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THE COEFFICIENTS OF QUASICONFORMALITY OF CONES IN n-SPACE

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1. Introduction. In this paper we extend some results of Gehring and Väisälä [6] to n-space. The outer coefficient of a cylinder and that of a convex cone have been obtained by them in 3-space. We show that their methods can be modified to obtain outer coefficients of increasing convex as well as nonconvex cones and include cylinder as a limiting case of a convex cone. The problems of characterizing domains with finite coefficients and that of determining these coefficients are rather complicated in spaces of dimension greater than 2. Some results in this direction have been obtained in [1], [2], [3], and [6].

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2. Notation. We refer to [10] for all definitions and notations not explicitly stated here.

For each positive integer p, we let Ω_p denote the p-dimensional Lebesgue measure of B^p and ω_{p-1} denote the $(p-1)$-dimensional Lebesgue measure of S^{p-1}. We observe that $\omega_{p-1}=p\Omega_p=2\pi^{p/2}/\Gamma_{p/2}$, where Γ is the classical Gamma function.

We let (r, θ, x_n) and (t, θ, φ) denote the cylindrical and spherical coordinates of $x=\sum_{i=1}^n x_i e_i$ in R^n. Here,

$$\theta = (\theta_1, \theta_2, \ldots, \theta_{n-2}), \quad r \geqq 0, \quad t \geqq 0, \quad 0 \leqq \varphi, \quad \theta_i \leqq \pi,$$

$1 \leqq i \leqq n-3$ and $0 \leqq \theta_{n-2} < 2\pi$. These coordinates are related by the formulas: $x_n = t\cos\varphi$, $r = t\sin\varphi$, $x_1 = r\cos\theta_1$, $x_2 = r\sin\theta_1\cos\theta_2$, $x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \ldots,$ $x_{n-2} = r\sin\theta_1\sin\theta_2\ldots\cos\theta_{n-2}$, $x_{n-1} = r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}$. A domain D in $\bar{R}^n$ is called a cone of angle α, $0<\alpha<\pi$, if D can be mapped conformally onto

$$C_\alpha = \{(t, \theta, \varphi)\colon 0 \leqq \varphi < \alpha\}, \tag{1}$$

while D is called a cylinder (cone of angle 0) if it can be conformally mapped

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onto

$$C_0 = \{(r, \theta, x_n)\colon\ r < 1\}. \tag{2}$$

If $0 \leqq \alpha \leqq \pi/2$, the cone is convex, while if $\pi/2 < \alpha < \pi$, it is nonconvex.

The inner, outer, and maximal coefficients of the ordered pair (D, D') of domains in $\bar{R}^n$, are defined as

$$\begin{aligned} K_I(D, D') &= \inf K_I(f), \quad K_O(D, D') = \inf K_O(f), \\ K(D, D') &= \inf K(f), \end{aligned} \tag{3}$$

where the infima are taken over all homeomorphisms f of D onto D'. It follows from (3) that

$$\begin{gathered} 1 \leqq K_I(D, D'), \quad K_O(D, D') \leqq \infty, \\ K_I(D, D') = K_O(D', D), \\ K_I(D, D') \leqq K_O^{n-1}(D, D'), \end{gathered} \tag{4}$$

and that the coefficients are finite if and only if there exists a quasiconformal mapping from D onto D'. In this case we say that D and D' are quasiconformally equivalent.

The following notation is used throughout the paper:

$$\begin{gathered} q(\varphi) = \int_0^{\varphi} (\sin u)^{\frac{2-n}{n-1}}\, du, \quad 0 \leqq \varphi \leqq \pi, \\ 0 < \alpha < \beta < \pi, \quad \text{given constants}, \\ c = q(\beta)/q(\alpha), \quad q(\varphi') = cq(\varphi), \quad \text{for} \quad 0 \leqq \varphi \leqq \alpha, \quad \text{and} \\ s(\varphi) = \frac{\sin \varphi'}{\sin \varphi} \quad \text{for} \quad 0 < \varphi \leqq \alpha, \quad s(0) = c^{n-1}. \end{gathered} \tag{5}$$

3. The results.

Lemma 1. *Given $0 < a < 1$ and $b > -1$, let f, g, h be functions on* $[0, \pi]$ *defined by*

$$f(t) = \int_0^t (\sin u)^b\, du,$$

$$g(t) = f^{-1}(af(t)),$$

and

$$h(t) = \frac{\sin t}{\sin g(t)} \quad \textit{for} \quad t \neq 0,$$

$h(0) = a^{-1/(b+1)}$. *Then h is continuous and decreasing.*

Proof. Continuity follows from L'Hospital's rule applied to the $(b+1)$-th power of $h(t)$ as $t \to 0$. Next for monotonicity, by differentiating $h(t)$ and simplifying, we get

$$h'(t) = \frac{\cos t \cos g(t)}{(\sin g(t))^{b+2}} G(t),$$

where $G(t)=(\sin g(t))^{b+1}\sec g(t)-a(\sin t)^{b+1}\sec t$, and $G'(t)=a(\sin t)^b(\tan^2 g(t)-$ $-\tan^2 t)$. Now, on $(0, \pi/2)$, since $g(t)<t$, it follows that G', G and h' are all negative. Next, on $[\pi/2, g^{-1}(\pi/2)]$, h is clearly decreasing. Finally, on $(g^{-1}(\pi/2), \pi)$, G' is positive, whence G and h' are negative. □

Theorem 1. *Given $0\leqq\alpha<\beta<\pi$, let D, D' be cones of angles α and β, respectively. Then*

$$K_O(D, D') \leqq \left(\frac{q(\beta)}{q(\alpha)}\right)^{n-2} \left(\frac{\sin\alpha}{\sin\beta}\right)^{\frac{n-2}{n-1}}, \tag{6}$$

where for $\alpha=0$, the right side is replaced by its limit as $\alpha\to 0$, that is,

$$q(\beta)^{n-2}\big((n-1)(\sin\beta)^{1/(n-1)}\big)^{2-n}. \tag{7}$$

Proof. We may assume that $D=C_\alpha$ and $D'=C_\beta$ as in (1). First let $\alpha>0$. Let $f_\alpha^\beta=f\colon C_\alpha\to C_\beta$ be defined by

$$\begin{gathered}(t', \theta, \varphi') = f(t, \theta, \varphi), \quad \text{where}\\ q(\varphi') = cq(\varphi), \quad \log t' = c\big(s(\alpha)\big)^{\frac{n-2}{n-1}}\log t,\end{gathered} \tag{8}$$

c, s as in (5). Then f is a diffeomorphism whose stretchings at a point (t, θ, φ) are proportional to

$$c\big(s(\alpha)\big)^{\frac{n-2}{n-1}}, \quad c\big(s(\varphi)\big)^{\frac{n-2}{n-1}} \quad \text{and} \quad s(\varphi),$$

where $s(\varphi)$ occurs $(n-2)$ times. From Lemma 1 it follows that the maximum of these stretchings is $c\big(s(\varphi)\big)^{(n-2)/(n-1)}$, whence

$$K_O(f) = c^{n-2}\big(s(\alpha)\big)^{\frac{2-n}{n-1}}, \tag{9}$$

and (6) follows for $\alpha>0$.

Next for $\alpha=0$ we use a limiting argument as follows. For each $j\in N$, let $f_{\beta/j}^\beta=f_j\colon C_{\beta/j}\to C_\beta$ be defined as in (8). Let S_j be the radial stretching of R^n given by $S_j(x)=\cot(\beta/j)x$ and T_j the translation $T_j(x)=x-\cot(\beta/j)e_n$. Then the sequence of mappings

$$T_j\circ S_j\circ f_j^{-1}\colon C_\beta \to C_{\beta/j}-\cot(\beta/j)e_n$$

converges c-uniformly ([10]) to a mapping $f^{-1}\colon C_\beta\to C_0$ and

$$K_O(f) = K_I(f^{-1}) \leqq \varliminf_{j\to\infty} K_O(f_j) = q(\beta)^{n-2}\big((n-1)(\sin\beta)^{1/(n-1)}\big)^{2-n},$$

and (6) follows for $\alpha=0$. □

We next proceed to prove that there is indeed equality in (6) for $0 \leqq \alpha < \beta \leqq \pi/2$ and for $\pi/2 \leqq \alpha < \beta < \pi$. For this we need the generalizations of some modulus estimates in [6] for curve families in a cylinder and in a cone.

Lemma 2. *Let $2 \leqq p \leqq n-1$ and let $\Gamma = \Gamma_P$ be the family of curves in $B^p(x, 1)$ joining its boundary $S^{p-1}(x, 1)$ to a given point $P \in B^p(x, 1)$ and let $\varrho \in F(\Gamma)$. Then*

$$\int_{R^p} \varrho^{p+1} \, dm_p \geqq p^{1-p} \Omega_p. \tag{10}$$

Proof. We may assume $P=0$. For each $y \in S^{p-1}$, let γ_y be the segment joining 0 and $S^{p-1}(x, 1)$ through y. Then Hölder's inequality yields

$$1 \leqq \left(\int_{\gamma_y} \varrho \, ds \right)^{p+1} \leqq \int_0^{l(\gamma_y)} \varrho^{p+1} t^{p-1} \, dt \left(\int_0^{l(\gamma_y)} t^{\frac{1-p}{p}} \, dt \right)^p$$

$$= l(\gamma_y) p^p \int_0^{l(\gamma_y)} \varrho^{p+1} t^{p-1} \, dt,$$

or

$$\int_0^{l(\gamma_y)} \varrho^{p+1} t^{p-1} \, dt \geqq p^{-p} l(\gamma_y)^{-1}.$$

Integrating with respect to y we get

$$\int_{R^p} \varrho^{p+1} \, dm_p \geqq p^{-1} \int_{S^{p-1}} l(\gamma_y)^{-1} \, dm_{p-1}.$$

On the other hand, Hölder's inequality again yields

$$\omega_{p-1}^{p+1} = \left(\int_{S^{p-1}} dm_{p-1} \right)^{p+1} \leqq \int_{S^{p-1}} l(\gamma_y)^{n-1} \, dm_{p-1} \left(\int_{S^{p-1}} l(\gamma_y)^{-1} \, dm_{p-1} \right)^p,$$

or

$$\int_{S^{p-1}} l(\gamma_y)^{-1} \, dm_{p-1} \geqq p \Omega_p.$$

Thus

$$\int_{R^p} \varrho^{p+1} \, dm_p \geqq p^{1-p} \Omega_p. \qquad \square$$

Corollary 1. *Given $0 < a < b$, let C be the finite part of the cylinder C_0, bounded by the planes $x_n = a$ and $x_n = b$, and let E be a connected set in C joining the bases of C. Let Γ be the family of curves in C joining E to the lateral surface of C. Then*

$$M(\Gamma) \geqq \omega_{n-2}(b-a)(n-1)^{1-n}, \tag{11}$$

with equality if E is the segment $\{te_n : a < t < b\}$.

Proof. Choose $\varrho\in F(\Gamma)$. For each $t\in(a,b)$, the plane $x_n=t$ has nonempty intersection with E and meets C in $B^{n-1}(te_n, 1)$. Thus (10) with $p=n-1$ yields

$$\int_{R^n}\varrho^n\,dm_n \geqq \int_a^b dt \int_{x_n=t}\varrho^n\,dm_{n-1} \geqq \omega_{n-2}(b-a)(n-1)^{1-n}.$$

Next if E is the segment $\{te_n\colon a<t<b\}$, the function, $\varrho(x)=r^{(2-n)/(n-1)}(n-1)^{-1}$ for $x=(r,\theta,x_n)\in C$ and $\varrho(x)=0$ for $x\notin C$, is in $F(\Gamma)$ and

$$\int_{R^n}\varrho^n\,dm_n = \omega_{n-2}(b-a)(n-1)^{1-n},$$

thus there is equality in (11). □

The proofs of the next lemma and its corollary are similar to those above and hence omitted ([6], [7]).

Lemma 3. *Given $0<\alpha\leqq\pi/2$, for $t>0$ let $T=C_\alpha\cap S^{n-1}(t)$ and $P\in T$. Let Γ be the family of curves in T joining P and $\bar{T}\cap\partial C_\alpha$. Then $\varrho\in F(\Gamma)$ implies*

$$\int_{S^{n-1}(t)}\varrho^n\,dm_{n-1} \geqq \omega_{n-2}\,q(\alpha)^{1-n}t^{-1}. \tag{12}$$

Corollary 2. *Given $0<\alpha\leqq\pi/2$, $0<a<b$, let C be the part of C_α bounded by $S^{n-1}(a)$ and $S^{n-1}(b)$. Let E be a connected set in C joining the spherical bases of C and let Γ be the family of curves in C joining E to the lateral surface of C. Then*

$$M(\Gamma) \geqq \omega_{n-2}\,q(\alpha)^{1-n}\log\left(\frac{b}{a}\right), \tag{13}$$

with equality if E is the segment $\{te_n\colon a<t<b\}$. Furthermore, the latter result holds for $0<\alpha<\pi$.

Lemma 4. *Suppose that $f\colon \bar{C}_0\setminus\{\infty\}\to\bar{C}_{\pi/2}\setminus\{0,\infty\}$ is a homeomorphism with*

$$\lim_{x_n\to-\infty} f(x) = 0, \qquad \lim_{x_n\to+\infty} f(x) = \infty, \tag{14}$$

and that f is K-quasiconformal in C_0. Then for each $a'>0$, the set $T=f^{-1}(S^{n-1}(a')\cap\bar{C}_{\pi/2})$ lies between two planes $x_n=a_1$ and $x_n=a_2$ where

$$0 \leqq a_2-a_1 \leqq AK^{1/(n-1)}, \quad A=A(n). \tag{15}$$

Proof. Let a_1, a_2 be the minimum and maximum of x_n, where $x\in T$. We may assume that $a_1<a_2$. Let C be the finite part of C_0 bounded by the bases $x_n=a_1$, $x_n=a_2$, let Γ be the family of curves in C joining these bases and let $\Gamma'=f(\Gamma)$. Then ([4], [10])

$$(1/2)H_n(1) \leqq M(\Gamma') \leqq KM(\Gamma) = K\Omega_{n-1}(a_2-a_1)^{1-n},$$

where $H_n(r)$ is the modulus of the Teichmüller ring

$$\bar{R}^n \setminus (C_1 \cup C_2), \quad C_1 = \{te_1 \colon -1 \leqq t \leqq 0\}, \quad C_2 = \{te_1 \colon r \leqq t \leqq \infty\}.$$

Thus (15) follows. □

We next show that equality holds in (6) for increasing convex cones.

Theorem 2. *Given* $0 \leqq \alpha < \beta \leqq \pi/2$, *let* D, D' *be cones of angles* α *and* β, *respectively. Then*

$$K_O(D, D') = \left(\frac{q(\beta)}{q(\alpha)}\right)^{n-2} \cdot \left(\frac{\sin\beta}{\sin\alpha}\right)^{\frac{2-n}{n-1}}. \tag{16}$$

Proof. Case (i): $\alpha = 0$, $\beta = \pi/2$. Let f be any quasiconformal mapping of C_0 onto $C_{\pi/2}$. Then f can be extended to a homeomorphism of $\bar{C}_0 \setminus \{\infty\}$ onto $\bar{C}_{\pi/2} \setminus \{0, \infty\}$ ([10]). Further, by composing with a Möbius transformation, we may assume that

$$\lim_{x_n \to -\infty} f(x) = 0 \quad \text{and} \quad \lim_{x_n \to +\infty} f(x) = \infty.$$

Now choose $0 < a' < b'$ and set $C' = (B^n(b') \setminus \bar{B}^n(a')) \cap C_{\pi/2}$,

$$E' = \{te_n \colon a' < t < b'\}, \quad T' = B^{n-1}(b') \setminus \bar{B}^{n-1}(a'), \quad S' = R^{n-1}.$$

Next let $\Gamma'_1 = \Delta(E', S'; C')$, $\Gamma'_2 = \Delta(S^{n-2}(a'), S^{n-2}(b'); T')$. Then (15) implies that f^{-1} maps $S^{n-1}(a') \cap C_{\pi/2}$ and $S^{n-1}(b') \cap C_{\pi/2}$ into $a_1 \leqq x_n \leqq a_2$ and $b_1 \leqq x_n \leqq b_2$, respectively, where

$$0 \leqq a_2 - a_1, \quad b_2 - b_1 \leqq AK(f)^{1/(n-2)}. \tag{17}$$

By choosing a' small enough we may also assume that $a_2 < b_1$. Then (11) yields

$$\frac{(b_1 - a_2)\omega_{n-2}}{(n-1)^{n-1}} \leqq M(\Gamma_1) \leqq K_O(f) M(\Gamma'_1) = K_O(f) \frac{\omega_{n-2}}{q(\pi/2)^{n-1}} \log(b'/a'),$$

and by the boundary correspondence property of f ([5], [7]) we get

$$\frac{\omega_{n-2}}{(b_2 - a_1)^{n-2}} \leqq M^S(\Gamma_2) \leqq K_O(f) M^{S'}(\Gamma'_2) = K_O(f)\omega_{n-2}(\log(b'/a'))^{2-n}.$$

Thus

$$K_O(f) \geqq \left(\frac{q(\pi/2)}{n-1}\right)^{n-2} \left(\frac{b_1 - a_2}{b_2 - a_1}\right)^{\frac{n-2}{n-1}}.$$

Now letting $a' \to 0$ and $b' \to \infty$ and using (17) it follows that

$$K_O(f) \geqq \left(\frac{q(\pi/2)}{n-1}\right)^{n-2}. \tag{18}$$

Combining this with (6), the result follows.

Case (ii): $0 \leqq \alpha < \beta \leqq \pi/2$. Let f be any quasiconformal mapping of $D = C_\alpha$ onto $D' = C_\beta$. Let $f_0^\alpha \colon C_0 \to C_\alpha$ and $f_\beta^{\pi/2} \colon C_\beta \to C_{\pi/2}$ be the mappings as in Theorem

1. Then $g=f_\beta^{\pi/2}\circ f\circ f_0^\alpha$ is a quasiconformal mapping of C_0 onto $C_{\pi/2}$ and from (9) and (18) it follows that

$$K_O(f)\geqq\left(\frac{q(\beta)}{q(\alpha)}\right)^{n-2}\left(\frac{\sin\alpha}{\sin\beta}\right)^{\frac{n-2}{n-1}},$$

which together with (6) yields (16). □

The next two lemmas will be needed in extending (16) for increasing non-convex cones. We omit their proofs since they are similar to those of the previous lemmas (see [6], [7]).

Lemma 5. *Given $0<\beta<\pi$, $0<a<b$, let Γ be the family of curves in $S=\partial C_\beta\cap B^n(b)\setminus\bar{B}^n(a)$ joining its boundary spheres. Then*

$$M^S(\Gamma)=\omega_{n-2}\left(\frac{\sin\beta}{\log(b/a)}\right)^{n-2}. \tag{19}$$

Lemma 6. *Given $0<\beta<\pi$, let $f\colon\bar{C}_{\pi/2}\to\bar{C}_\beta$ be a homeomorphism, $f(0)=0$, $f(\infty)=\infty$ and let f be K-quasiconformal in $C_{\pi/2}$. Then for each $a'>0$, the set $f^{-1}(S^{n-1}(a')\cap\bar{C}_\beta)$ lies in $\bar{B}^n(a_2)\setminus B^n(a_1)$, where*

$$1\leqq a_2/a_1\leqq A,\quad A=A(n,\beta,K).$$

Theorem 3. *Given $\pi/2\leqq a<\beta<\pi$, let D, D' be cones of angles α,β, respectively. Then*

$$K_O(D,D')=\left(\frac{q(\beta)}{q(\alpha)}\right)^{n-2}\left(\frac{\sin\alpha}{\sin\beta}\right)^{\frac{n-2}{n-1}}. \tag{20}$$

Proof. As in the proof of Theorem 2 we consider two cases.
Case (i): Let $\alpha=\pi/2$. We may assume that $D=C_{\pi/2}$, $D'=C_\beta$. Let f be any quasiconformal mapping of D onto D'. Then f can be extended to a homeomorphism of $\bar{D}$ onto $\bar{D}'$ ([10]) with $f(0)=0$, $f(\infty)=\infty$. Now choose $0<a'<b'$ and set $C'=D'\cap B^n(b')\setminus\bar{B}^n(a')$, $E'=\{te_n\colon a'<t<b'\}$, $T'=\partial D'\cap B^n(b')\setminus\bar{B}^n(a')$, $S'=\partial D'$, $\Gamma_1'=\Delta(E',S';C')$, $\Gamma_2'=\Delta(S^{n-1}(a'), S^{n-1}(b'); T')$. Next f^{-1} maps $\bar{D}'\cap S^{n-1}(a')$ and $\bar{D}'\cap S^{n-1}(b')$ into $\bar{B}^n(a_2)\setminus B^n(a_1)$ and $\bar{B}^n(b_2)\setminus B^n(b_1)$, respectively, where

$$1\leqq a_2/a_1,\quad b_2/b_1\leqq A. \tag{21}$$

By choosing a' small enough, we may assume that $a_2<b_1$. Then Corollary 2 yields

$$\frac{\omega_{n-2}}{q(\pi/2)^{n-2}}\log\left(\frac{b_1}{a_2}\right)\leqq M(\Gamma_1)\leqq K_O(f)M(\Gamma_1')=K_O(f)\frac{\omega_{n-2}}{q(\beta)^{n-1}}\log\left(\frac{b'}{a'}\right),$$

and by the boundary correspondence property of f ([5], [7]) we get

$$\frac{\omega_{n-2}}{(\log(b_2/a_1))^{n-2}}\leqq M^S(\Gamma_2)\leqq K_O(f)M^{S'}(\Gamma_1')=K_O(f)\,\omega_{n-2}\left(\frac{\sin\beta}{\log(b'/a')}\right)^{n-2}.$$

Thus combining the above two inequalities, letting $a'\to 0$ and using (21), we get

$$K_O(f) \geqq \left(\frac{q(\beta)}{q(\pi/2)}\right)^{n-2} (\sin\beta)^{\frac{2-n}{n-1}}. \tag{22}$$

Hence (6) and (22) yield (20) for the case $\alpha=\pi/2$.

Case (ii): Let $\pi/2<\alpha<\beta<\pi$. As before we may assume that $D=C_\alpha$, $D'=C_\beta$. Let f be any quasiconformal mapping of D onto D' and let $f^\alpha_{\pi/2}\colon C_{\pi/2}\to C_\alpha$, be the mapping as in Theorem 1. Then $g=f^\alpha_{\pi/2}\circ f$ is a quasiconformal mapping of $C_{\pi/2}$ onto C_β, whence from (9) and (22), it follows that

$$K_O(f) \geqq \left(\frac{q(\beta)}{q(\alpha)}\right)^{n-2} \left(\frac{\sin\alpha}{\sin\beta}\right)^{\frac{n-2}{n-1}},$$

which together with (6) yields (20). □

Remark. If $0\leqq\alpha<\beta\leqq\pi/2$ or $\pi/2\leqq\alpha<\beta<\pi$, then (9), (16) and (20), imply that the mapping $f^\beta_\alpha\colon C_\alpha\to C_\beta$, is extremal for the outer coefficient $K_O(C_\alpha, C_\beta)$. For $\alpha<\pi/2<\beta$, the problem is still open.

Given a domain D in $\bar{R}^n$, a point $P\in\partial D$ is said to be a cone point for D of angle α, $0<\alpha<\pi$, if there exists a neighborhood V of P and a cone G of vertex P, angle α, such that $V\cap D=V\cap G$. Theorems 2 and 3 together with the fact that a cone is ray like at its vertex yield sharp lower bounds for outer dilatation of mappings of a class of domains. This result is analogous to Theorem 9 in [6] and Theorem 40.3 in [10].

Theorem 4. *Let D, D' be domains in $\bar{R}^n$ which have cone points P, Q of angles α, β, respectively, where $0<\alpha<\beta<\pi$. Let f be a homeomorphism of D onto D' such that Q is a cluster point of f at P. Then*

$$K_O(f) \geqq \left(\frac{q(\beta)}{q(\alpha)}\right)^{n-2} \left(\frac{\sin\alpha}{\sin\beta}\right)^{\frac{n-2}{n-1}}, \tag{23}$$

and the bound is sharp.

In the above discussion we have only considered the outer coefficient for increasing cones. In view of (4) we get analogous results for the inner coefficient for decreasing cones. However, the problem of determining the inner coefficients for increasing cones is still open. Of course, rough upper and lower bounds for this case can be obtained by obvious n-dimensional analogues of Theorem 9.2 in [6] and Theorem 3.2 in [11].

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