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THE SPHERICAL DERIVATIVE AND NORMAL FUNCTIONS (II)

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P. Lappan in his paper *The spherical derivative and normal functions* [2] poses three questions concerning the properties of higher spherical derivatives of a meromorphic function. The purpose of this paper is to answer the first two questions; our techniques are of use in establishing the normality or non-normality of a function and its derivatives. Throughout the paper $(f^{(j)})^{\#}(z)$ will denote the j-th spherical derivative of f, i.e.

$$(f^{(j)})^{\#}(z) = |f^{(j+1)}(z)|/(1+|f^{(j)}(z)|^2),$$

where $f^{(j)}(z)$ denotes the j-th ordinary derivative of f.

Lappan's main result extends a result of Yamashita [3] and provides the focal point for his three open questions.

Theorem 1. *Let f be a normal meromorphic function in $|z|<1$. Then for each positive integer n there exists a finite constant $c_n(f)$ such that*

$$\sup_{|z|<1} (1-|z|^2)^n \prod_{j=0}^{n-1} (f^{(j)})^{\#}(z) \leqq c_n(f).$$

The first question posed by Lappan concerns the existence of a normal meromorphic function in $|z|<1$ such that the sequence $c_n(f)$ of Theorem 1 is unbounded. He asserts that it is conceivable that the sequence $c_n(f)$ is bounded for each fixed normal meromorphic function f, but we have been able to show by use of a counterexample that this need not happen. Intuition would suggest that such a counterexample must be a meromorphic function having its poles approaching $|z|=1$, its higher derivatives non-normal, and the sup of $(1-|z|^2)^n \prod_{j=0}^{n-1} (f^{(j)})^{\#}(z)$ approached as $|z|\to 1$. However, we have found functions $f(z)$, one of which we will present as a counterexample, such that $f(z)$ and each of its derivatives are bounded (hence normal) analytic functions in $|z|<1$ and $\prod_{j=0}^{n-1} (f^{(j)})^{\#}(0)$, which is bounded above by $c_n(f)$, is unbounded as $n\to\infty$.

Example 1. Let $\{n_k\}$ be the following inductively defined sequence of integers: $n_1=2$ and for $k>1$ n_k equals the first integer greater than n_{k-1} such that

$$(1) \qquad \frac{n_k!}{(\log n_k)^{n_k}} > n_k 2^{n_k} \prod_{j=1}^{k-1} \frac{n_j!}{(\log n_j)^{n_j}}.$$

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Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$, where $a_n=(n!)^{-1}$ if $n\neq n_k$ for any k and $a_n=(\log n)^{-n}$ if $n=n_k$ for some k. Since $\limsup |a_n|^{1/n}=0$ we see that $f(z)$ is *entire*; hence, $f(z)$ and each of its derivatives is a bounded (thus normal) analytic function in $|z|<1$. A simple computation shows that for sufficiently large values of $n=n_k$ we have

$$\begin{aligned}\prod_{j=0}^{n_k-1} (f^{(j)})^{\#}(0) &= \frac{|f^{(n_k)}(0)|}{1+|f(0)|^2} \prod_{j=1}^{n_k-1} \frac{|f^{(j)}(0)|}{1+|f^{(j)}(0)|^2} \\ &\geqq \frac{|f^{(n_k)}(0)|}{2^{n_k}} \prod_{j=1}^{n_k-1} \frac{1}{|f^{(j)}(0)|} \\ &= \frac{|f^{(n_k)}(0)|}{2^{n_k}} \cdot \frac{1}{|f^{(n_1)}(0) f^{(n_2)}(0) \dots f^{(n_{k-1})}(0)|} \\ &= \frac{n_k!}{(\log n_k)^{n_k}} \cdot \frac{1}{2^{n_k}} \cdot \prod_{j=1}^{k-1} \frac{(\log n_j)^{n_j}}{(n_j)!} > n_k,\end{aligned}$$

which diverges as $n\to\infty$.

Lappan's second question concerns the possibility of identifying the normality of a function or one of its derivatives in terms of the associated sequence of spherical derivatives. In particular, does a partial converse of Theorem 1 hold, that is, if a meromorphic function f has the property that there exists a positive integer n_0 and a finite constant Q such that

$$\sup_{|z|<1} (1-|z|^2)^{n_0} \prod_{k=0}^{n_0-1} (f^{(k)})^{\#}(z) \leqq Q,$$

then must one of the functions $f, f', f'', \dots, f^{(n_0)}$ be normal. The answer is no. Campbell and Piranian [1] have shown that the function

$$F(z) = 2(1-z)e^{\frac{2+z}{1-z}} + e^{\frac{z-1}{z+1}} \tag{2}$$

has the property that it and each of its derivatives and integrals is a non-normal function. (The non-normality of F, F', F'' can be easily verified by computing the radial and tangential limits near $z=\pm 1$). We now prove that

$$\sup_{|z|<1} (1-|z|^2)^2 \prod_{k=0}^{1} (F^{(k)})^{\#}(z) \equiv \sup_{|z|<1} (1-|z|^2)^2 \frac{|F'(z)|}{1+|F(z)|^2} \cdot \frac{|F''(z)|}{1+|F'(z)|^2} \tag{3}$$

is finite. Differentiation yields

$$F'(z) = 2(2+z)(1-z)^{-1} e^{\frac{2+z}{1-z}} + 2(z+1)^{-2} e^{\frac{z-1}{z+1}},$$

$$F''(z) = 18(1-z)^{-3} e^{\frac{2+z}{1-z}} - 4z(z+1)^{-4} e^{\frac{z-1}{z+1}}.$$

An application of the triangle inequality yields

$$\begin{aligned}|F'(z)F''(z)| &\leqq 36\,|2+z|\,|1-z|^{-4} \left|e^{\frac{2+z}{1-z}}\right|^2 + \\ &+ 8\,|z|\,|z+1|^{-6} \left|e^{\frac{z-1}{z+1}}\right|^2 + \left|e^{\frac{2+z}{1-z}}\right| \left|e^{\frac{z-1}{z+1}}\right| |36(z+1)^2 - 8(2+z)z(1-z)^2|\,|z+1|^{-4}\,|1-z|^{-3}.\end{aligned} \tag{4}$$

Therefore by (3) and (4) the quantity $(1-|z|^2)^2 \prod_{k=0}^{1} (F^{(k)})^{\#}(z)$ will have a finite sup in $|z|<1$ if A, B, and C are finite, where

$$A = \sup_{|z|<1} \frac{(1-|z|^2)^2 |e^{\frac{2+z}{1-z}}|^2 |1-z|^{-4}}{(1+|F(z)|^2)(1+|F'(z)|^2)},$$

$$B = \sup_{|z|<1} \frac{(1-|z|^2)^2 |e^{\frac{2+z}{1-z}}|\, |e^{\frac{z-1}{z+1}}|\, |z+1|^{-4} |1-z|^{-3}}{(1+|F(z)|^2)(1+|F'(z)|^2)},$$

$$C = \sup_{|z|<1} \frac{(1-|z|^2)^2 |e^{\frac{z-1}{z+1}}|^2 |z+1|^{-6}}{(1+|F(z)|^2)(1+|F'(z)|^2)}.$$

Let $\{z_n\}$ be a sequence of points in $|z|<1$, such that $A_{f^2}(z_n) \to A$, where

$$A_{f^2}(z) = \frac{(1-|z|^2)^2 |1-z|^{-4} |e^{\frac{2+z}{1-z}}|^2}{(1+|F(z)|^2)(1+|F'(z)|^2)}.$$

By passing to a subsequence we may assume $z_n \to z_0$, $|z_0| \leqq 1$. If $z_0 \neq 1$, then A is clearly finite. If $z_0 = 1$ then by passing to a subsequence we may assume that

$$\left|e^{\frac{2+z_n}{1-z_n}}/(1-z_n)\right| \to \alpha,$$

α a finite or infinite quantity. Since

$$A_{f^2}(z) \leqq \frac{4|e^{\frac{2+z}{1-z}}/(1-z)|^2}{1+|2(2+z)e^{\frac{2+z}{1-z}}/(1-z)+2(z+1)^{-2}e^{\frac{z-1}{z+1}}|^2}$$

we see that $A \leqq 4\alpha^2$ if α is finite and $A_{f^2}(z_n)$ is bounded if $\alpha = \infty$.

An identical argument holds for C upon using the quantity $|e^{(z-1)/(z+1)}/(z+1)^2|$, as also holds for B upon using $e^{(2+z)/(1-z)}/(1-z)$ or $e^{(z-1)/(z+1)}/(z+1)^2$ depending on whether a subsequence of z_n clusters at $+1$ or -1, respectively.

Thus the finiteness of $\sup_{|z|<1} (1-|z|^2)^2 \prod_{k=0}^{1} (F^{(k)})^{\#}(z)$ does not imply the normality of F, F' or F''.

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