A NOTE ON HURWITZ'S ZETA-FUNCTION

R. BALASUBRAMANIAN

The aim of this paper is to prove an asymptotic formula for the mean square of Hurwitz's zeta-function, defined by

\[ \zeta(s; \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} \quad \text{in} \quad \Re s > 1 \]

and its analytic continuation in \( \Re s \geq 1 \). The result is as follows:

Theorem. If \( \zeta_1(s; \alpha) = \zeta(s; \alpha) - 1/\alpha^s \), then for \( t \geq 30 \),

\[ \int_{0}^{1} |\zeta_1(1/2 + it, \alpha)|^2 \, dx = \log t + O(\log \log t). \]

This improves the result of Koksma and Lekkerkerker [2] which states that

\[ \int_{0}^{1} |\zeta_1(1/2 + it, \alpha)|^2 \, dx = O(\log t). \]

It is very likely that the error term \( O(\log \log t) \) of the theorem can be improved. A modification of our proof gives an asymptotic formula for \( \sum_{\chi} |L(s, \chi)|^2 \), which improves a theorem of Gallagher [1] in some range of \( t \). It will form the subject matter of another paper.

We prove the theorem by establishing six lemmas.

Lemma 1. We have

\[ \zeta_1(1/2 + it, \alpha) = \sum_{1 \leq n \leq T} \frac{1}{(n+\alpha)^{1/2 + it}} + O(t^{-1/2}), \]

where \( T \) is the nearest integer to \( t \).

Proof. This is well known.

Lemma 2. We have

\[ \int_0^1 |\zeta_1(1/2 + it, \alpha)|^2 \, d\alpha = \log t + O(J_1 + J_2 + J_3 - J_4 - J_5 - J_6) + O(1), \]

where

\[ J_1 = \sum_{k \geq T^{1/2}} \frac{(T - k + 1)^{1/2}(T + 1)^{1/2} e^{-it \log (T - k + 1) - \log (T + 1)}}{tk}, \]

\[ J_2 = \sum_{T^{1/2} < k \leq T^{1/2} \log^2 T} \frac{(T - k + 1)^{1/2}(T + 1)^{1/2} e^{-it \log (T - k + 1) - \log (T + 1)}}{tk}, \]

\[ J_3 = \sum_{T^{1/2} \log^2 T < k \leq T} \frac{(T - k + 1)^{1/2}(T + 1)^{1/2} e^{-it \log (T - k + 1) - \log (T + 1)}}{tk}, \]

\[ J_4 = \sum_{k = 1}^T \frac{(1 + k)^{1/2} e^{it \log (1 + k)}}{tk}, \]

\[ J_5 = \frac{1}{2} \sum_{k = 1}^T \int_1^{T - k + 1} \frac{v^{-1/2}(v + k)^{1/2} e^{-it \log (v + k)}}{tk} \, dv, \]

and

\[ J_6 = \frac{1}{2} \sum_{k = 1}^T \int_1^{T - k + 1} \frac{(v + k)^{-1/2} v^{1/2} e^{-it \log (v + k)}}{tk} \, dv. \]

Proof. Using Lemma 1, we have

\[ \int_0^1 |\zeta_1(1/2 + it, \alpha)|^2 \, d\alpha = \int_0^1 \sum_{n \leq T} \sum_{m \leq T} \frac{1}{(n + \alpha)^{1/2 + it}} \frac{1}{(m + \alpha)^{1/2 - it}} \, d\alpha + O(1), \]

and the terms corresponding to \( m = n \) give the main term \( \log t \), with an error \( O(1) \).

The terms \( m \neq n \) give

\[ \int_0^1 \sum_{n \leq T} \sum_{m \neq n} \frac{1}{(n + \alpha)^{1/2 + it}} \frac{1}{(m + \alpha)^{1/2 - it}} \, d\alpha \]

\[ = \int_0^1 \sum_{m \leq T} \sum_{n \neq m} \frac{1}{(n + \alpha)^{1/2 + it}} \frac{1}{(m + \alpha)^{1/2 - it}} \, d\alpha \]

\[ + \int_0^1 \sum_{m \leq T} \sum_{n \neq m} \frac{1}{(n + \alpha)^{1/2 + it}} \frac{1}{(m + \alpha)^{1/2 - it}} \, d\alpha. \]
It is sufficient to consider one of the terms. We obtain

\[ \int_0^1 \sum_{m \equiv T} \sum_{n < m} \frac{1}{(n + \alpha)^{1/2} + it} \frac{1}{(m + \alpha)^{1/2} - it} \, d\alpha \]

\[ = \sum_{k=1}^T \sum_{n=1}^{T-k} \int_0^1 \frac{1}{(n + \alpha)^{1/2} + it} \frac{1}{(n + k + \alpha)^{1/2} - it} \, d\alpha \]

\[ = \sum_{k=1}^T \sum_{n=1}^{T-k} \int_{n}^{n+1} \frac{1}{v^{1/2 + it} (v + k)^{1/2} - it} \, dv \]

\[ = \sum_{k=1}^T \int_1^{T-k+1} \frac{1}{v^{1/2} (v + k)^{1/2} e^{-it \log \log (v + k)}} \, dv \]

\[ = -i \sum_{k=1}^T \frac{1}{kt} \left[ \frac{1}{v^{1/2} (v + k)^{1/2} e^{-it \log \log (v + k)}} \right]_1^{T-k+1} \]

\[ + i \sum_{k=1}^T \frac{1}{kt} \int_1^{T-k+1} \left( (1/2) v^{-1/2} (v + k)^{1/2} + (1/2) (v + k)^{-1/2} v^{1/2} \right) e^{-it \log \log (v + k)} \, dv \]

\[ = -i(J_1 + J_2 + J_3 - J_4 - J_5 - J_6). \]

This proves the lemma.

Lemma 3. If \( J_4 \) and \( J_2 \) are as defined in Lemma 2, then,

\[ J_4 = O(1), \quad J_2 = O(\log \log T). \]

**Proof.** Trivial.

Lemma 4. If \( J_5 \) and \( J_6 \) are as defined in Lemma 2, then

\[ J_5 = O(1), \quad J_6 = O(1). \]

**Proof.** The result follows, using Lemma 4.3 (p. 61) of Titchmarsh [3].

Lemma 5. If \( J_1 \) is as defined in Lemma 2, then \( J_1 = O(1) \).

**Proof.** In \( J_1 \), \( \log (T-k+1) - \log (T+1) \) can be replaced by \( -k/(T+1) \) with a small error. Hence

\[ J_1 = \sum_{k \geq T^{1/2}} \frac{(T-k+1)^{1/2} (T+1)^{1/2} e^{i tk/(T+1)}}{tk} + O(1). \]

Since the partial sums of \( \sum_k e^{itk/(T+1)} \) are bounded, the result follows from Abel’s partial summation formula.
Lemma 6. If $J_3$ is as defined in Lemma 2, then

$$J_3 = O(1).$$

Proof. We apply Theorem 5.9 (p. 90) of Titchmarsh [3] to get a good bound for the sums

$$\sum_{X^2 \leq k \leq Y} e^{-it \log (T - k + 1)}$$

where $Y \leq X$; and $T^{1/2} \log^3 T \leq X \leq T/100$, and use Abel's partial summation formula to prove that

$$\sum_{T^{1/2} \log^3 T \leq k \leq T/100} (T - k + 1)^{1/2} (T + 1)^{1/2} e^{-it \log (T - k + 1) - \log (T + 1)}$$

tk

is $O(1)$. We observe that

$$\sum_{T/100 \leq k \leq T} (T - k + 1)^{1/2} (T + 1)^{1/2} e^{-it \log (T - k + 1) - \log (T + 1)}$$

tk

is $O(1)$. This proves the lemma.

The theorem follows from Lemmas 2 to 6.

Acknowledgement. The author wishes to express his thanks to Prof. K. Ramachandra for his encouragement and for checking the manuscript.

References


Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
India

Received 21 February 1978