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ON THE OUTER COEFFICIENT OF QUASICONFORMALITY OF A CONVEX DIHEDRAL WEDGE

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1. Introduction

Let D and D' be domains in R^3 and $f\colon D \to D'$ a homeomorphism. The numbers

$$K_I(f) = \sup_{\Gamma} \frac{M(f\Gamma)}{M(\Gamma)}, \quad K_O(f) = \sup_{\Gamma} \frac{M(\Gamma)}{M(f\Gamma)}$$

are called the inner and the outer dilatation of f. There $M(\Gamma)$ and $M(f\Gamma)$ are the moduli of the curve families Γ and $f\Gamma$ and the suprema are taken over all families which lie in D. The mapping f is quasiconformal if the maximal dilatation $\max\left(K_I(f),\ K_O(f)\right)$ is finite. Further, the inner and outer coefficients of the quasiconformality of D with respect to D' are defined by the numbers

$$K_I(D, D') = \inf_f K_I(f), \quad K_O(D, D') = \inf_f K_O(f),$$

where f runs through all homeomorphisms $f\colon D \to D'$. The case when D' is a ball or a half space is of particular interest.

The problem of determining the coefficients of quasiconformality and the corresponding extremal mappings is fairly difficult and has been only solved for a few domains. In this paper we shall consider the outer coefficient problem in the case where D is a convex dihedral wedge in R^3 and D' is a half space, still with certain additional local conditions for the mapping at one arbitrary edge point. Gehring and Väisälä have solved the corresponding problem for the inner coefficient [2], p. 43 (see also Väisälä [3], p. 133[1]). As for the theory of n-dimensional quasiconformal mappings we refer to [3].

[1] The definitions of dilatations in [2] are not exactly the same as in this paper and in [3].

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2. Additional conditions

Let (r, ψ, φ) be spherical coordinates in R^3, where the polar angle φ is measured from the positive half of the x_3-axis. A domain in R^3 is called a dihedral *wedge* of angle α, $0<\alpha\leqq 2\pi$, if it can be mapped by means of a similarity transformation onto the domain

$$D_\alpha = \{(r, \psi, \varphi)|r > 0,\ 0 < \psi < \alpha,\ 0 < \varphi < \pi\}.$$

The inverse image of the x_3-axis under this mapping is said to be the *edge* of the dihedral wedge.

We consider now the subclass W of homeomorphisms $f\colon \bar{D}_\alpha \to \bar{D}_\pi$, $0<\alpha<\pi$, $f(0)=0$, whose restrictions $f|D_\alpha$ are quasiconformal mappings onto D_π and which satisfy the following conditions A and B at the origin.

A. *There is a polar angle* φ_0, $0<\varphi_0<\pi/2$, *such that the limit*

$$\lim_{t\to 0+} f(te)/t = k(e) \neq 0, \infty$$

exists for every $e \in \bar{D}_\alpha$ *with* $0\leqq(e, e_3)\leqq\varphi_0$, *where* (e, e_3) *denotes the actual angle between the vectors* e *and* e_3.

Let $g_n\colon \bar{D}_\alpha \to \bar{D}_\pi$ be the sequence, defined by

$$g_n(x) = nf(x/n), \quad n = 1, 2, \ldots .$$

We extend g_n to a quasiconformal mapping of R^3 onto itself. First we map D_α onto D_π by the folding h, $h(r, \psi, \varphi)=(r, \pi\psi/\alpha, \varphi)$. Then $g_n \circ h^{-1}$ will be extended to a quasi-conformal mapping $h_1\colon R^3 \to R^3$ by reflection. Finally, we define a mapping $h_2\colon R^3\to R^3$ by $h_2(r, \psi, \varphi)=(r, \psi', \varphi)$, where

$$\psi' = \begin{cases} \alpha\psi/\pi & \text{for } \ 0 \leqq \psi \leqq \pi \\ \alpha + \dfrac{2\pi-\alpha}{\pi}(\psi-\pi) & \text{for } \ \pi \leqq \psi \leqq 2\pi. \end{cases}$$

Then $\hat{g}_n = h_1 \circ h_2^{-1}\colon R^3 \to R^3$ is quasiconformal and $\hat{g}_n|\bar{D}_\alpha = g_n$.

Because $(k(e), k(e_3))>0$ for a fixed f and every e, $0<(e, e_3)\leqq\varphi_0$ (see [1], Theorem 3.3), the sequence $\hat{g}_n$ is by [3], 19.4 and 20.5 a normal family and thus has a subsequence $\hat{g}_i$, $i\in J\subset N$, which converges to the limit function $\hat{g}\colon R^3\to R^3$ uniformly on every compact subset of R^3. By the condition A $\hat{g}$ is a homeomorphism, since $\hat{g}(0)=0$ and $\hat{g}_n(e_3)=nf(e_3/n)\to k(e_3)$; see [3], 21.3. By [3], 37.2, $g=\hat{g}|D_\alpha$ is quasiconformal and

$$K_O(g) \leqq \varliminf_{n\to\infty} K_O(g_n) = K_O(f).$$

The condition A implies that for every $e\in\bar{D}_\alpha$, $0\leqq(e, e_3)\leqq\varphi_0$, g maps the ray $\{te|t>0\}$ onto the ray $\{tk(e)|t>0\}$ linearly, $g(te)=tk(e)$. We set the following additional condition for f:

$$\text{B.} \quad \lim_{\varepsilon \to 0+} \Big(\max_{\substack{(e, e_3)=\varepsilon \\ e \in D_\alpha}} (k(e), k(e_3)) \Big/ \min_{\substack{(e, e_3)=\varepsilon \\ e \in D_\alpha}} (k(e), k(e_3)) \Big) = 1.$$

This condition as well is local for f at the origin. We denote

$$K_O(D_\alpha, D_\pi; W) = \inf \{K_O(f) | f \in W\}$$

and prove the following theorem.

Theorem 1. $K_O(D_\alpha, D_\pi; W) = \pi/\alpha$.

For $\varepsilon > 0$ let $f \in W$ be a mapping such that

$$K_O(f) < K_O(D_\alpha, D_\pi; W) + \varepsilon$$

and g the limit mapping, associated with f by the above process. Then $K_O(g) \leqq K_O(f)$ and thus also

$$K_O(g) < K_O(D_\alpha, D_\pi; W) + \varepsilon. \tag{1}$$

Next, to prove the Theorem 1 we consider the following curve families in D_α.

3. The curve families Γ and Γ_1

We denote

$$D_\alpha(r_1, r_2; \beta, \varphi_0) = \{(r, \psi, \varphi) | r_1 < r < r_2, \ 0 < \psi < \alpha, \ \beta < \varphi < \varphi_0\},$$

where $r_1 > 0$, $0 < \beta < \varphi_0$. Let Γ be the family of all arcs joining the plane parts

$$T_0 = \overline{D_\alpha(r_1, r_2; \beta, \varphi_0)} \cap \{\psi = 0\}$$

and

$$T_\alpha = \overline{D_\alpha(r_1, r_2; \beta, \varphi_0)} \cap \{\psi = \alpha\}$$

of the boundary $\partial D_\alpha(r_1, r_2; \beta, \varphi_0)$ in $D_\alpha(r_1, r_2; \beta, \varphi_0)$.

Let $\varrho \in F(\Gamma)$ be arbitrary, i.e. $\int_\gamma \varrho \, ds \geqq 1$ when $\gamma \in \Gamma$. Choose $y \in T_0$ and consider the horizontal circular arc

$$\gamma_y = \{(r, \psi, \varphi) | r = |y|, \ 0 \leqq \psi \leqq \alpha, \ \varphi = (y, e_3)\},$$

where (y, e_3) is the angle between the vectors y and e_3. By Hölder's inequality we obtain

$$1 \leqq \int_0^\alpha \varrho r \sin \varphi \, d\psi = \int_0^\alpha \varrho r^{2/3} \sin^{1/3} \varphi (r^{1/3} \sin^{2/3} \varphi) \, d\psi$$

$$\leqq \Big(\int_0^\alpha \varrho^3 r^2 \sin \varphi \, d\psi \Big)^{1/3} \Big(\int_0^\alpha r^{1/2} \sin \varphi \, d\psi \Big)^{2/3}.$$

Hence

$$\int_0^{\alpha} \varrho^3 r^2 \sin\varphi \, d\psi \geqq 1/(\alpha^2 r \sin^2\varphi)$$

and consequently

$$\int_{D_\alpha(r_1, r_2;\, \beta, \varphi_0)} \varrho^3 \, dV = \int_{r_1}^{r_2} dr \int_{\beta}^{\varphi_0} d\varphi \int_0^{\alpha} \varrho^3 r^2 \sin\varphi \, d\psi$$

$$\geqq \int_{r_1}^{r_2} dr \int_{\beta}^{\varphi_0} \frac{d\varphi}{\alpha^2 r \sin^2\varphi} = (1/\alpha^2) \frac{\sin(\varphi_0 - \beta)}{\sin\varphi_0 \sin\beta} \log(r_2/r_1).$$

From the above it follows that

$$M(\Gamma) \geqq \frac{1}{\alpha^2} \cdot \frac{\sin(\varphi_0 - \beta)}{\sin\varphi_0 \sin\beta} \log(r_2/r_1).$$

Let Γ_1 be the arc family joining the spheres $|x| = r_1$ and $|x| = r_2$ in the set

$$D_\alpha(r_1, r_2;\, \beta) = \{(r, \psi, \varphi) | r_1 < r < r_2,\ 0 < \psi < \alpha,\ 0 < \varphi < \beta\}.$$

Then

$$M(\Gamma_1) = \frac{\alpha}{2\pi} \cdot \frac{\pi(2 - 2\cos\beta)}{(\log(r_2/r_1))^2} = \frac{\alpha(1 - \cos\beta)}{(\log(r_2/r_1))^2}.$$

In particular,

$$\varliminf_{\beta \to 0} M(\Gamma)^2 M(\Gamma_1) \geqq 1/2\alpha^3.$$

We shall next consider the g-images Γ' and Γ_1' of the families Γ and Γ_1.

4. Upper bounds for $M(\Gamma')$ and $M(\Gamma_1')$

We may assume that $k(e_3)$ lies on the positive x_3-axis. Let

$$\begin{aligned} \underline{r}_i &= \min\{|g(x)| \,\big|\, |x| = r_i,\ x \in \bar{D}_\alpha\} \\ \bar{r}_i &= \max\{|g(x)| \,\big|\, |x| = r_i,\ x \in \bar{D}_\alpha\}, \end{aligned} \qquad (i = 1, 2)$$

where r_1 and r_2 are the radii of $D_\alpha(r_1, r_2;\, \beta, \varphi_0)$ and $D_\alpha(r_1, r_2;\, \beta)$. Furthermore we denote

$$D_\alpha(\beta, \varphi_0) = \{(r, \psi, \varphi) | r > 0,\ 0 < \psi < \alpha,\ \beta < \varphi < \varphi_0\}$$

and

$$E = \big(gD_\alpha(\beta, \varphi_0)\big) \cap \big(B^3(\bar{r}_2) \setminus \overline{B^3(\underline{r}_1)}\big).$$

Define

$$\varrho(r, \psi, \varphi) = \begin{cases} 1/(\pi r \sin\varphi) & \text{for } (r, \psi, \varphi) \in E \\ 0 & \text{otherwise.} \end{cases}$$

We can suppose φ_0 so small that for the above chosen mapping f $(k(e), e_3)<\pi$ when $(e, e_3)\leqq\varphi_0$. Then

$$\int_\gamma \varrho\, ds \geqq \int_0^\pi (r\sin\varphi/(\pi r\sin\varphi))\, d\psi = 1$$

for all $\gamma\in\Gamma'$, because $ds\geqq r\sin\varphi\, d\psi$ for every curve element. Hence

$$M(\Gamma')\leqq\int_{R^3}\varrho^3\, dV=\int_E \frac{dV}{\pi^3 r^3\sin^3\varphi}=\frac{1}{\pi^3}\int_{\underline{r}_1}^{\bar{r}_2} dr\int_{E\cap S^2(r)}\frac{dA}{r^3\sin^3\varphi}.$$

Denote by $C_r(\beta, \varphi_0)$ the subset

$$\{(r, \psi, \varphi)\,|\,0\leqq\psi\leqq\pi,\ \Phi\leqq\varphi\leqq\Phi'\}$$

of $S^2(r)$, where

$$\Phi=\min_{(e, e_3)=\beta}(k(e), e_3)$$

and Φ' is defined such that the area of $C_r(\beta, \varphi_0)$ is equal to $m_2(E\cap S^2(r))$. There Φ and Φ' are independent of r, since g maps each ray $\{te\,|\,t>0\}$, $(e, e_3)=\beta$, onto a ray. From the inequalities

$$\int_{G(r)}\frac{dA}{\sin^3\varphi}\leqq\int_{G(r)}\frac{dA}{\sin^3\Phi'}=\int_{H(r)}\frac{dA}{\sin^3\Phi'}\leqq\int_{H(r)}\frac{dA}{\sin^3\varphi},$$

where

$$G(r)=E\cap S^2(r)\setminus C_r(\beta, \varphi_0),$$

$$H(r)=C_r(\beta, \varphi_0)\setminus E\cap S^2(r),$$

it follows that

$$(2)\qquad M(\Gamma')\leqq\frac{1}{\pi^3}\int_{\underline{r}_1}^{\bar{r}_2} dr\int_{C_r(\beta, \varphi_0)}\frac{dA}{r^3\sin^3\varphi}=\frac{1}{\pi^3}\int_{\underline{r}_1}^{\bar{r}_2}\frac{dr}{r}\int_0^\pi d\psi\int_\Phi^{\Phi'}\frac{d\varphi}{\sin^2\varphi}$$

$$=\frac{1}{\pi^2}\cdot\frac{\sin(\Phi'-\Phi)}{\sin\Phi\sin\Phi'}\log(\bar{r}_2/\underline{r}_1)\leqq\frac{1}{\pi^2\sin\Phi}\log(\bar{r}_2/\underline{r}_1).$$

To obtain a similar estimate for $M(\Gamma_1')$, we denote

$$D_\alpha(\beta)=\{(r, \psi, \varphi)\,|\,r>0,\ 0\leqq\psi\leqq\alpha,\ 0\leqq\varphi\leqq\beta\},$$

$$A_r(\beta)=(gD_\alpha(\beta))\cap S^2(r).$$

Let r_1 be so small that $\bar{r}_1<\underline{r}_2$. Choose

$$\varrho(r, \psi, \varphi)=\begin{cases}1/(r\log(\underline{r}_2/\bar{r}_1)) & \text{for}\quad (r, \psi, \varphi)\in gD_\alpha(\beta)\cap(\overline{B^3(\underline{r}_2)}\setminus B^3(\bar{r}_1))\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\int_\gamma \varrho\, ds \geqq \int_{\bar r_1}^{\underline{r}_2} \frac{dr}{r \log(\underline{r}_2/\bar r_1)} = 1$$

for every $\gamma \in \Gamma_1'$, i.e. $\varrho \in F(\Gamma_1')$, and consequently

$$M(\Gamma_1') \leqq \int_{R^3} \varrho^3\, dV = \int_{\bar r_1}^{\underline{r}_2} dr \int_{A_r(\beta)} \frac{dA}{r^3 \log^3(\underline{r}_2/\bar r_1)} = (1/\log^3(\underline{r}_2/\bar r_1)) \int_{\bar r_1}^{\underline{r}_2} \frac{m_2(A_r(\beta))}{r^2}\frac{dr}{r}.$$

If we denote

$$\bar\Phi = \max_{x \in A_r(\beta)} (x, e_3) = \max_{e \in D_\alpha(\beta)} (k(e), e_3),$$

then

(3) $$M(\Gamma_1') \leqq (\log(\underline{r}_2/\bar r_1))^{-3} \int_{\bar r_1}^{\underline{r}_2} \pi(1-\cos\bar\Phi)\, dr/r = 2\pi(\sin(\bar\Phi/2))^2(\log(\underline{r}_2/\bar r_1))^{-2}.$$

5. Proof of Theorem 1

Keeping r_1 and r_2 fixed we let now $\beta \to 0$. By the condition B, the estimates (2) and (3) imply

(4) $$\overline{\lim_{\beta\to 0}}\, M(\Gamma')^2 M(\Gamma_1') \leqq \frac{1}{2\pi^3}(\log(\bar r_2/\underline{r}_1)/\log(\underline{r}_2/\bar r_1))^2.$$

Since the extended mapping $\hat g\colon R^3 \to R^3$ is quasiconformal, $\overline{\lim}_{r_1\to 0}(\bar r_1/\underline{r}_1)$ is finite and with a fixed r_2

$$\lim_{r_1\to 0}(\log(\bar r_2/\underline{r}_1)/\log(\underline{r}_2/\bar r_1)) = 1.$$

The above estimate (4) is valid for every pair $r_1, r_2 > 0$ with small r_1, and letting $r_1 \to 0$ with fixed r_2 we obtain

$$\overline{\lim_{r_1\to 0}}\,(\overline{\lim_{\beta\to 0}}\, M(\Gamma')^2 M(\Gamma_1')) \leqq 1/2\pi^3.$$

Thus the inequalities $M(\Gamma) \leqq K_O(g) M(\Gamma')$ and $M(\Gamma_1) \leqq K_O(g) M(\Gamma_1')$ imply

$$K_O(g) \geqq \pi/\alpha.$$

Because $\varepsilon > 0$ in (1) is arbitrary, it follows

$$K_O(D_\alpha, D_\pi;\, W) \geqq \pi/\alpha.$$

The example $f \in W, f(r, \varphi, z) = (r, \pi\varphi/\alpha, \pi z/\alpha)$, given in cylindrical coordinates, shows that

$$K_O(D_\alpha, D_\pi;\, W) \leqq \pi/\alpha.$$

Thus the theorem is proved.

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