DENSITIES OF MEASURES ON THE REAL LINE

PERTTI MATTILA

1. Introduction. Suppose that $\mu$ is an outer measure on the real line $R$ such that $\mu(R) > 0$ and all Borel sets are $\mu$ measurable. Let $h: (0, \infty) \to (0, \infty)$ be a non-decreasing function with $\lim_{r \to 0} h(r) = 0$. These assumptions on $\mu$ and $h$ will be made throughout the whole paper. The upper and lower $h$-densities of $\mu$ at $a \in R$ are defined by

$$D(\mu, a) = \limsup_{r \downarrow 0} \frac{\mu[a-r, a+r]}{h(2r)},$$

$$D(\mu, a) = \liminf_{r \downarrow 0} \frac{\mu[a-r, a+r]}{h(2r)}.$$ 

If they are equal, their common value is called the $h$-density of $\mu$ at $a$, and it is denoted by $D(\mu, a)$. We shall also consider one-sided densities of $\mu$. The upper and lower right $h$-densities of $\mu$ are defined by

$$D^+(\mu, a) = \limsup_{r \downarrow 0} \frac{\mu[a, a+r]}{h(r)},$$

$$D^+(\mu, a) = \liminf_{r \downarrow 0} \frac{\mu[a, a+r]}{h(r)}.$$ 

The upper and lower left $h$-densities $D^-(\mu, a)$ and $D^-(\mu, a)$ are defined similarly as the upper and lower limits of the ratios $\mu[a-r, a]/h(r)$. The results of this paper are usually stated and proved for right densities, but their obvious analogues hold for left densities as well.

The main results are Theorems 8 and 11. They state that if $\mu$ satisfies certain homogeneity conditions in terms of $h$-densities, then it is absolutely continuous with respect to the Lebesgue measure $L^1$. More precisely, $\mu$ is absolutely continuous if either $0 < D(\mu, a) < \infty$ for $\mu$ a.e. $a \in R$ or $0 < D^+(\mu, a) = D^+(\mu, a) < \infty$ for $\mu$ a.e. $a \in R$. These results characterize absolutely continuous measures of $R$ through their density properties.

In Corollaries 9 and 13 to Theorems 8 and 11 we obtain results on the densities of measures which are singular with respect to the Lebesgue measure. Similar results for $s$-dimensional Hausdorff measures, $0 < s < 1$, have been proved by Besicovitch in [1] and [2].

2. Remarks. (1) The results of this paper are false if \( \lim_{r \to 0} h(r) = 0 \) as the example where \( \mu \) is a Dirac measure shows.

(2) In the following proofs we shall usually have the situation where some of the densities defined in Introduction is finite \( \mu \) a.e. This always implies that \( \mu \{ a \} = 0 \) for all \( a \in \mathbb{R} \).

3. Lemma. Let \( A \subset \mathbb{R} \). If for every \( a \in A \) there is \( r > 0 \) such that \( (a, a + r) \subset A \), then \( A \) is a Borel set.

Proof. Let \( A_n \) be the set of all \( a \in [-n, n] \cap A \) for which \( \sup \{ r : (a, a + r) \subset A \} > 1/n \). Then \( A = \bigcup_{n=1}^{\infty} A_n \). Define

\[
 b_1 = \sup A_n, \quad a_1 = \inf [b_1 - 1/n, b_1] \cap A_n,
\]

\[
 b_k = \sup (-\infty, b_k - 1/n) \cap A_n, \quad a_k = \inf [b_k - 1/n, b_k] \cap A_n,
\]

for \( k = 2, \ldots, m \), where the process terminates when \( (a, a + r) \subset A \) is an interval with end points \( a_k \) and \( b_k \), and \( A_n = \bigcup_{k=1}^{m} I_k \).

It follows that \( A \) is a Borel set.

4. Theorem. The densities \( \bar{D}(\mu, \cdot), \bar{D}(\mu, \cdot), \bar{D}^+(\mu, \cdot), \bar{D}^-(\mu, \cdot), \bar{D}^+(\mu, \cdot), \bar{D}^-(\mu, \cdot) \) are Borel functions.

Proof. We prove, for example, that \( \bar{D}^+(\mu, \cdot) \) is a Borel function. We first show that given \( 0 < r < \infty, f : a \mapsto \mu[a, a + r] \) is a Borel function. Express the interior of the set \( \{ a : f(a) = \infty \} \) as \( \bigcup_{j=1}^{\infty} I_j \), where \( I_j \)'s are open disjoint intervals and set

\[ A = \mathbb{R} \setminus \bigcup_{j=1}^{\infty} \text{Cl} I_j. \]

Let \( a \in A \) such that \( f(a) < \alpha \). Then, by the definition of \( A \), there is \( b \in (a, a + r) \) such that \( f(b) < \infty \). Hence \( \mu[a, b + r] \leq \alpha + f(b) < \infty \) and

\[ \limsup_{c \downarrow a} f(c) \leq \lim_{c \downarrow a} \mu[a, c + r] = f(a) < \alpha. \]

Therefore we can find \( s > 0 \) such that \( f(c) < \alpha \) for \( c \in (a, a + s) \). By Lemma 3 the set \( \{ a \in A : f(a) < \alpha \} \) is then a Borel set. Hence \( f|_A \) is a Borel function. Since \( f(a) < \infty \) for at most countably many \( a \in \mathbb{R} \setminus A \), \( f|_{\mathbb{R} \setminus A} \) is also a Borel function. Thus \( f \) is a Borel function.

Since \( h \) is non-decreasing, the set \( D \) consisting of all points of discontinuity of \( h \) and of all positive rational numbers is countable. If \( r > 0 \) and \( r \notin D \), then for any \( \varepsilon > 0 \) there is \( s \in D \) such that \( r < s < r + \varepsilon \) and \( \mu[a, a + r]/h(r) \approx \mu[a, a + s]/h(s) + \varepsilon. \) Hence

\[ \bar{D}^+(\mu, a) = \limsup_{r \to 0} \mu[a, a + r]/h(r), \]

from which the assertion follows.
If $E \subset R$ the restriction measure $\mu \restriction E$ is defined by $(\mu \restriction E)(A) = \mu(E \cap A)$ for $A \subset R$.

5. Theorem. If $E \subset R$ is a Borel set and $\bar{D}(\mu, a) < \infty$ for $\mu$ a.e. $a \in E$ or $\bar{D}^+(\mu, a) < \infty$ for $\mu$ a.e. $a \in E$, then

$$D(\mu \restriction (R \setminus E), a) = \bar{D}^+(\mu \restriction (R \setminus E), a) = 0 \text{ for } \mu \text{ a.e. } a \in E.$$  

Proof. We prove the theorem under the assumption $\bar{D}^+(\mu, a) < \infty$ for $\mu$ a.e. $a \in E$. The case $\bar{D}(\mu, a) < \infty$ can be handled similarly. For $n = 1, 2, \ldots$ let

$$E_n = \{a \in E : \mu[a, a+r] \leq nh(r) \text{ for } 0 < r \leq 1/n\}.$$  

Then $\mu(E \setminus \bigcup_{n=1}^{\infty} E_n) = 0$. The assumption $\bar{D}^+(\mu, a) < \infty$ for $\mu$ a.e. $a \in E$ implies that $\mu\{a\} = 0$ for all $a \in E$; therefore $\mu$ almost all of $E_n$ can be covered with countably many open intervals each of finite $\mu$-measure. Let $I$ be one such interval and $F$ a closed subset of $I \cap E_n$. To prove that $D(\mu \restriction (R \setminus E), a) = 0$ for $\mu$ a.e. $a \in E$, it is then sufficient to show that $D(\mu \restriction (R \setminus E), a) = 0$ for $\mu$ a.e. $a \in F$, since any Borel set of finite measure can be approximated from within by a closed subset (see, for example [3, 2.2.2 (1)]).

To do this, let $\varepsilon > 0$ and denote

$$A_\varepsilon = \{a \in F : \bar{D}(\mu \restriction (R \setminus E), a) > \varepsilon\}.$$  

By [3, 2.2.2 (1)] there exists a closed set $C \subset I \setminus E$ such that $\mu((I \setminus E) \setminus C) < \varepsilon^2$. For each $a \in A_\varepsilon$, there is $0 < r(a) < 1/2n$ such that $[a-r(a), a+r(a)] \subset I \setminus C$ and $\mu([a-r(a), a+r(a)] \setminus E) \geq nh(2r(a))$. By Besicovitch covering theorem [3, 2.8.14] we can find a sequence $(a_i, r_i) = (a_i, r(a))$ of such pairs such that $A_\varepsilon \subset \bigcup_{i=1}^{\infty} [a_i-r_i, a_i+r_i]$ and at most $k$ of the intervals $[a_i-r_i, a_i+r_i]$ may have a point in common, where $k$ is an absolute constant. Letting $b_i = \min [a_i-r_i, a_i+r_i] \subset F$, we have

$$\mu([a_i-r_i, a_i+r_i] \cap A_\varepsilon) \leq \mu[b_i, b_i+2r_i] \leq nh(2r_i).$$  

We obtain

$$\mu(A_\varepsilon) \leq \sum_{i=1}^{\infty} \mu([a_i-r_i, a_i+r_i] \cap A_\varepsilon) \leq n \sum_{i=1}^{\infty} h(2r_i) < (n/\varepsilon) \sum_{i=1}^{\infty} \mu([a_i-r_i, a_i+r_i] \setminus E) \leq (kn/\varepsilon) \mu((I \setminus C) \setminus E) < kn\varepsilon,$$

and

$$\mu\{a \in E : \bar{D}^+(\mu \restriction (R \setminus E), a) > 0\} = \lim_{\varepsilon \to 0} \mu(A_\varepsilon) = 0.$$  

To show that $\bar{D}^+(\mu \restriction (R \setminus E), a) = 0$ for $\mu$ a.e. $a \in E$, we may proceed as above, but this time applying the Besicovitch covering theorem to intervals $[a-r(a)/2, a+r(a)/2]$ such that $\mu([a, a+r(a)] \setminus E) > nh(r(a))$. This completes the proof.

6. Corollary. If $E \subset R$ is a Borel set and $\bar{D}(\mu, a) < \infty$ for $\mu$ a.e. $a \in E$ or $\bar{D}^+(\mu, a) < \infty$ for $\mu$ a.e. $a \in E$, then $\bar{D}(\mu \restriction E, a) = \bar{D}(\mu, a), \bar{D}^+(\mu \restriction E, a) = \bar{D}^+(\mu, a), \bar{D}^+(\mu \restriction E, a) = \bar{D}^+(\mu, a), \bar{D}^+(\mu \restriction E, a) = \bar{D}^+(\mu, a)$ for $\mu$ a.e. $a \in E$. 

7. Theorem. \( \overline{D}(\mu, a) \leq \overline{D}^+(\mu, a) = \overline{D}^-(\mu, a) \leq 2 \overline{D}(\mu, a) \) for \( \mu \) a.e. \( a \in R \).

Proof. To prove the inequality \( \overline{D}(\mu, a) \leq \overline{D}^+(\mu, a) \), denote \( E_t = \{a: \overline{D}^+(\mu, a) \leq t\} \) for \( 0 < t < \infty \). Fix \( t \) and let \( \varepsilon > 0 \). For \( n = 1, 2, \ldots \), set

\[
E_{t,n} = \{a \in E_t: \mu[a, a+r] \leq (t+\varepsilon)h(r) \text{ for } 0 < r < 1/n\} \cap [-n, n].
\]

Then \( \mu(E_{t,n}) < \infty \) and \( E_t = \bigcup_{n=1}^{\infty} E_{t,n} \). Let \( F \) be a closed subset of \( E_{t,n} \). By Theorem 5, \( D(\mu \mathbb{L}(R \setminus F), a) = 0 \) for \( \mu \) a.e. \( a \in F \). Take such a point \( a \) and let \( 0 < r_0 < 1/2n \) be such that

\[
\mu([a-r, a+r] \setminus F) \leq \varepsilon h(2r) \text{ for } 0 < r < r_0.
\]

Let \( 0 < r < r_0 \) and \( b = \min [a-r, a] \cap F \). Then

\[
\mu[a-r, a+r] \leq \mu([a-r, a+r] \setminus F) + \mu[b, b+2r] \leq (t+2\varepsilon)h(2r),
\]

whence \( \overline{D}(\mu, a) \leq t+2\varepsilon \). By [3, 2.2.2 (1)] this implies that \( \overline{D}(\mu, a) \leq t+2\varepsilon \) for \( \mu \) a.e. \( a \in E_{t,n} \). Since this holds for all \( \varepsilon > 0 \) and \( n = 1, 2, \ldots \), we obtain

\[
\mu\{a: \overline{D}^+(\mu, a) \leq t, \overline{D}(\mu, a) > t\} = 0
\]

for \( 0 < t < \infty \). Since \( \{a: \overline{D}(\mu, a) > \overline{D}^+(\mu, a)\} \) is the union of the sets

\[
\{a: \overline{D}^+(\mu, a) \leq t, \overline{D}(\mu, a) > t\}
\]

when \( t \) runs through the positive rational numbers, we obtain \( \overline{D}(\mu, a) \leq \overline{D}^+(\mu, a) \) for \( \mu \) a.e. \( a \in R \).

To prove the inequality \( \overline{D}^+(\mu, a) \leq 2\overline{D}(\mu, a) \), denote \( E_t = \{a: \overline{D}^-(\mu, a) \leq t\} \) for \( 0 < t < \infty \). Fix \( t \) and let \( \varepsilon > 0 \). Let \( n \) be a positive integer and \( F \) a closed subset of

\[
E_{t,n} = \{a \in E_t: \mu[a-r, a+r] \leq (t+\varepsilon)h(2r) \text{ for } 0 < r < 1/n\} \cap [-n, n].
\]

Suppose that \( a \in F \) and \( \overline{D}^+(\mu \mathbb{L}(R \setminus F), a) = 0 \). By Theorem 5 this is true for \( \mu \) a.e. \( a \in F \). Then there is \( 0 < r_0 \leq 1/n \) such that \( \mu([a, a+r] \setminus F) < \varepsilon h(r) \) for \( 0 < r < r_0 \). Let \( 0 < r < r_0 \). If there is \( b \in [a+r/2, a+r] \cap F \), then

\[
\mu[a, a+r] \leq \mu[a-r/2, a+r/2] + \mu[b-r/2, b+r/2] \leq 2(t+\varepsilon)h(r).
\]

Otherwise \( [a+r/2, a+r] \subseteq [a, a+r] \setminus F \), and the same inequality follows. Hence \( \overline{D}^+(\mu, a) \leq 2(t+\varepsilon) \). The proof can be completed as in the first part.

To prove the inequality \( \overline{D}^-(\mu, a) \leq \overline{D}^+(\mu, a) \), let

\[
E_{s,t} = \{a: \overline{D}^+(\mu, a) \leq t < s \leq \overline{D}^-(\mu, a)\}
\]

for \( 0 < t < s < \infty \) and let \( 0 < \varepsilon < (s-t)/3 \). Let \( n \) be a positive integer and \( F \) a closed subset of

\[
E_{s,t,n} = \{a \in E_{s,t}: \mu[a, a+r] \leq (t+\varepsilon)h(r) \text{ for } 0 < r < 1/n\} \cap [-n, n].
\]
Densities of measures on the real line

Suppose that $a \in F$ and $\overline{D}^{-}(\mu \mathbb{I}(R \setminus F), a) = 0$, which again holds for $\mu$ a.e. $a \in F$. Then there is $0 < r < 1/n$ such that

$$\mu([a-r, a] \setminus F) < \varepsilon h(r), \ \mu[a-r, a] > (s-\varepsilon) h(r).$$

Let $b = \min [a-r, a] \cap F$. Then

$$(t+\varepsilon) h(r) \equiv \mu[b, a] \equiv \mu[a-r, a] - \mu([a-r, a] \setminus F) > (s-2\varepsilon) h(r),$$

and $s-t < 3\varepsilon$. This contradicts the choice of $\varepsilon$, and it follows that $\mu(F) = 0$. By a similar argument as in the first part of the proof, we obtain $\overline{D}^{-}(\mu, a) = \overline{D}^{+}(\mu, a)$ for $\mu$ a.e. $a \in R$.

The opposite inequality is proved in the same way, and the theorem follows.

We say that $\mu$ is absolutely continuous if $L^1(A) = 0$ implies $\mu(A) = 0$, and that $\mu$ is singular if there is a set $E \subset R$ such that $L^1(E) = 0$ and $\mu(R \setminus E) = 0$.

8. Theorem. If $\overline{D}^{+}(\mu, a) < \infty$ and $\overline{D}^{+}(\mu, a) > 0$ for $\mu$ a.e. $a \in R$, then $\mu$ is absolutely continuous.

Proof. Using [3, 2.2.2 (1)] we find $0 < d < 1$, $0 < r_0 < \infty$ and a closed set $F \subset R$ such that $\mu(F) > 0$ and

$$dh(r) \equiv \mu[a, a+r] \leq h(r)/d \ \text{for} \ 0 < r < r_0, \ a \in F.$$ 

Making $r_0$ smaller if necessary, we use Theorem 5 to obtain $a \in F$ such that

$$\mu([a, a+r] \setminus F) \equiv (d^3/8) h(r) \ \text{for} \ 0 < r < r_0.$$ 

Let $r_i > 0$, $0 < \sum_{i=1}^{k} r_i < s < r_0$. Choose a positive integer $m$ such that $s = m \sum_{i=1}^{k} r_i < 2s$. Then there are points $a_i, j \in F$, $i = 1, \ldots, k$, $j = 1, \ldots, m$, such that

$$[a, a+s] \cap F \subset \bigcup_{i,j} [a_i, a_i + r_i].$$

Then

$$dh(s) \equiv \mu[a, a+s] \equiv \mu([a, a+s] \setminus F) + \sum_{i,j} \mu[a_i, j, a_i + r_i]$$

$$\equiv (d/2) h(s) + (m/d) \sum_{i=1}^{k} h(r_i) < (d/2) h(s) + \left(2s \left[\sum_{i=1}^{k} r_i\right]\right) \sum_{i=1}^{k} h(r_i),$$

and

$$\sum_{i=1}^{k} h(r_i) > (d^3/4) h(s) \sum_{i=1}^{k} r_i/s.$$ 

Take now $0 < r < r_0/4$ and $r_0/2 \leq s < r_0$. Write

$$(a, a+s) \setminus F = \bigcup_{i=1}^{\infty} (a_i, a_i + r_i),$$
where the intervals \((a_i, a_i+r_i)\) are disjoint and \(r_1 \equiv r_2 \equiv \ldots\). Suppose that \(r_1 \equiv r\) and let \(k\) be the largest integer such that \(r_k \equiv r\). Since \(a_i \in F\) for all \(i\), we have

\[
d \sum_{i=1}^{k} h(r_i) \leq \sum_{i=1}^{k} \mu(a_i, a_i+r_i) \leq \mu([a, a+s] \cap F) < (d^3/8)h(s).
\]

Combining this with (1) we get

\[
(d^3/4)h(s) \sum_{i=1}^{k} r_i/s < (d^3/8)h(s)
\]

and

\[
\sum_{i=1}^{k} r_i < s/2.
\]

Define \(b_1 = a\), \(b_j = \min\{b_{j-1}+r, a+s\} \cap F\), \(j = 2, \ldots, n\), where the process stops when \(a+s < b_j + r\) or \([b_j + r, a+s] \cap F = 0\). Then \((a, a+s) \setminus \bigcup_{i=1}^{k} (a_i, a_i+r_i) \subset \bigcup_{i=1}^{n} [b_i, b_i+2r]\), since \(r_i < r\) for \(i > k\). Hence

\[
s/2 \leq L^1((a, a+s) \setminus \bigcup_{i=1}^{k} (a_i, a_i+r_i)) \leq 2nr,
\]

and \(n \equiv s/4r\). This is true also if \(r_1 < r\). Thus we have

\[
h(s)/d \equiv \mu(a, a+s) \equiv \sum_{i=1}^{n} \mu(b_i, b_i+r) \equiv n \, dh(r) \equiv s \, dh(r)/4r,
\]

which gives

\[
h(r) \equiv 4rh(s)/(d^2 s) \equiv (8h(r_0)/(d^2 r_0))r.
\]

Since this holds for all \(0 < r < r_0/4\), the assertion follows from the assumption \(\bar{D}^+(\mu, a) < \infty\) for \(\mu\) a.e. \(a \in R\).

9. Corollary. If \(\mu\) is singular and \(\bar{D}^+(\mu, a) < \infty\) for \(\mu\) a.e. \(a \in R\), then \(D^+(\mu, a) = 0\) for \(\mu\) a.e. \(a \in R\).

Proof. If this is not true, there exists a Borel set \(E \subset R\) such that \(\mu(E) > 0\) and \(D^+(\mu, a) > 0\) for \(a \in E\). By Corollary 6, \(D^+(\mu \cdot E, a) > 0\) for \(\mu\) a.e. \(a \in E\), and Theorem 8 implies that \(\mu \cdot E\) is absolutely continuous. This is impossible, since \(\mu\), and hence \(\mu \cdot E\), is singular.

10. Theorem. If \(E \subset R\) and \(D^+(\mu, a) = 0\) for \(\mu\) a.e. \(a \in E\), then (with the agreement that \(0 \cdot \infty = \infty\))

\[
D(\mu, a) \equiv (\limsup_{r_i \to 0} h(r)/h(2r))\bar{D}(\mu, a) \quad \text{for \(\mu\) a.e. \(a \in E\).}
\]

This can be proved with the help of Theorem 5 by the same method as Theorem 5 in [1]. We omit the details.

11. Theorem. If \(0 < D(\mu, a) < \infty\) for \(\mu\) a.e. \(a \in R\), then \(\mu\) is absolutely continuous.
Proof. Suppose $\mu$ is not absolutely continuous. Then there is a Borel set $E \subseteq \mathbb{R}$ such that $\mu(E) = 0$ and $\mu \perp E$ is singular. Hence by Corollary 6 we may assume that $\mu$ is singular. To simplify the notation, we write $g(r) = h(2r)$.

If $\lim_{r \to 0} g(r)/g(2r) < 1$, we derive a contradiction from 7, 9, and 10. Therefore we assume that there is a sequence $r_i \to 0$ such that $\lim_{i \to \infty} g(r_i)/g(2r_i) = 1$. Setting $E_k = \{ x \in E : 1/k \leq D(\mu, x) \leq k \}$ for $k = 1, 2, \ldots$, we fix $k$ such that $\mu(E_k) > 0$. Let $0 < \varepsilon < 1/k$. We use the notation $B(x, r) = [x-r, x+r]$. There are $1/k \leq \lambda \leq k$, $0 < r_0 \leq \infty$ and a closed set $F \subseteq E$ such that $\mu(F) > 0$ and

$$(\lambda - \varepsilon)g(r) \leq \mu B(x, r) \leq (\lambda + \varepsilon)g(r) \quad \text{for} \quad x \in F, \quad 0 < r \leq r_0.$$ 

By Theorem 5 there are $x \in F$ and $i$ such that $2r_i \leq r_0$, $g(2r_i) \leq (1 + \varepsilon)g(r_i)$ and

$$\mu(B(x, r_i) \setminus F) < \varepsilon g(r_i).$$

Then

$$\mu(B(x, 2r_i) \setminus B(x, r_i)) = \mu B(x, 2r_i) - \mu B(x, r_i)$$

$$\leq (\lambda + \varepsilon)g(2r_i) - (\lambda - \varepsilon)g(r_i) \leq (1 + \varepsilon)(\lambda + \varepsilon) - (\lambda - \varepsilon)g(r_i) < (3 + k)\varepsilon g(r_i).$$

Denote

$$a = \min \{ x - r_i, x \} \cap F, \quad b = \max \{ x, x + r_i \} \cap F,$$

$$c = \max \{ a, (a+b)/2 \} \cap F, \quad d = \min \{ (a+b)/2, b \} \cap F,$$

$$r = b - a, \quad s = c - a, \quad t = b - d.$$

We may assume, without loss of generality, that $t \leq s$. Then

$$B(a, r-t) \cap B(b, r-s) \subseteq (B(x, r_i) \setminus F) \cup \{ c, d \},$$

whence

$$\mu(B(a, r-t) \cap B(b, r-s)) \leq \varepsilon g(r_i)$$

and

$$\mu(B(a, r-t) \cup B(b, r-s)) = \mu B(a, r-t) + \mu B(b, r-s)$$

$$- \mu(B(a, r-t) \cap B(b, r-s)) \leq (\lambda - \varepsilon)g(r-t) + (\lambda - \varepsilon)g(r-s) - \varepsilon g(r_i).$$

On the other hand

$$(B(a, r-t) \cup B(b, r-s)) \setminus B(a, r) \subseteq (B(x, r_i) \setminus F) \cup (B(x, 2r_i) \setminus B(x, r_i)),$$

whence

$$\mu(B(a, r-t) \cup B(b, r-s)) \leq \mu B(a, r) + \mu ((B(a, r-t) \cup B(b, r-s)) \setminus B(a, r))$$

$$\leq (\lambda + \varepsilon)g(r) + (4 + k)\varepsilon g(r_i).$$

Since $r-s \leq r-t$, we obtain combining the above inequalities

$$2(\lambda - \varepsilon)g(r-s) \leq (\lambda - \varepsilon)(g(r-s) + g(r-t)) \leq (\lambda + \varepsilon)g(r) + (5 + k)\varepsilon g(r_i).$$

From the inclusion

$$B(a, r) \setminus B(c, r-s) \subseteq (B(x, r_i) \setminus F) \cup (B(x, 2r_i) \setminus B(x, r_i))$$
we deduce
\((\lambda - \varepsilon)g(r) \leq \mu B(a, r) \leq \mu B(c, r-s) + \mu (B(a, r) \setminus B(c, r-s))\)
\[\leq (\lambda + \varepsilon)g(r-s) + (4 + k)\varepsilon g(r_i)\].

Hence
\[
2(\lambda - \varepsilon)g(r-s)
\leq (\lambda + \varepsilon)(\lambda - \varepsilon)^{-1}g(r-s) + (4 + k)\varepsilon(\lambda + \varepsilon)(\lambda - \varepsilon)^{-1}g(r_i) + (5 + k)\varepsilon g(r_i).
\]

Since \(\frac{r}{2} \leq r-s, \frac{1}{k} \leq \lambda \leq k\) and \(k\) does not depend on \(\varepsilon\) (whereas \(\lambda\) may), we obtain
\[g(r/2) \equiv o(\varepsilon)g(r_i),\]
where \(o(\varepsilon) \to 0\) as \(\varepsilon \downarrow 0\). Finally, we use the inclusion \(B(x, r_i) \cap F \subset B(a, s) \cup B(b, t)\) and the inequalities \(s \leq r/2, t \leq r/2\) to obtain
\[(\lambda - 2\varepsilon)g(r_i) \leq \mu B(x, r_i) - \mu (B(x, r_i) \setminus F) = \mu (B(x, r_i) \cap F)\]
\[\leq \mu B(a, s) + \mu B(b, t) \equiv (\lambda + \varepsilon)g(s) + (\lambda + \varepsilon)g(t)\]
\[\leq 2(\lambda + \varepsilon)g(r/2) \equiv 2(\lambda + \varepsilon)o(\varepsilon)g(r_i),\]
and
\[1/k - 2\varepsilon \leq \lambda - 2\varepsilon \leq 2(\lambda + \varepsilon)o(\varepsilon) \leq 2(k + \varepsilon)o(\varepsilon),\]
which gives a contradiction when \(\varepsilon \downarrow 0\).

12. Corollary. If \(0 < D(\mu, a) < \infty\) for \(\mu\) a.e. \(a \in \mathbb{R}\), then the limit \(l = \lim_{r \downarrow 0} h(r)/r\) exists, \(0 < l < \infty\), and
\[\mu(A) = l \int_A D(\mu, x) dL^1 x\]
for all \(L^1\) measurable sets \(A \subset \mathbb{R}\).

Proof. Since \(\mu\) is absolutely continuous, there exists an \(L^1\) integrable function \(f\) such that \(0 < f(x) < \infty\) for \(\mu\) a.e. \(x \in \mathbb{R}\) and \(\mu(A) = \int_A f dL^1\) for all \(L^1\) measurable sets \(A \subset \mathbb{R}\). By Lebesgue's theorem
\[\lim_{r \downarrow 0} \mu[x-r, x+r]/(2r) = f(x)\] for \(L^1\) a.e. \(x \in \mathbb{R}\).

Thus
\[\frac{h(r)}{r} = \frac{\mu[x-r/2, x+r/2]}{r} \to \frac{f(x)}{D(\mu, x)}\] as \(r \downarrow 0\),
and
\[f(x) = lD(\mu, x)\]
for \(\mu\) a.e. \(x \in \mathbb{R}\).

13. Corollary. If \(\mu\) is singular and \(0 < D(\mu, a) < \infty\) for \(\mu\) a.e. \(a \in \mathbb{R}\), then \(D(\mu, a) < D(\mu, a)\) for \(\mu\) a.e. \(a \in \mathbb{R}\).

14. Remark. It follows as in the proof of 12 that if \(\mu\) is absolutely continuous, then \(0 < D^+(\mu, a) = D^+(\mu, a) = D(\mu, a) < \infty\) for \(\mu\) a.e. \(a \in \mathbb{R}\) with \(h(r) = r\). Thus the sufficient conditions in Theorems 8 and 11 are also in a sense necessary.
15. Remark. To generalize Theorem 11 to the Euclidean $n$-space $\mathbb{R}^n$ is an interesting and difficult problem. A reasonable conjecture seems to be the following:

If $\varphi$ is an outer measure over $\mathbb{R}^n$ such that Borel sets are $\varphi$ measurable and $0<\lim_{r \to 0} \varphi\{y: |x-y| \leq r\}/h(r) < \infty$ for $\varphi$ a.e. $x \in \mathbb{R}^n$, then there exist a positive integer $m$ and a countably $(H^m, m)$ rectifiable (see [3, 3.2.14]) set $E \subset \mathbb{R}^n$ such that $\varphi$ is absolutely continuous with respect to $H^m LE$. Here $H^m$ is the $m$-dimensional Hausdorff measure.

This conjecture is true by the results of Marstrand [4] and Moore [5] in the case where $h(r) = r^s$ for some $0<s<2$. Then it follows that $m=s=1$. For $s \geq 2$ the question is open.

References


University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

Received 21 April 1978