Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 3, 1977, 311-316

A SPECIAL CASE OF SCHÖNFLIES THEOREM FOR QUASICONFORMAL MAPPINGS IN n-SPACE

DAVID B. GAULD and M. K. VAMANAMURTHY

1. Introduction. Gehring [5] first proved extension theorems for quasiconformal mappings in n-space using the ideas of Mazur [7] and Morse [8]. We have recently [4] given an alternative proof of Gehring's Theorem 1 [5] using the ideas of Brown [2]. Näätänen [9] has obtained estimates for the dilatation of Gehring's extension under an additional hypothesis. In this note we construct an extension for this special case, obtain estimates for the dilatation of the extended mapping and finally apply this to the case of an important plane mapping as in [9].

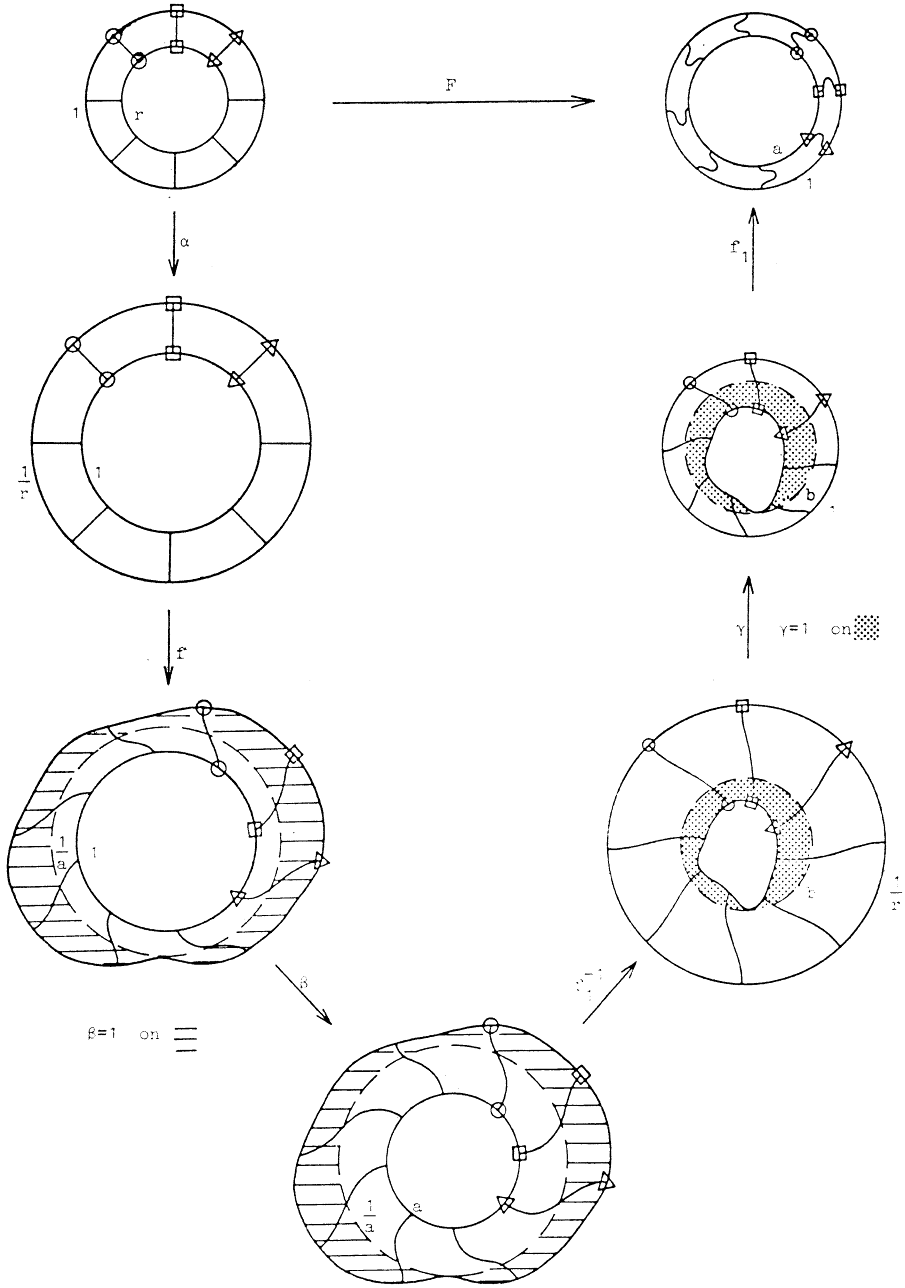
We thank Professor Gehring for helpful discussions. The work was completed during the second author's visit to the University of Michigan while on leave from the University of Auckland. Thanks are due to both the Universities.

2. Notation. We let $\boldsymbol{R}^n$, $\overline{\boldsymbol{R}}^n=\boldsymbol{R}^n\cup\{\infty\}$, denote, respectively, the n-dimensional $(n\geqq 2)$ euclidean and Möbius spaces. For $x=(x_1, x_2, \ldots, x_n)=\sum_{j=1}^n x_j e_j \in \boldsymbol{R}^n$, we let $|x|$ denote the usual euclidean norm. In $\boldsymbol{R}^2$ we also use the complex notation $z=x_1+ix_2$, $i=\sqrt{-1}$. For $E\subset\overline{\boldsymbol{R}}^n$ we let $\overline{E}$, ∂E, $\operatorname{int} E$, denote, respectively, the closure, boundary, and interior of E, with respect to $\overline{\boldsymbol{R}}^n$. For $a\in\boldsymbol{R}^n$, $r>0$, $B^n(a;r)$ is the open ball with center a, radius r and $S^{n-1}(a;r)=\partial B^n(a;r)$. In particular, $B^n(r)=B^n(0;r)$, $B^n=B^n(1)$, $S^{n-1}(r)=S^{n-1}(0;r)$, $S^{n-1}=S^{n-1}(1)$. For $p=0, 1, 2, \ldots$, we let ω_p denote the p-dimensional Lebesgue measure of S^p. We observe that $\omega_0=2$, $\omega_1=2\pi$, $\omega_2=4\pi, \ldots$ and in general, $\omega_p= =2\pi^{(p+1)/2}/\Gamma((p+1)/2)$, where Γ denotes the classical Gamma function. For other definitions and notations we refer to [11].

3. Results.

Theorem 1. *Let $\overline{B}_2\subset B_1$, $\overline{D}_2\subset D_1$, where B_1, B_2, D_2 are balls and D_1, a Jordan domain in $\overline{\boldsymbol{R}}^n$. Let $B=B_1-\overline{B}_2$, $A=D_1-\overline{D}_2$ and $f\colon \overline{B}\to\overline{A}$, a homeomorphism such that $f(\partial B_i)=\partial D_i$, $i=1, 2$. Then there exists a homeomorphism F of $\overline{B}_1$ onto $\overline{D}_1$ such that $F(x)=f(x)$, for all x in $\overline{B}$. Further, if f is quasiconformal, then so is F.*

doi:10.5186/aasfm.1977.0307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Proof. By performing preliminary Möbius transformations we may assume that $B_1=B^n(1/r)$, $0<r<1$, and $B_2=D_2=B^n$. We let $F(x)=f(x)$, for x in $\bar{B}_1-B_2$. Next, to define F on B_2, we use a construction (see the figure) similar to that of Lemma 9 in [4]. Let f_1 be the reflection of f in ∂B_2, that is

$$f_1(x)=\begin{cases} f(x) & \text{for } \ 1\leqq |x|\leqq 1/r,\\ f(x|x|^{-2})\left|f(x|x|^{-2})\right|^{-2} & \text{for } \ r\leqq |x|\leqq 1.\end{cases}$$

Let

$$a=\max\{|f_1(x)|:\ |x|=r\}$$
$$b=\max\{|f_1^{-1}(x)|:\ |x|=a\}.$$

Let α, β and γ be radial mappings of $\boldsymbol{R}^n$ defined as follows:

$$\alpha(x)=x/r,$$

$$\beta(x)=\begin{cases} ax|x| & \text{for } \ |x|\leqq 1/a,\\ x & \text{for } \ |x|\geqq 1/a,\end{cases}$$

$$\gamma(x)=\begin{cases} r^{\frac{\log b}{\log(rb)}}|x|^{-\frac{\log r}{\log(rb)}}x & \text{for } \ |x|\geqq b,\\ x & \text{for } \ |x|\leqq b.\end{cases}$$

Define the mapping F on $\bar{B}_2-B^n(r)$ by

$$F=f_1\circ\gamma\circ f_1^{-1}\circ\beta\circ f\circ\alpha.$$

Then $F(x)=f(x)$ for $x\in\partial B_2$ and $F(S^{n-1}(r))=S^{n-1}(a)$.

Finally, the mapping F on $\bar{B}^n(r)$ is obtained by reflection. Moreover, if f is quasiconformal, so is F, since all the auxiliary mappings are quasiconformal. □

In the next theorem we obtain estimates for the maximal dilatation of the above extension F.

Theorem 2. *Let the mappings f and F be as in Theorem 1 and let $K(f)=K$. Then $K(F)=K^*$ satisfies*

(1) $$K\leqq K^*\leqq K^3(2M)^{n-1}C^{2^{1-n}C^{M^{1-n}}},$$

where

$$C=C(n,K)=\exp\left((n-1)K\frac{\omega_{n-1}}{\omega_{n-2}}q^{n-1}\right),$$

$$M=\operatorname{mod} B=\log\frac{1}{r},\quad \textit{and}\quad q=\int_0^{\pi/2}(\sin t)^{\frac{2-n}{n-1}}\,dt.$$

Moreover, for large values of M, (i.e. $r\to 0$), we have

(2) $$K^*\leqq 4^{n-1}K^5\left(\frac{M}{M-(K^{2/(n-1)}+K^{1/(n-1)})\log\lambda}\right)^{n-1},$$

where $\lambda=\lambda(n)$ is a positive constant.

Proof. Let the notation be as in Theorem 1. Since F extends f, clearly $K \leqq K^*$. Next, $K(f_1)=K(f)=K$, $K(\alpha)=1$, $K(\beta)=2^{n-1}$, and $K(\gamma)=(\log(rb)/\log b)^{n-1}$. Thus

$$K \leqq K^* \leqq 2^{n-1}K^3(\log(rb)/\log b)^{n-1} \leqq 2^{2(n-1)}K^3M^{n-1}(\log(1/b))^{1-n}. \tag{3}$$

The result follows once we eliminate b in the above inequality. For this we show that

$$\log\frac{1}{b} \geqq 2\exp(-KC_1 \cdot 2^{1-n}\exp((n-1)KC_1M^{1-n})), \tag{4}$$

and

$$\log\frac{1}{b} \geqq MK^{\frac{2}{1-n}} - (1+K^{\frac{1}{1-n}})\log\lambda, \tag{5}$$

where $C_1=C_1(n)$ and $\lambda=\lambda(n)$ are positive constants. We observe that (5) is non-trivial only when M is so large that the right side of (5) is positive.

By the extremal property of the Grötzsch ring [3] we get

$$\log\frac{1}{a} \leqq K^{\frac{1}{n-1}} \operatorname{mod} R_G(b), \tag{6}$$

where $R_G(b)$ is the ring

$$B^n - \{te_1 \colon 0 \leqq t \leqq b\}.$$

Next by Lemma 9.9 in [3] and Corollary 1 in [1] we get

$$\log\frac{1}{a} \leqq K^{\frac{1}{n-1}} \min\left[\log\frac{\lambda}{b}, \left(\frac{C_1}{\log((1+b)/(1-b))}\right)^{\frac{1}{n-1}}\right], \tag{7}$$

where $\lambda = \lim_{t\to 0} t\exp(\operatorname{mod} R_G(t))$ and $C_1=(\omega_{n-1}/\omega_{n-2})q^{n-1}$.

Similarly

$$M = \log\frac{1}{r} \leqq K^{\frac{1}{n-1}} \min\left[\log\frac{\lambda}{a}, \left(\frac{C_1}{\log((1+a)/(1-a))}\right)^{\frac{1}{n-1}}\right]. \tag{8}$$

From (7),

$$\log\frac{1+b}{1-b} \leqq \frac{KC_1}{(\log(1/a))^{n-1}},$$

hence

$$\log\frac{1}{b} \geqq \log\coth\frac{KC_1}{2(\log(1/a))^{n-1}} \geqq 2\exp\left(-\frac{KC_1}{(\log(1/a))^{n-1}}\right)$$

since $\log\coth x \geqq 2\exp(-2x)$ for $x>0$. Similarly from (8) we have

$$\log\frac{1}{a} \geqq 2\exp\left(-\frac{KC_1}{M^{n-1}}\right),$$

so that

$$\log\frac{1}{b} \geqq 2\exp(-KC_1 \cdot 2^{1-n}\exp(KC_1M^{1-n}(n-1))),$$

which is (4).

Next (5) follows from (7) and (8). Now (3) and (4) imply (1) while (3) and (5) yield (2). This completes the proof of Theorem 2.

Remark. The construction of F in the proof of Theorem 1 does not require f to be one to one on $S^{n-1}(1/r)$ provided it extends to a homeomorphism of $f^{-1}(\bar{B}^n(1/a))-B_2$ onto $\bar{B}^n(1/a)-D_2$. Hence under these conditions the construction of F is valid even when $f(\partial B_1)$ is not the boundary of a Jordan domain. In particular, on reflection, the construction applies to the case where $n=2$ and f is the conformal mapping of the plane annulus $r<|z|\leqq 1$ onto the ring $\bar{B}_2$ minus the slit $[-a, a]$ of the real axis [10].

Theorem 3. *Let f be the above plane conformal mapping and F the extension given by the proof of Theorem* 1. *Then the maximal dilatation K^* of F satisfies*

$$K^* \leqq 2MC^{(1/2)C^{1/M}} \quad \textit{for} \quad 0<r<1, \tag{9}$$

and

$$K^* \leqq \frac{4M}{M-\log 16} \quad \textit{for} \quad 0<r<\frac{1}{16} \tag{10}$$

where $M=\log(1/r)$, $C=\exp(\pi^2/2)$.

Proof. Follows from (1) and (2) by putting $n=2$, $K=1$ and $\lambda=4$ (see [6]).

Remark. Given a quasiconformal mapping f as in Theorem 1, let $\hat{K}(f)$ denote the infimum of the dilatations of all possible quasiconformal extensions F of f satisfying Theorem 1. In [9], Näätänen defines the function $\varphi(n, K, M)$ to be the supremum of the numbers $\hat{K}(f)$ as f ranges through all mappings with dilatation $\leqq K$ and shows that it is a decreasing function of M. The estimates in Theorem 2 show that $K\leqq\lim_{M\to\infty}\varphi(n, K, M)\leqq 4^{n-1}K^5$. However, as $M\to 0$, the upper bound for $\varphi(n, K, M)$ in [9] is better than that in our Theorem 2.

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University of Auckland
Department of Mathematics
Auckland
New Zealand

University of Michigan
Department of Mathematics
Ann Arbor, Michigan 48109
USA

University of Auckland
Department of Mathematics
Auckland
New Zealand

Received 7 November 1977