A SPECIAL CASE OF SCHÖNFLIES THEOREM FOR QUASICONFORMAL MAPPINGS IN $n$-SPACE

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1. Introduction. Gehring [5] first proved extension theorems for quasiconformal mappings in $n$-space using the ideas of Mazur [7] and Morse [8]. We have recently [4] given an alternative proof of Gehring's Theorem 1 [5] using the ideas of Brown [2]. Näätänen [9] has obtained estimates for the dilatation of Gehring's extension under an additional hypothesis. In this note we construct an extension for this special case, obtain estimates for the dilatation of the extended mapping and finally apply this to the case of an important plane mapping as in [9].

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2. Notation. We let $\mathbb{R}^n$, $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, denote, respectively, the $n$-dimensional ($n \geq 2$) euclidean and Möbius spaces. For $x = (x_1, x_2, \ldots, x_n) = \sum_{j=1}^n x_j e_j \in \mathbb{R}^n$, we let $|x|$ denote the usual euclidean norm. In $\mathbb{R}^3$ we also use the complex notation $z = x_1 + i x_2$, $i = \sqrt{-1}$. For $E \subset \overline{\mathbb{R}}^n$ we let $\overline{E}$, $\partial E$, int $E$, denote, respectively, the closure, boundary, and interior of $E$, with respect to $\overline{\mathbb{R}}^n$. For $a \in \mathbb{R}^n$, $r > 0$, $B^n(a; r)$ is the open ball with center $a$, radius $r$ and $S^{n-1}(a; r) = \partial B^n(a; r)$. In particular, $B^n(r) = B^n(0; r)$, $B^n = B^n(1)$, $S^{n-1}(r) = S^{n-1}(0; r)$, $S^{n-1} = S^{n-1}(1)$.

For $p = 0, 1, 2, \ldots$, we let $\omega_p$ denote the $p$-dimensional Lebesgue measure of $S^p$. We observe that $\omega_0 = 2$, $\omega_1 = 2\pi$, $\omega_2 = 4\pi$, ... and in general, $\omega_p = 2\pi^{(p+1)/2}/\Gamma((p+1)/2)$, where $\Gamma$ denotes the classical Gamma function. For other definitions and notations we refer to [11].

3. Results.

Theorem 1. Let $\overline{B}_2 \subset B_1$, $\overline{D}_2 \subset D_1$, where $B_1$, $B_2$, $D_2$ are balls and $D_1$, a Jordan domain in $\overline{\mathbb{R}}^n$. Let $B = B_1 - \overline{B}_2$, $A = D_1 - \overline{D}_2$ and $f: \overline{B} \to \overline{A}$, a homeomorphism such that $f(\partial B_i) = \partial D_i$, $i = 1, 2$. Then there exists a homeomorphism $F$ of $\overline{B}_1$ onto $\overline{D}_1$ such that $F(x) = f(x)$, for all $x$ in $\overline{B}_1$. Further, if $f$ is quasiconformal, then so is $F$.

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Proof. By performing preliminary Möbius transformations we may assume that $B_1 = B^n(1/r)$, $0 < r < 1$, and $B_2 = D_2 = B^n$. We let $F(x) = f(x)$, for $x$ in $B_1 - B_2$. Next, to define $F$ on $B_2$, we use a construction (see the figure) similar to that of Lemma 9 in [4]. Let $f_1$ be the reflection of $f$ in $\partial B_2$, that is

$$f_1(x) = \begin{cases} f(x) & \text{for } 1 \leq |x| \leq 1/r, \\ f(x|x|^{-2}) |f(x|x|^{-2})|^{-2} & \text{for } r \leq |x| \leq 1. \end{cases}$$

Let

$$a = \max \{ |f_1(x)| : |x| = r \},$$

$$b = \max \{ |f_1^{-1}(x)| : |x| = a \}.$$

Let $\alpha, \beta$ and $\gamma$ be radial mappings of $R^n$ defined as follows:

$$\alpha(x) = x/r,$$

$$\beta(x) = \begin{cases} ax|x| & \text{for } |x| \leq 1/a, \\ x & \text{for } |x| \geq 1/a, \end{cases}$$

$$\gamma(x) = \begin{cases} r \log|\beta| |x|^{-\log|\beta|} & \text{for } |x| \leq b, \\ x & \text{for } |x| \geq b. \end{cases}$$

Define the mapping $F$ on $\overline{B}_2 - B^n(r)$ by

$$F = f_1 \circ \gamma \circ f_1^{-1} \circ \beta \circ f \circ \alpha.$$ 

Then $F(x) = f(x)$ for $x \in \partial B_2$ and $F(S^n(r)) = S^n(a)$.

Finally, the mapping $F$ on $\overline{B}^n(r)$ is obtained by reflection. Moreover, if $f$ is quasiconformal, so is $F$, since all the auxiliary mappings are quasiconformal. □

In the next theorem we obtain estimates for the maximal dilatation of the above extension $F$.

**Theorem 2.** Let the mappings $f$ and $F$ be as in Theorem 1 and let $K(f) = K$. Then $K(F) = K^*$ satisfies

$$K \equiv K^* \equiv K^3(2M)^{n-1} C^{2-n} C^M^{1-n},$$

where

$$C = C(n, K) = \exp \left( (n-1)K \frac{M}{\log q(n-1)} \right),$$

$$M = \text{mod } B = \log \frac{1}{r}, \text{ and } q = \int_0^{\pi/2} (\sin t)^{2-n} dt.$$ 

Moreover, for large values of $M$, (i.e. $r \to 0$), we have

$$K^* \equiv 4^{n-1} K^5 \left( \frac{M}{M - (K^2/n-1) + K^{1/(n-1)}) \log \lambda} \right)^{n-1},$$

where $\lambda = \lambda(n)$ is a positive constant.
Proof. Let the notation be as in Theorem 1. Since $F$ extends $f$, clearly $K \equiv K^*$. Next, $K(v) = K(f) = K$, $K(\alpha) = 1$, $K(\beta) = 2^{n-1}$, and $K(\gamma) = (\log (rb) / \log b)^{n-1}$. Thus

$$K \leq K^* \leq 2^{n-1} K^* (\log (rb) / \log b)^{n-1} \leq 2^{2(n-1)} K^* M^{n-1} (\log (1/b))^{1-n}.$$ (3)

The result follows once we eliminate $b$ in the above inequality. For this we show that

$$\log \frac{1}{b} \geq 2 \exp (-KC_1 \cdot 2^{1-n} \exp ((n-1)KC_1 M^{1-n})), (4)$$

and

$$\log \frac{1}{b} \geq MK^{1-n} - (1 + K^{1-n}) \log \lambda, (5)$$

where $C_1 = C_1(n)$ and $\lambda = \lambda(n)$ are positive constants. We observe that (5) is non-trivial only when $M$ is so large that the right side of (5) is positive.

By the extremal property of the Grötzsch ring [3] we get

$$\log \frac{1}{a} \equiv K^{n-1} \mod R_G(b), (6)$$

where $R_G(b)$ is the ring $B^a - \{te_1: 0 \leq t \leq b\}$.

Next by Lemma 9.9 in [3] and Corollary 1 in [1] we get

$$\log \frac{1}{a} \leq K^{n-1} \min \left[ \log \frac{\lambda}{b} \cdot \left( \frac{C_1}{\log ((1+b)/(1-b))} \right)^{n-1} \right], (7)$$

where $\lambda = \lim_{t \to 0} t \exp (\mod R_G(t))$ and $C_1 = (\omega_{n-1}/\omega_{n-2}) q^{n-1}$.

Similarly

$$M = \log \frac{1}{r} \leq K^{n-1} \min \left[ \log \frac{\lambda}{a} \cdot \left( \frac{C_1}{\log ((1+a)/(1-a))} \right)^{n-1} \right]. (8)$$

From (7),

$$\log \frac{1+b}{1-b} \equiv \frac{KC_1}{(\log (1/a))^{n-1}},$$

hence

$$\log \frac{1}{b} \equiv \log \coth \frac{KC_1}{2(\log (1/a))^{n-1}} \leq 2 \exp \left( -\frac{KC_1}{(\log (1/a))^{n-1}} \right)$$

since $\log \coth x \equiv 2 \exp (-2x)$ for $x > 0$. Similarly from (8) we have

$$\log \frac{1}{a} \equiv 2 \exp \left( -\frac{KC_1}{M^{n-1}} \right),$$

so that

$$\log \frac{1}{b} \equiv 2 \exp \left( -KC_1 \cdot 2^{1-n} \exp (KC_1 M^{1-n}(n-1)) \right),$$

which is (4).
Next (5) follows from (7) and (8). Now (3) and (4) imply (1) while (3) and (5) yield (2). This completes the proof of Theorem 2.

Remark. The construction of \( F \) in the proof of Theorem 1 does not require \( f \) to be one to one on \( S^{n-1}(1/r) \) provided it extends to a homeomorphism of \( f^{-1}(\overline{B}^n(1/a))-B_2 \) onto \( \overline{B}^n(1/a)-D_2 \). Hence under these conditions the construction of \( F \) is valid even when \( f(\partial B_2) \) is not the boundary of a Jordan domain. In particular, on reflection, the construction applies to the case where \( n=2 \) and \( f \) is the conformal mapping of the plane annulus \( r<|z|\leq 1 \) onto the ring \( \overline{B}_2 \) minus the slit \([−a,a]\) of the real axis [10].

Theorem 3. Let \( f \) be the above plane conformal mapping and \( F \) the extension given by the proof of Theorem 1. Then the maximal dilatation \( K^* \) of \( F \) satisfies

\[
K^* \leq 2MC^{(1/2)c^{4/r}} \quad \text{for} \quad 0 < r < 1, \tag{9}
\]

and

\[
K^* \leq \frac{4M}{M-\log 16} \quad \text{for} \quad 0 < r < \frac{1}{16}, \tag{10}
\]

where \( M = \log (1/r), \ C = \exp (\pi^2/2). \)

Proof. Follows from (1) and (2) by putting \( n=2, \ K=1 \) and \( \lambda=4 \) (see [6]).

Remark. Given a quasiconformal mapping \( f \) as in Theorem 1, let \( \hat{K}(f) \) denote the infimum of the dilatations of all possible quasiconformal extensions \( F \) of \( f \) satisfying Theorem 1. In [9], Näätänen defines the function \( \varphi(n, K, M) \) to be the supremum of the numbers \( \hat{K}(f) \) as \( f \) ranges through all mappings with dilatation \( \leq K \) and shows that it is a decreasing function of \( M \). The estimates in Theorem 2 show that \( K \leq \lim_{M \to \infty} \varphi(n, K, M) \leq 4^{n-1}K^5. \) However, as \( M \to 0, \) the upper bound for \( \varphi(n, K, M) \) in [9] is better than that in our Theorem 2.

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