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DENSITIES OF MEASURES ON THE REAL LINE

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1. Introduction. Suppose that μ is an outer measure on the real line R such that $\mu(R)>0$ and all Borel sets are μ measurable. Let $h\colon (0,\infty)\to(0,\infty)$ be a non-decreasing function with $\lim_{r\downarrow 0} h(r)=0$. These assumptions on μ and h will be made throughout the whole paper. The *upper* and *lower h-densities* of μ at $a\in R$ are defined by

$$\bar{D}(\mu, a) = \limsup_{r\downarrow 0} \mu[a-r, a+r]/h(2r),$$

$$\underline{D}(\mu, a) = \liminf_{r\downarrow 0} \mu[a-r, a+r]/h(2r).$$

If they are equal, their common value is called the *h-density* of μ at a, and it is denoted by $D(\mu, a)$. We shall also consider one-sided densities of μ. The *upper* and *lower right h-densities* of μ are defined by

$$\bar{D}^+(\mu, a) = \limsup_{r\downarrow 0} \mu[a, a+r]/h(r),$$

$$\underline{D}^+(\mu, a) = \liminf_{r\downarrow 0} \mu[a, a+r]/h(r).$$

The *upper* and *lower left h-densities* $\bar{D}^-(\mu, a)$ and $\underline{D}^-(\mu, a)$ are defined similarly as the upper and lower limits of the ratios $\mu[a-r, a]/h(r)$. The results of this paper are usually stated and proved for right densities, but their obvious analogues hold for left densities as well.

The main results are Theorems 8 and 11. They state that if μ satisfies certain homogeneity conditions in terms of h-densities, then it is absolutely continuous with respect to the Lebesgue measure L^1. More precisely, μ is absolutely continuous if either $0<D(\mu, a)<\infty$ for μ a.e. $a\in R$ or $0<\underline{D}^+(\mu, a)\leqq\bar{D}^+(\mu, a)<\infty$ for μ a.e. $a\in R$. These results characterize absolutely continuous measures of R through their density properties.

In Corollaries 9 and 13 to Theorems 8 and 11 we obtain results on the densities of measures which are singular with respect to the Lebesgue measure. Similar results for s-dimensional Hausdorff measures, $0<s<1$, have been proved by Besicovitch in [1] and [2].

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2. Remarks. (1) The results of this paper are false if $\lim_{r\downarrow 0} h(r)>0$ as the example where μ is a Dirac measure shows.

(2) In the following proofs we shall usually have the situation where some of the densities defined in Introduction is finite μ a.e. This always implies that $\mu\{a\}=0$ for all $a\in R$.

3. Lemma. *Let $A\subset R$. If for every $a\in A$ there is $r>0$ such that $(a, a+r)\subset A$, then A is a Borel set.*

Proof. Let A_n be the set of all $a\in[-n, n]\cap A$ for which $\sup\{r\colon (a, a+r)\subset A\}> 1/n$. Then $A=\bigcup_{n=1}^{\infty} A_n$. Define

$$b_1 = \sup A_n, \quad a_1 = \inf\,[b_1-1/n, b_1]\cap A_n,$$

$$b_k = \sup\,(-\infty, b_{k-1}-1/n]\cap A_n, \quad a_k = \inf\,[b_k-1/n, b_k]\cap A_n,$$

$k=2, \dots, m$, where the process terminates when $(-\infty, b_k-1/n]\cap A_n=\emptyset$. For each k, $I_k=[b_k-1/n, b_k]\cap A_n$ is an interval with end points a_k and b_k, and $A_n=\bigcup_{k=1}^{m} I_k$. It follows that A is a Borel set.

4. Theorem. *The densities $\bar{D}(\mu, \ )$, $\underline{D}(\mu, \ )$, $\bar{D}^+(\mu, \ )$, $\underline{D}^+(\mu, \ )$, $\bar{D}^-(\mu, \ )$, $\underline{D}^-(\mu, \ )$ are Borel functions.*

Proof. We prove, for example, that $\bar{D}^+(\mu, \ )$ is a Borel function. We first show that given $0<r<\infty$, $f\colon a\mapsto\mu[a, a+r]$ is a Borel function. Express the interior of the set $\{a\colon f(a)=\infty\}$ as $\bigcup_{j=1}^{\infty} I_j$, where I_j's are open disjoint intervals and set

$$A = R\setminus\bigcup_{j=1}^{\infty} \mathrm{Cl}\, I_j.$$

Let $\alpha\in R$, $a\in A$ such that $f(a)<\alpha$. Then, by the definition of A, there is $b\in(a, a+r)$ such that $f(b)<\infty$. Hence $\mu[a, b+r]\leqq\alpha+f(b)<\infty$ and

$$\limsup_{c\downarrow a} f(c) \leqq \lim_{c\downarrow a} \mu[a, c+r] = f(a) < \alpha.$$

Therefore we can find $s>0$ such that $f(c)<\alpha$ for $c\in(a, a+s)$. By Lemma 3 the set $\{a\in A\colon f(a)<\alpha\}$ is then a Borel set. Hence $f|A$ is a Borel function. Since $f(a)<\infty$ for at most countably many $a\in R\setminus A$, $f|R\setminus A$ is also a Borel function. Thus f is a Borel function.

Since h is non-decreasing, the set D consisting of all points of discontinuity of h and of all positive rational numbers is countable. If $r>0$ and $r\notin D$, then for any $\varepsilon>0$ there is $s\in D$ such that $r<s<r+\varepsilon$ and $\mu[a, a+r]/h(r)\leqq\mu[a, a+s]/h(s)+\varepsilon$. Hence

$$\bar{D}^+(\mu, a) = \limsup_{\substack{r\downarrow 0 \\ r\in D}} \mu[a, a+r]/h(r),$$

from which the assertion follows.

If $E \subset R$ the restriction measure $\mu \llcorner E$ is defined by $(\mu \llcorner E)(A) = \mu(E \cap A)$ for $A \subset R$.

5. Theorem. *If* $E \subset R$ *is a Borel set and* $\bar{D}(\mu, a) < \infty$ *for* μ *a.e.* $a \in E$ *or* $\bar{D}^+(\mu, a) < \infty$ *for* μ *a.e.* $a \in E$, *then*

$$D(\mu \llcorner (R \setminus E), a) = \bar{D}^+(\mu \llcorner (R \setminus E), a) = 0 \quad \textit{for } \mu \textit{ a.e. } \quad a \in E.$$

Proof. We prove the theorem under the assumption $\bar{D}^+(\mu, a) < \infty$ for μ a.e. $a \in E$. The case $\bar{D}(\mu, a) < \infty$ can be handled similarly. For $n = 1, 2, \ldots$ let

$$E_n = \{a \in E \colon \mu[a, a+r] \leqq nh(r) \text{ for } 0 < r \leqq 1/n\}.$$

Then $\mu(E \setminus \bigcup_{n=1}^{\infty} E_n) = 0$. The assumption $\bar{D}^+(\mu, a) < \infty$ for μ a.e. $a \in E$ implies that $\mu\{a\} = 0$ for all $a \in E$; therefore μ almost all of E_n can be covered with countably many open intervals each of finite μ-measure. Let I be one such interval and F a closed subset of $I \cap E_n$. To prove that $D(\mu \llcorner (R \setminus E), a) = 0$ for μ a.e. $a \in E$, it is then sufficient to show that $D(\mu \llcorner (R \setminus E), a) = 0$ for μ a.e. $a \in F$, since any Borel set of finite measure can be approximated from within by a closed subset (see, for example [3, 2.2.2 (1)]).

To do this, let $\varepsilon > 0$ and denote

$$A_\varepsilon = \{a \in F \colon \bar{D}(\mu \llcorner (R \setminus E), a) > \varepsilon\}.$$

By [3, 2.2.2 (1)] there exists a closed set $C \subset I \setminus E$ such that $\mu((I \setminus E) \setminus C) < \varepsilon^2$. For each $a \in A_\varepsilon$, there is $0 < r(a) < 1/2n$ such that $[a - r(a), a + r(a)] \subset I \setminus C$ and $\mu([a - r(a), a + r(a)] \setminus E) > \varepsilon h(2r(a))$. By Besicovitch covering theorem [3, 2.8.14] we can find a sequence $(a_i, r_i) = (a_i, r(a_i))$ of such pairs such that $A_\varepsilon \subset \bigcup_{i=1}^{\infty} [a_i - r_i, a_i + r_i]$ and at most k of the intervals $[a_i - r_i, a_i + r_i]$ may have a point in common, where k is an absolute constant. Letting $b_i = \min\, [a_i - r_i, a_i + r_i] \cap F$, we have

$$\mu([a_i - r_i, a_i + r_i] \cap A_\varepsilon) \leqq \mu[b_i, b_i + 2r_i] \leqq nh(2r_i).$$

We obtain

$$\mu(A_\varepsilon) \leqq \sum_{i=1}^{\infty} \mu([a_i - r_i, a_i + r_i] \cap A_\varepsilon) \leqq n \sum_{i=1}^{\infty} h(2r_i)$$

$$< (n/\varepsilon) \sum_{i=1}^{\infty} \mu([a_i - r_i, a_i + r_i] \setminus E) \leqq (kn/\varepsilon)\, \mu((I \setminus C) \setminus E) < kn\varepsilon,$$

and

$$\mu\{a \in E \colon \bar{D}^+(\mu \llcorner (R \setminus E), a) > 0\} = \lim_{\varepsilon \downarrow 0} \mu(A_\varepsilon) = 0.$$

To show that $\bar{D}^+(\mu \llcorner (R \setminus E), a) = 0$ for μ a.e. $a \in E$, we may proceed as above, but this time applying the Besicovitch covering theorem to intervals $[a - r(a)/2, a + r(a)/2]$ such that $\mu([a, a + r(a)] \setminus E) > \varepsilon h(r(a))$. This completes the proof.

6. Corollary. *If* $E \subset R$ *is a Borel set and* $\bar{D}(\mu, a) < \infty$ *for* μ *a.e.* $a \in E$ *or* $\bar{D}^+(\mu, a) < \infty$ *for* μ *a.e.* $a \in E$, *then* $\bar{D}(\mu \llcorner E, a) = \bar{D}(\mu, a)$, $\underline{D}(\mu \llcorner E, a) = \underline{D}(\mu, a)$, $\bar{D}^+(\mu \llcorner E, a) = \bar{D}^+(\mu, a)$, $\underline{D}^+(\mu \llcorner E, a) = \underline{D}^+(\mu, a)$ *for* μ *a.e.* $a \in E$.

7. Theorem. $\bar{D}(\mu, a) \leqq \bar{D}^+(\mu, a) = \bar{D}^-(\mu, a) \leqq 2\bar{D}(\mu, a)$ *for* μ *a.e.* $a \in R$.

Proof. To prove the inequality $\bar{D}(\mu, a) \leqq \bar{D}^+(\mu, a)$, denote $E_t = \{a: \bar{D}^+(\mu, a) \leqq t\}$ for $0 < t < \infty$. Fix t and let $\varepsilon > 0$. For $n = 1, 2, \ldots$, set

$$E_{t,n} = \{a \in E_t: \mu[a, a+r] \leqq (t+\varepsilon)h(r) \text{ for } 0 < r < 1/n\} \cap [-n, n].$$

Then $\mu(E_{t,n}) < \infty$ and $E_t = \bigcup_{n=1}^{\infty} E_{t,n}$. Let F be a closed subset of $E_{t,n}$. By Theorem 5, $D(\mu \llcorner (R \setminus F), a) = 0$ for μ a.e. $a \in F$. Take such a point a and let $0 < r_0 \leqq 1/2n$ be such that

$$\mu([a-r, a+r] \setminus F) \leqq \varepsilon h(2r) \quad \text{for} \quad 0 < r < r_0.$$

Let $0 < r < r_0$ and $b = \min [a-r, a] \cap F$. Then

$$\mu[a-r, a+r] \leqq \mu([a-r, a+r] \setminus F) + \mu[b, b+2r] \leqq (t+2\varepsilon)h(2r),$$

whence $\bar{D}(\mu, a) \leqq t + 2\varepsilon$. By [3, 2.2.2 (1)] this implies that $\bar{D}(\mu, a) \leqq t + 2\varepsilon$ for μ a.e. $a \in E_{t,n}$. Since this holds for all $\varepsilon > 0$ and $n = 1, 2, \ldots$, we obtain

$$\mu\{a: \bar{D}^+(\mu, a) \leqq t, \bar{D}(\mu, a) > t\} = 0$$

for $0 < t < \infty$. Since $\{a: \bar{D}(\mu, a) > \bar{D}^+(\mu, a)\}$ is the union of the sets

$$\{a: \bar{D}^+(\mu, a) \leqq t, \ \bar{D}(\mu, a) > t\}$$

when t runs through the positive rational numbers, we obtain $\bar{D}(\mu, a) \leqq \bar{D}^+(\mu, a)$ for μ a.e. $a \in R$.

To prove the inequality $\bar{D}^+(\mu, a) \leqq 2\bar{D}(\mu, a)$, denote $E_t = \{a: \bar{D}(\mu, a) \leqq t\}$ for $0 < t < \infty$. Fix t and let $\varepsilon > 0$. Let n be a positive integer and F a closed subset of

$$E_{t,n} = \{a \in E_t: \mu[a-r, a+r] \leqq (t+\varepsilon)h(2r) \text{ for } 0 < r < 1/n\} \cap [-n, n].$$

Suppose that $a \in F$ and $\bar{D}^+(\mu \llcorner (R \setminus F), a) = 0$. By Theorem 5 this is true for μ a.e. $a \in F$. Then there is $0 < r_0 \leqq 1/n$ such that $\mu([a, a+r] \setminus F) < \varepsilon h(r)$ for $0 < r < r_0$. Let $0 < r < r_0$. If there is $b \in [a+r/2, a+r] \cap F$, then

$$\mu[a, a+r] \leqq \mu[a-r/2, a+r/2] + \mu[b-r/2, b+r/2] \leqq 2(t+\varepsilon)h(r).$$

Otherwise $[a+r/2, a+r] \subset [a, a+r] \setminus F$, and the same inequality follows. Hence $\bar{D}^+(\mu, a) \leqq 2(t+\varepsilon)$. The proof can be completed as in the first part.

To prove the inequality $\bar{D}^-(\mu, a) \leqq \bar{D}^+(\mu, a)$, let

$$E_{s,t} = \{a: \bar{D}^+(\mu, a) \leqq t < s \leqq \bar{D}^-(\mu, a)\}$$

for $0 < t < s < \infty$ and let $0 < \varepsilon < (s-t)/3$. Let n be a positive integer and F a closed subset of

$$E_{s,t,n} = \{a \in E_{s,t}: \mu[a, a+r] \leqq (t+\varepsilon)h(r) \text{ for } 0 < r < 1/n\} \cap [-n, n].$$

Suppose that $a \in F$ and $\bar{D}^-(\mu \llcorner (R \setminus F), a)=0$, which again holds for μ a.e. $a \in F$. Then there is $0<r<1/n$ such that

$$\mu([a-r, a] \setminus F) < \varepsilon h(r), \ \mu[a-r, a] > (s-\varepsilon) h(r).$$

Let $b = \min [a-r, a] \cap F$. Then

$$(t+\varepsilon) h(r) \geqq \mu[b, a] \geqq \mu[a-r, a] - \mu([a-r, a] \setminus F) > (s-2\varepsilon) h(r),$$

and $s-t<3\varepsilon$. This contradicts with the choice of ε, and it follows that $\mu(F)=0$. By a similar argument as in the first part of the proof, we obtain $\bar{D}^-(\mu, a) \leqq \bar{D}^+(\mu, a)$ for μ a.e. $a \in R$.

The opposite inequality is proved in the same way, and the theorem follows.

We say that μ is absolutely continuous if $L^1(A)=0$ implies $\mu(A)=0$, and that μ is singular if there is a set $E \subset R$ such that $L^1(E)=0$ and $\mu(R \setminus E)=0$.

8. Theorem. *If* $\bar{D}^+(\mu, a)<\infty$ *and* $\underline{D}^+(\mu, a)>0$ *for* μ *a.e.* $a \in R$, *then* μ *is absolutely continuous.*

Proof. Using [3, 2.2.2 (1)] we find $0<d<1$, $0<r_0<\infty$ and a closed set $F \subset R$ such that $\mu(F)>0$ and

$$d h(r) \leqq \mu[a, a+r] \leqq h(r)/d \quad \text{for} \quad 0<r<r_0, \quad a \in F.$$

Making r_0 smaller if necessary, we use Theorem 5 to obtain $a \in F$ such that

$$\mu([a, a+r] \setminus F) \leqq (d^3/8) h(r) \quad \text{for} \quad 0<r<r_0.$$

Let $r_i>0$, $0<\sum_{i=1}^k r_i<s<r_0$. Choose a positive integer m such that $s<m\sum_{i=1}^k r_i<2s$. Then there are points $a_{i,j} \in F$, $i=1, \dots, k$, $j=1, \dots, m$, such that

$$[a, a+s] \cap F \subset \bigcup_{i,j} [a_{i,j}, a_{i,j}+r_i].$$

Then

$$d h(s) \leqq \mu[a, a+s] \leqq \mu([a, a+s] \setminus F) + \sum_{i,j} \mu[a_{i,j}, a_{i,j}+r_i]$$

$$\leqq (d/2) h(s) + (m/d) \sum_{i=1}^k h(r_i) < (d/2) h(s) + \left(2s\Big/\left(d \sum_{i=1}^k r_i\right)\right) \sum_{i=1}^k h(r_i),$$

and

(1) $$\sum_{i=1}^k h(r_i) > (d^2/4) h(s) \sum_{i=1}^k r_i/s.$$

Take now $0<r<r_0/4$ and $r_0/2 \leqq s<r_0$. Write

$$(a, a+s) \setminus F = \bigcup_{i=1}^{\infty} (a_i, a_i+r_i),$$

where the intervals (a_i, a_i+r_i) are disjoint and $r_1 \geqq r_2 \geqq \ldots$. Suppose that $r_1 \geqq r$ and let k be the largest integer such that $r_k \geqq r$. Since $a_i \in F$ for all i, we have

$$d \sum_{i=1}^{k} h(r_i) \leqq \sum_{i=1}^{k} \mu[a_i, a_i+r_i] \leqq \mu([a, a+s] \setminus F) < (d^3/8) h(s).$$

Combining this with (1) we get

$$(d^3/4) h(s) \sum_{i=1}^{k} r_i/s < (d^3/8) h(s)$$

and

$$\sum_{i=1}^{k} r_i < s/2.$$

Define $b_1=a$, $b_j=\min\,[b_{j-1}+r, a+s] \cap F$, $j=2, \ldots, n$, where the process stops when $a+s<b_j+r$ or $[b_j+r, a+s] \cap F=\emptyset$. Then $(a, a+s) \setminus \bigcup_{i=1}^{k} (a_i, a_i+r_i) \subset \bigcup_{i=1}^{n} [b_i, b_i+2r]$, since $r_i<r$ for $i>k$. Hence

$$s/2 \leqq L^1\Big((a, a+s) \setminus \bigcup_{i=1}^{k} (a_i, a_i+r_i)\Big) \leqq 2nr,$$

and $n \geqq s/4r$. This is true also if $r_1<r$. Thus we have

$$h(s)/d \geqq \mu[a, a+s] \geqq \sum_{i=1}^{n} \mu[b_i, b_i+r] \geqq n\, dh(r) \geqq s\, dh(r)/4r,$$

which gives

$$h(r) \leqq 4rh(s)/(d^2 s) \leqq \big(8h(r_0)/(d^2 r_0)\big) r.$$

Since this holds for all $0<r<r_0/4$, the assertion follows from the assumption $\bar{D}^+(\mu, a)<\infty$ for μ a.e. $a \in R$.

9. Corollary. *If μ is singular and $\bar{D}^+(\mu, a)<\infty$ for μ a.e. $a \in R$, then $\underline{D}^+(\mu, a)=0$ for μ a.e. $a \in R$.*

Proof. If this is not true, there exists a Borel set $E \subset R$ such that $\mu(E)>0$ and $\underline{D}^+(\mu, a)>0$ for $a \in E$. By Corollary 6, $\underline{D}^+(\mu \llcorner E, a)>0$ for μ a.e. $a \in E$, and Theorem 8 implies that $\mu \llcorner E$ is absolutely continuous. This is impossible, since μ, and hence $\mu \llcorner E$, is singular.

10. Theorem. *If $E \subset R$ and $\underline{D}^+(\mu, a)=0$ for μ a.e. $a \in E$, then (with the agreement that $0 \cdot \infty=\infty$)*

$$\underline{D}(\mu, a) \leqq \big(\limsup_{r \downarrow 0} h(r)/h(2r)\big) \bar{D}(\mu, a) \quad \textit{for } \mu \textit{ a.e.} \quad a \in E.$$

This can be proved with the help of Theorem 5 by the same method as Theorem 5 in [1]. We omit the details.

11. Theorem. *If $0<D(\mu, a)<\infty$ for μ a.e. $a \in R$, then μ is absolutely continuous.*

Proof. Suppose μ is not absolutely continuous. Then there is a Borel set $E\subset R$ such that $\mu(E)>0$ and $\mu \llcorner E$ is singular. Hence by Corollary 6 we may assume that μ is singular. To simplify the notation, we write $g(r)=h(2r)$.

If $\limsup_{r\downarrow 0} g(r)/g(2r)<1$, we derive a contradiction from 7, 9 and 10. Therefore we assume that there is a sequence $r_i\downarrow 0$ such that $\lim_{i\to\infty} g(r_i)/g(2r_i)=1$. Setting $E_k=\{x\in E\colon 1/k\leqq D(\mu, x)\leqq k\}$ for $k=1, 2, \dots$, we fix k such that $\mu(E_k)>0$. Let $0<\varepsilon<1/k$. We use the notation $B(x, r)=[x-r, x+r]$. There are $1/k\leqq\lambda\leqq k$, $0<r_0<\infty$ and a closed set $F\subset E$ such that $\mu(F)>0$ and

$$(\lambda-\varepsilon)g(r) \leqq \mu B(x, r) \leqq (\lambda+\varepsilon)g(r) \quad \text{for} \quad x\in F, \quad 0<r\leqq r_0.$$

By Theorem 5 there are $x\in F$ and i such that $2r_i\leqq r_0$, $g(2r_i)\leqq(1+\varepsilon)g(r_i)$ and

$$\mu\big(B(x, r_i)\setminus F\big) < \varepsilon g(r_i).$$

Then

$$\mu\big(B(x, 2r_i)\setminus B(x, r_i)\big) = \mu B(x, 2r_i)-\mu B(x, r_i)$$
$$\leqq (\lambda+\varepsilon)g(2r_i)-(\lambda-\varepsilon)g(r_i) \leqq \big((1+\varepsilon)(\lambda+\varepsilon)-(\lambda-\varepsilon)\big)g(r_i) < (3+k)\varepsilon g(r_i).$$

Denote

$$a = \min\,[x-r_i, x]\cap F, \quad b = \max\,[x, x+r_i]\cap F,$$
$$c = \max\,[a, (a+b)/2]\cap F, \quad d = \min\,[(a+b)/2, b]\cap F,$$
$$r = b-a, \quad s = c-a, \quad t = b-d.$$

We may assume, without loss of generality, that $t\leqq s$. Then

$$B(a, r-t)\cap B(b, r-s)\subset \big(B(x, r_i)\setminus F\big)\cup\{c, d\},$$

whence

$$\mu\big(B(a, r-t)\cap B(b, r-s)\big) \leqq \varepsilon g(r_i)$$

and

$$\mu\big(B(a, r-t)\cup B(b, r-s)\big) = \mu B(a, r-t)+\mu B(b, r-s)$$
$$-\mu\big(B(a, r-t)\cap B(b, r-s)\big) \geqq (\lambda-\varepsilon)g(r-t)+(\lambda-\varepsilon)g(r-s)-\varepsilon g(r_i).$$

On the other hand

$$\big(B(a, r-t)\cup B(b, r-s)\big)\setminus B(a, r)\subset \big(B(x, r_i)\setminus F\big)\cup\big(B(x, 2r_i)\setminus B(x, r_i)\big),$$

whence

$$\mu\big(B(a, r-t)\cup B(b, r-s)\big) \leqq \mu B(a, r)+\mu\big(\big(B(a, r-t)\cup B(b, r-s)\big)\setminus B(a, r)\big)$$
$$\leqq (\lambda+\varepsilon)g(r)+(4+k)\varepsilon g(r_i).$$

Since $r-s\leqq r-t$, we obtain combining the above inequalities

$$2(\lambda-\varepsilon)g(r-s) \leqq (\lambda-\varepsilon)\big(g(r-s)+g(r-t)\big) \leqq (\lambda+\varepsilon)g(r)+(5+k)\varepsilon g(r_i).$$

From the inclusion

$$B(a, r)\setminus B(c, r-s)\subset \big(B(x, r_i)\setminus F\big)\cup\big(B(x, 2r_i)\setminus B(x, r_i)\big)$$

we deduce

$$(\lambda-\varepsilon)g(r) \leqq \mu B(a, r) \leqq \mu B(c, r-s)+\mu\big(B(a, r)\setminus B(c, r-s)\big)$$
$$\leqq (\lambda+\varepsilon)g(r-s)+(4+k)\varepsilon g(r_i).$$

Hence

$$2(\lambda-\varepsilon)g(r-s)$$
$$\leqq (\lambda+\varepsilon)^2(\lambda-\varepsilon)^{-1}g(r-s)+(4+k)\varepsilon(\lambda+\varepsilon)(\lambda-\varepsilon)^{-1}g(r_i)+(5+k)\varepsilon g(r_i).$$

Since $r/2 \leqq r-s$, $1/k \leqq \lambda \leqq k$ and k does not depend on ε (whereas λ may), we obtain

$$g(r/2) \leqq o(\varepsilon)g(r_i),$$

where $o(\varepsilon)\to 0$ as $\varepsilon\downarrow 0$. Finally, we use the inclusion $B(x, r_i)\cap F\subset B(a, s)\cup B(b, t)$ and the inequalities $s\leqq r/2$, $t\leqq r/2$ to obtain

$$(\lambda-2\varepsilon)g(r_i) \leqq \mu B(x, r_i)-\mu\big(B(x, r_i)\setminus F\big) = \mu\big(B(x, r_i)\cap F\big)$$
$$\leqq \mu B(a, s)+\mu B(b, t) \leqq (\lambda+\varepsilon)g(s)+(\lambda+\varepsilon)g(t)$$
$$\leqq 2(\lambda+\varepsilon)g(r/2) \leqq 2(\lambda+\varepsilon)o(\varepsilon)g(r_i),$$

and

$$1/k-2\varepsilon \leqq \lambda-2\varepsilon \leqq 2(\lambda+\varepsilon)o(\varepsilon) \leqq 2(k+\varepsilon)o(\varepsilon),$$

which gives a contradiction when $\varepsilon\downarrow 0$.

12. Corollary. *If* $0<D(\mu, a)<\infty$ *for* μ *a.e.* $a\in R$, *then the limit* $l=\lim_{r\downarrow 0} h(r)/r$ *exists,* $0<l<\infty$, *and*

$$\mu(A) = l \int_A D(\mu, x)\, dL^1 x$$

for all L^1 *measurable sets* $A\subset R$.

Proof. Since μ is absolutely continuous, there exists an L^1 integrable function f such that $0<f(x)<\infty$ for μ a.e. $x\in R$ and $\mu(A)=\int_A f\,dL^1$ for all L^1 measurable sets $A\subset R$. By Lebesgue's theorem

$$\lim_{r\downarrow 0} \mu[x-r, x+r]/(2r) = f(x) \quad \text{for } L^1 \text{ a.e.} \quad x\in R.$$

Thus

$$\frac{h(r)}{r} = \frac{\mu[x-r/2, x+r/2]}{r}\cdot\frac{h(r)}{\mu[x-r/2, x+r/2]} \to \frac{f(x)}{D(\mu, x)} \quad \text{as} \quad r\downarrow 0,$$

and

$$f(x) = lD(\mu, x)$$

for μ a.e. $x\in R$.

13. Corollary. *If* μ *is singular and* $0<\bar{D}(\mu, a)<\infty$ *for* μ *a.e.* $a\in R$, *then* $\underline{D}(\mu, a)<\bar{D}(\mu, a)$ *for* μ *a.e.* $a\in R$.

14. Remark. It follows as in the proof of 12 that if μ is absolutely continuous, then $0<\underline{D}^+(\mu, a)=\bar{D}^+(\mu, a)=D(\mu, a)<\infty$ for μ a.e. $a\in R$ with $h(r)=r$. Thus the sufficient conditions in Theorems 8 and 11 are also in a sense necessary.

15. Remark. To generalize Theorem 11 to the Euclidean n-space R^n is an interesting and difficult problem. A reasonable conjecture seems to be the following:

If φ is an outer measure over R^n such that Borel sets are φ measurable and $0<\lim_{r\downarrow 0}\varphi\{y\colon |x-y|\leqq r\}/h(r)<\infty$ for φ a.e. $x\in R^n$, then there exist a positive integer m and a countably (H^m, m) rectifiable (see [3, 3.2.14]) set $E\subset R^n$ such that φ is absolutely continuous with respect to $H^m \llcorner E$. Here H^m is the m-dimensional Hausdorff measure.

This conjecture is true by the results of Marstrand [4] and Moore [5] in the case where $h(r)=r^s$ for some $0<s<2$. Then it follows that $m=s=1$. For $s\geqq 2$ the question is open.

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