_k)_x(z-y/2)/2+(u_k)_y(z-y/2). \end{aligned}$$

Lemma 4. *There is a* k_0 *such that for* $k\geqq k_0$ $f_k\colon H\to H$ *is a bijection.*

Proof. One easily sees that f_k is one to one if and only if the pair of equations

$$\alpha_k(x+iy)=a\tag{10}$$

$$\beta_k(x+iy)=b\tag{11}$$

has one and only one solution for all a, b with $a>b$. But (10) and (11) define for $x<\varphi^{-1}(2a)$ resp. $x>\varphi^{-1}(2b)$ a decreasing resp. increasing curve $y=g_k(x)$, $y=h_k(x)$. These intersect certainly if $-g_k'$ and h_k' are bounded above. But

$$-g_k'(x) = (1/2+(u_k)_y(z+y/2)/(u_k)_x(z+y/2))^{-1},$$

and

$$h_k'(x) = (1/2-(u_k)_y(z-y/2)/(u_k)_x(z-y/2))^{-1},$$

so that the assertion follows from Lemma 2.

We have shown that f_k is — for k large enough — a real-analytic homeomorphism with correct boundary values. Next we want to estimate the maximal dilatation of f_k. Denote the ratios $(\alpha_k)_y/(\alpha_k)_x$, $(\beta_k)_y/(\beta_k)_x$, $(\alpha_k)_x/(\beta_k)_x$, evaluated at z, by ξ, η, ζ, respectively. If $\varrho'>\varrho$ is arbitrary we can, by Lemmas 2 and 3, fix a $k\geqq k_0$ such that ξ and η lie between $(1+\varrho')^{-1}$ and $\varrho'(1+\varrho')^{-1}$ for all $z\in H$. We denote f_k by f and its dilatation quotient at z by D. As in [2] we compute

$$(12)\qquad D+D^{-1} = \frac{1}{2r(\xi+\eta)}\left(\left(\zeta(1+\xi^2)+\zeta^{-1}(1+\eta^2)\right)(1+r^2)+2(1-\xi\eta)(1-r^2)\right).$$

The right hand side of (12) is invariant under the transformation $(\xi, \eta, \zeta) \mapsto (\eta, \xi, \zeta^{-1})$. Hence it suffices to consider the case $\xi\geqq\eta$. With this condition one easily sees that for $\xi, \eta,$ and r fixed, the right hand side of (12) is maximized by $\zeta=\varrho'$. It follows that

$$D+D^{-1} \leqq a(\xi, \eta)r+b(\xi, \eta)r^{-1} = F(\xi, \eta, r),$$

where

$$a(\xi, \eta) = \frac{(\varrho'-1)^2+(\varrho'\xi+\eta)^2}{2\varrho'(\xi+\eta)}$$

and

$$b(\xi, \eta) = \frac{(\varrho'+1)^2+(\varrho'\xi-\eta)^2}{2\varrho'(\xi+\eta)}.$$

The problem is to minimize with respect to r the maximum of F in the triangle $T=\{(\xi, \eta)|(\varrho'+1)^{-1}\leqq\eta\leqq\xi\leqq\varrho'(\varrho'+1)^{-1}\}$. A routine computation shows that F can attain its maximum in T only at the vertices of T. Denote

$$F\left((\varrho'+1)^{-1}, (\varrho'+1)^{-1}, r\right) = F_1(r) = a_1 r+b_1 r^{-1},$$
$$F\left(\varrho'(\varrho'+1)^{-1}, (\varrho'+1)^{-1}, r\right) = F_2(r) = a_2 r+b_2 r^{-1},$$
$$F\left(\varrho'(\varrho'+1)^{-1}, \varrho'(\varrho'+1)^{-1}, r\right) = F_3(r) = a_3 r+b_3 r^{-1},$$

where

$$a_1 = \frac{\varrho'^3-\varrho'^2+2}{4\varrho'}, \qquad b_1 = \frac{\varrho'^4+4\varrho'^3+7\varrho'^2+2\varrho'+2}{4\varrho'^2+4\varrho'},$$

$$a_2 = \frac{\varrho'^4+1}{\varrho'^3+2\varrho'^2+\varrho'}, \qquad b_2 = \varrho'+\varrho'^{-1},$$

$$a_3 = \frac{2\varrho'^3-\varrho'+1}{4\varrho'^2}, \qquad b_3 = \frac{2\varrho'^4+2\varrho'^3+7\varrho'^2+4\varrho'+1}{4\varrho'^3+4\varrho'^2}.$$

Every F_i is minimized at $r_i=(b_i/a_i)^{1/2}$ and the only poossible positive root r_{ij} of the equation $F_i(r)=F_j(r)$, $i\neq j$, satisfies

$$r_{ij}^2 = (b_i-b_j)/(a_j-a_i).$$

It is clear that $\max\{F_1, F_2, F_3\}$ is minimized by some r_i or some r_{ij}. We observe that $b_1>b_2>b_3$ for all $\varrho'>1$. We first compare F_1 and F_2. A computation shows that $r_1>r_2$ for all $\varrho'>1$. It follows that $\max\{F_1, F_2\}$ is minimized at r_2 if $r_{12}\leqq r_2$, at r_{12} if $r_2<r_{12}<r_1$ and at r_1 if $r_1\leqq r_{12}$ or r_{12} is imaginary. The first alternative is found out to hold for $\varrho'\leqq\varrho_2=1.387\,808\ldots$, the second for $\varrho_2<\varrho'<$ $<\varrho_1=1.969\,351\ldots$, and the third for $\varrho'\geqq\varrho_1$. Further calculations show that r_{23} is real only for $\varrho'<1.959\,413\ldots<\varrho_1$. Thus if $\varrho'>\varrho_1$, then $F_1(r)>F_3(r)$ for all r. On the other hand the inequalities $r_2<r_{23}$ and $r_{12}<r_{23}$ are found out to hold for all $\varrho'>1$ for which r_{23} is real. It follows, in particular, that $F_2(r_2)>F_3(r_2)$ and $F_2(r_{12})>F_3(r_{12})$ for $1<\varrho'<\varrho_1$.

Altogether, we have shown that the maximal dilatation K of f satisfies the inequalities

$$K+K^{-1} \leqq F_2(r_2) = 2(a_2b_2)^{1/2}, \quad \text{for} \quad \varrho' \leqq \varrho_2,$$

$$K+K^{-1} \leqq F_1(r_{12}) = \frac{a_2b_1-a_1b_2}{((a_2-a_1)(b_1-b_2))^{1/2}},$$

for $\varrho_2<\varrho'<\varrho_1$, and

$$K+K^{-1} \leqq F_1(r_1) = 2(a_1b_1)^{1/2}, \quad \text{for} \quad \varrho_1 \leqq \varrho'.$$

Now if $\varrho'<\varrho_2$ the inequality $K<\varrho'^{3/2}$ holds if

$$4a_2b_2 < \varrho'^3+\varrho'^{-3}+2.$$

A computation shows that the inequality is indeed valid. For $\varrho'>\varrho_1$ the inequality $K<c\varrho'^2$ is satisfied if

$$4a_1b_1 < c^2\varrho'^4+c^{-2}\varrho'^{-4}+2. \tag{13}$$

If (13) is written explicitly one sees that it is satisfied by any $c>1/2$ for ϱ' large enough and for all $\varrho'>\varrho_1$ e.g. by $c=3/4$. The computations are rather complicated in the case $\varrho_2<\varrho'<\varrho_1$. It turns out that the inequality $K<\varrho'^{3/2}$ holds for $\varrho'<\varrho_0=1.925\,057\ldots$ while for $\varrho_0\leqq\varrho'<\varrho_1$ the inequality $K<c\varrho'^2$ holds with $c=3/4$. Since $\varrho'>\varrho$ was arbitrary, the proof of the theorem is completed.

Remark. The above estimation of K is valid, with ϱ' replaced by ϱ, for the original extension of Beurling and Ahlfors. As pointed out in the proof, actually $K<c(\varrho)\varrho^2$ with $\lim_{\varrho\to\infty} c(\varrho)=1/2$. In view of the universal bound $K<8\varrho$, due to Reed [5], this is, however, of little interest.

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