A CONFORMAL SELF-MAP WHICH FIXES THREE POINTS IS THE IDENTITY

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Recently I. Lieb raised the question of fixed points of conformal self-maps of plane domains with arbitrary connectivity. He conjectured the following

Theorem. Let $A$ be a plane domain and $f: A \rightarrow A$ a conformal map. If $f$ has three fixed points, then $f$ is the identity map.

In fact, this theorem is an immediate consequence of a result by B. Maskit [2], which states that one may associate with $A$ another domain $A'$, conformally equivalent to $A$, such that all conformal self-maps of $A'$ are Möbius transformations. Maskit's proof is rather elaborate, as it depends heavily on the structure of the group of conformal self-maps of $A$. In this note we attempt to give a straightforward and relatively short proof of the theorem.

We first remark that there exist domains of arbitrarily high connectivity allowing conformal self-maps with two fixed points, other than the identity. As an example one may consider any domain $A$ obtained from the plane when the origin and an arbitrary closed set of points on the unit circle, symmetric with respect to the real axis and not containing the points $-1$ and $1$, are deleted. Then $f, f(z) = 1/z$, maps $A$ onto itself and fixes $-1$ and $1$.

If $A$ is simply or doubly connected, consideration of all possible standard domains and their conformal self-maps [3, pp. 226—236] shows that the theorem is true in this case. In the sequel, therefore, we may assume that $A$ is at least triply connected. The unit disc $D$ can then be taken to be the universal covering surface of $A$, and the cover transformation group $G$ of $D$ relative to $A$ is a non-elementary properly discontinuous group of Möbius transformations.

In what follows we let $f$ be a fixed conformal self-map of $A$ and assume that $f(w_0) = w_0$ for some $w_0$ in $A$. Denote the projection map of $D$ onto $A$ by $h$; there is no loss of generality in assuming $h(0) = w_0$. Also, we may fix a lifting $\tilde{f}: D \rightarrow D$ of $f$ such that $\tilde{f}(0) = 0$. Denote the $n$-th iterate of $\tilde{f}$ by $\tilde{f}^n$. We then have

Lemma. There exists a natural number $p$ such that $\tilde{f}^p = \text{id}$.

Proof. Evidently $\tilde{f}(z) = \exp(ia)z$ for some real $a$. Choose a $g_0$ in $G$, $g_0 \neq \text{id}$. Then $g_n = \tilde{f}^n \circ g_0 \circ (\tilde{f}^n)^{-1}$ is in $G$. If the number of distinct $g_n$'s is infinite, a sequence $(g_n)$ converges to a conformal $g$. The discontinuity of $G$ makes this impossible.

Consequently $g_n = g_{n+p}$ for some $n$ and $p \geq 1$. In particular, then, $\exp \left( i a (n+p) \right) g_0(0) = \exp \left( i a n \right) g_0(0)$. But 0 is not a fixed point of $g_0$, and we may cancel by $\exp \left( i a n \right) g_0(0)$ to obtain the assertion of the Lemma.

To proceed in the proof of the Theorem, let us denote by $F$ the set of points $z$ in $D$, such that $h(z)$ is a fixed point of $f$, but $h(z) \neq w_0$. Assume $F$ to be non-empty. The set $F$ has a positive Euclidean distance $r$ from 0. Choose $z_1$ in $F$ such that $\left| z_1 \right| = r$, and let $p$ be the smallest natural number such that $f^p = \text{id}$. Set $z_k = f(z_{k-1})$, $k = 2, \ldots, p$, and join every $z_k$ to 0 by the line segment $\bar{z}_k$. Then $w_k = h(\bar{z}_k)$ is a path joining $w_1 = h(z_1)$ to $w_0$. Moreover, $w_k$ is a Jordan arc. For assume $h(t_1) = h(t_2)$, where $t_1, t_2 \in \bar{z}_k$, and $|t_1| < |t_2|$. Then $w_0$ and $w_1$ are joined by $h(\beta)$, where $\beta$ consists of the segment $(0, t_1)$ and a circular arc congruent modulo $G$ with the segment $(t_2, z_k)$. The hyperbolic length of $\beta$ is strictly smaller than that of $\bar{z}_k$, but the Euclidean distance of 0 from the other end-point of $\beta$ is at least $r$. This is in contradiction with the fact that rays issuing from the origin are geodesics in the hyperbolic metric.

A similar argument shows that for $k \neq j$, $\bar{z}_k$ and $\bar{z}_j$ cannot meet except at $w_0$ and $w_1$. Assume $h(t_k) = h(t_j)$ with $t_k \in \bar{z}_k$, $t_j \in \bar{z}_j$. Because $h(t_k)$ is not a fixed point of $f$, we may suppose $|t_k| < |t_j|$. It follows that $w_0$ and $w_1$ are joined by $h(\gamma)$, where $\gamma$ is composed of the segment $(0, t_0)$ and a circular arc congruent modulo $G$ with the segment $(t_j, z_j)$. Again, the hyperbolic length of $\gamma$ is strictly less than that of $\bar{z}_k$, while the Euclidean distance of 0 from the other end-point of $\gamma$ is at least $r$.

To complete the proof, assume $w_2$ is a fixed point of $f$, distinct from $w_0$ and $w_1$. One may join $w_2$, which does not lie on any $\bar{z}_k$, to $w_0$ by an arc $\alpha$ which does not meet any $\bar{z}_k$. Then $\alpha$ lies in the Jordan domain $B$ (of the extended plane) bounded by, say, $\bar{z}_k$ and $\bar{z}_j$, where $\bar{z}_k$ and $\bar{z}_j$ are adjacent segments. Consider the lifting $\tilde{\alpha}$ of $\alpha$, with initial point 0. Since $h$ is a local homeomorphism at 0, we see that $f(\tilde{\alpha})$ must emerge from $w_0$ into the complement of $B$. Then $f(\tilde{\alpha})$ must have a point in common either with $\bar{z}_k$ or $\bar{z}_j$. But this is impossible, since $f$ carries the set of arcs $\bar{z}_k$ onto itself [4].

Remark. Another proof for the Theorem above has been found by K. Leschinger (Bonn) [1].

References


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