Let $R$ be a thin horizontal elastic plate clamped along its border. We denote by $\beta_R(z, \zeta)$ the deflection of $R$ at $z \in R$ under a point load at $\zeta \in R$, that is, the biharmonic Green’s function on $R$ with the pole at $\zeta$. The function is characterized by $A_2^2 \beta_R(z, \zeta) = 2\pi \delta_\zeta$ on $R$, with $\delta_\zeta$ the Dirac measure at $\zeta$, and by the conditions $\beta_R(z, \zeta)/\partial n_z = 0$ at the boundary $\partial R$ of $R$ (e.g., Bergman—Schiffer [1]). Accordingly, it is customary to assume that the boundary $\partial R$ of $R$ relative to the complex plane $C$ is smooth. If $R$ is an arbitrary plane region, a natural procedure is to define $\beta_R(z, \zeta)$ for $z \in R$ as the directed limit

\[ \lim_{\Omega \to R} \beta_\Omega(z, \zeta) \]  

where $\{\Omega\}$ is the directed set of regular subregions, i.e., relatively compact subregions $\Omega$ of $R$ with smooth boundaries $\partial \Omega$. We denote by $O_\beta$ the family of plane regions $R$ for which (1) is divergent for some $\zeta \in R$. The purpose of the present paper is to give a complete characterization of $O_\beta$ as follows:

1°. A plane region $R \in O_\beta$ if and only if the complement $C - R$ of $R$ does not contain any noncollinear triple of points (and hence e.g. $C - \{0, 1, \imath\} \notin O_\beta$).

2°. If a plane region $R \notin O_\beta$, then $\beta_R(z, \zeta) = \lim_{\Omega \to R} \beta_\Omega(z, \zeta)$ is symmetric and continuous on $R \times R$, the convergence is uniform on every compact subset of $R \times R$, and $z \to \beta_R(z, \zeta)$ is biharmonic on $R - \zeta$.

3°. There exist plane regions $R$ which are “unstable” in the sense that (1) is divergent for some $(z, \zeta) \in R \times R$ but convergent for some other $(z, \zeta) \in R \times R$. Such unstable regions $R$ are characterized by the existence of a line $l(R)$ such that $C - R$ is a proper subset of $l(R)$ consisting of at least two points, and (1) is divergent at e.g., $(\zeta, \zeta)$ for any $\zeta \notin l(R)$ and convergent at every $(z, \zeta) \in (l(R) \times l(R)) \cap (R \times R)$.

We denote by $H_\alpha(R)$ the closed subspace of $L_\alpha(R)$ consisting of square integrable harmonic functions on $R$. To prove $1° - 3°$, we shall make essential use of the follow-
ing results obtained in [3]. Let \( H(\Omega) \) be the class of harmonic functions on \( \Omega \) and denote by \((\cdot,\cdot)_{\Omega}\) the inner product in \( L_{2}(\Omega) \). The function \( H_{\Omega}(z,\zeta)=\Delta_{z}\beta_{\Omega}(z,\zeta) \) is referred to as the \( \beta\)-density on \( \Omega \), characterized by \( H_{\Omega}(z,\zeta)+\log|z-\zeta|\in H(\Omega) \) as a function of \( z \), and by \( (H_{\Omega}(\cdot,\zeta),\mu)_{\Omega}=0 \) for every \( u\in H_{2}(\Omega) \). Then \( \beta_{\Omega}(z,\zeta)=(H_{\Omega}(\cdot,\cdot),H_{\Omega}(\cdot,z),H_{\Omega}(\cdot,\zeta))_{\Omega} \) and

\[
(2) \quad |\beta_{\Omega}(z,\zeta)-\beta_{\Omega}(\zeta,z)| \equiv \|H_{\Omega}(\cdot,\cdot)-H_{\Omega}(\cdot,z)\|_{R} \cdot \|H_{\Omega}(\cdot,z)-H_{\Omega}(\cdot,\zeta)\|_{R}
\]
on \( \Omega\times\Omega \) for every regular subregion \( \Omega' \) with \( \Omega\subset\Omega'\subset R \); here \( \|\cdot\|_{R} \) is the norm in \( L_{2}(R) \), and we have set \( H_{\Omega}(z,\zeta)=0 \) for \( (z,\zeta)\notin \Omega\times\Omega \). In particular, we have

\[
(3) \quad \beta_{\Omega}(\zeta,\zeta)-\beta_{\Omega}(\zeta,\zeta) = \|H_{\Omega}(\cdot,\cdot)-H_{\Omega}(\cdot,\zeta)\|_{R}^{2} = \|H_{\Omega}(\cdot,\zeta)\|_{R}^{2} + \|H_{\Omega}(\cdot,\zeta)\|_{R}^{2} - \|H_{\Omega}(\cdot,\zeta)\|_{R}^{2}.
\]
Thus the limit (1) exists if and only if

\[
(4) \quad \lim_{\|\Omega\|_{R}} \|H_{\Omega}(\cdot,\zeta)\|_{R}^{2} < +\infty.
\]

This in turn is equivalent to the existence of an \( H_{R}(\cdot,\zeta) \) on \( R \) such that \( H_{R}(z,\zeta)+\log|z-\zeta|\in H(R) \) as a function of \( z \) and \( (H_{R}(\cdot,\zeta),u)_{R}=0 \) for every \( u\in H_{2}(R) \); in this case, \( \lim_{\|\Omega\|_{R}} \|H_{\Omega}(\cdot,\zeta)-H_{\Omega}(\cdot,\zeta)\|_{R}=0 \) and \( \beta_{R}(z,\zeta)=(H_{R}(\cdot,z),H_{R}(\cdot,\zeta))_{R} \).

We will see that the orthogonal complement \( H_{2}(R)_{\zeta}^{\perp} \) of \( H_{2}(R) \) in \( H_{2}(R-\zeta) \) is either \{0\} or \( RH_{R}(\cdot,\zeta) \), where \( R \) is the field of real numbers. Accordingly, the essential point is to determine the pairs \((R,\zeta)\) of plane regions \( R \) and their points \( \zeta\in R \) such that \( \dim H_{2}(R-\zeta)=\dim H_{2}(R)+1 \). Thus we are led to study the Hilbert space \( H_{2}(R) \). It is locally bounded and therefore has a reproducing kernel \( h_{R}(z,\zeta) \) characterized by \( u(\zeta)=\langle u,h_{R}(\cdot,\zeta)\rangle_{R} \) for every \( u\in H_{2}(R) \). It is seen that \( h_{R}(\cdot,\zeta)=\Delta_{\zeta}H_{R}(\cdot,\zeta) \) if \( H_{R}(\cdot,\zeta) \) exists (cf. e.g. Garabedian [2]); we will, however, not make use of this fact in the present work.

In nos. 1—5 we study the dimension of \( H_{2}(R) \) and give a complete characterization of those plane regions \( R \) for which \( \dim H_{2}(R)=0 \). We then proceed to \( H_{2}(R-\zeta) \) and, in nos. 6—7, characterize those plane regions \( R \) for which \( \dim H_{2}(R-\zeta)=\dim H_{2}(R)+1 \) for every \( \zeta\in R \), for some \( \zeta\in R \), or for no \( \zeta\in R \). For the first case we study, in nos. 8—11, the continuity of \( H_{R}(\cdot,\zeta) \) and the uniformity of the convergence \( H_{\Omega}(\cdot,\zeta)\to H_{R}(\cdot,\zeta) \) with respect to \( \zeta \). That assertions 1°—3° follow from these considerations will be briefly discussed in the final no. 12.

We close this introduction by stressing once more that the class \( O_{\beta} \) is not conformally invariant and not even invariant under Möbius transformations. In fact, the regions \( C-\{0,1,2\}\notin O_{\beta} \) and \( C-\{0,1,2\}\notin O_{\beta} \) are equivalent by the Möbius transformation \((z,0,1,2)=(w,0,1,2) \).

1. Suppose \( u(z) \) is harmonic in a punctured disk \( \Delta_{0}(\zeta,\rho):0<|z-\zeta|<\rho \) about a point \( \zeta\in C \) (the finite complex plane). Then \( u(z) \) has the Laurent expansion

\[
u(z)=\text{Re}\left(-c\log(z-\zeta)+\sum_{n=-\infty}^{\infty} a_{n}(z-\zeta)^{n}\right)\]
in $\Delta_0(\zeta, \varrho)$, with $c \in \mathbb{R}$ (the field of real numbers) and $a_n \in \mathbb{C}$. It is readily seen that $u$ is square integrable in $\Delta_0(\zeta, \varrho)$ ($\varrho \in (0, \varrho)$) if and only if $a_n = 0$ for every negative $n$:

$$\begin{align*}
(5) \quad u(z) &= \Re \left( -c \log(z - \zeta) + \sum_{n=0}^{\infty} a_n (z - \zeta)^n \right).
\end{align*}$$

Next suppose $u(z)$ is harmonic in a punctured disk $\Delta_0(\infty, \varrho)$ ($\varrho < |z| < +\infty$) about the point $\infty$ at infinity. Then the Laurent expansion of $u(z)$ is given by

$$\begin{align*}
(6) \quad u(z) &= \Re \left( \sum_{n=-\infty}^{\infty} a_n z^{-n} \right),
\end{align*}$$

and in this case $u$ is also harmonic at $\infty$ with $u(\infty) = 0$.

For convenience we denote by $l_{\zeta}(z)$ the normalized logarithmic pole $-\log |z - \zeta|$ at $\zeta \in \mathbb{C}$. The Laurent expansion of $l_{\zeta}(z)$ is

$$\begin{align*}
(7) \quad l_{\zeta}(z) &= \Re \left( -\log z + \sum_{n=1}^{\infty} \frac{\varrho^n}{n} z^{-n} \right)
\end{align*}$$
in $\Delta_0(\infty, |\varrho|)$. Note that the coefficient of $z^{-1}$ is $\zeta$.

2. Let $\zeta$ be the set of $m$ distinct points $\zeta_j$ in $\mathbb{C}$ ($j = 1, \ldots, m$) and consider the region $R_\zeta = \mathbb{C} - \zeta$. The matrix

$$\begin{align*}
(8) \quad A = A(\zeta) = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\Re \zeta_1 & \Re \zeta_2 & \cdots & \Re \zeta_m \\
\Im \zeta_1 & \Im \zeta_2 & \cdots & \Im \zeta_m
\end{pmatrix}
\end{align*}$$

associated with the region $R_\zeta$ will be instrumental in our reasoning. We shall also use the column vector $t$ whose components are $t_1, t_2, \ldots, t_m$ in $\mathbb{R}$. Let $S = S(\zeta)$ be the vector space of solution vectors $t$ of the equation $At = 0$ where $0$ is the transpose of $(0, 0, 0)$. Then

$$\dim S(\zeta) = m - \text{rank } A(\zeta).$$

With each column vector $t$ we associate $h_t = \sum_{j=1}^{m} t_j l_{\zeta_j}$. We will show that $t \mapsto h_t$ is a linear bijection: $S \to H_2(R_2)$, so that

$$\dim H_2(R_2) = m - \text{rank } A(\zeta).$$

First we prove that $h_t \in H_2(R_2)$ if $t \in S$. It is clear that $h_t$ belongs to $H(R_2)$ and is square integrable over some $\Delta_0(\zeta_j, \varrho_j)$ for every $j = 1, \ldots, m$. By (7) we see that

$$\begin{align*}
h_t(z) &= \Re \left( -\left( \sum_{j=1}^{m} t_j \right) \log z + \left( \sum_{j=1}^{m} \zeta_j t_j \right) z^{-1} + \sum_{n=2}^{\infty} a_m z^{-n} \right)
\end{align*}$$
where \( \alpha_n = n^{-1} \sum_{j=1}^{m} \varepsilon_j^n t_j \). Since \( At = 0 \), the coefficients of \( \log z \) and \( z^{-1} \) of \( h_t(z) \) must vanish and we obtain \( h_t(z) = \text{Re} \left( \sum_{n=2}^{\infty} \alpha_n z^{-n} \right) \), which by (6) shows that \( h_t \) is also square integrable over some \( A_0(\infty, \phi) \). Since \( C - A_0(\infty, \phi) \cap \sum_{j=1}^{m} A_0(\varepsilon_j, \eta_j) \) is compact we finally conclude that \( h_t \in H_2(\mathbb{R}^a) \), that is, \( t \mapsto h_t \) is a well defined mapping: \( S \to H_2(\mathbb{R}^a) \). Since \( \{l_{\varepsilon_j}\}_{j=1}^{m} \) is a linearly independent family, we see that \( t \mapsto h_t \) is a linear injection of \( S \) into \( H_2(\mathbb{R}^a) \).

Next we prove that it is surjective. Choose an arbitrary \( u \in H_2(\mathbb{R}^a) \). By (5) we have

\[
u(z) = \text{Re} \left( -t_j \log(z - \zeta_j) + \sum_{n=0}^{\infty} a_{jn}(z - \zeta_j)^n \right)
\]
in a certain \( A_0(\zeta_j, \eta_j) \) \((j = 1, \ldots, m)\). This determines the column vector \( t \) whose components are \( t_1, \ldots, t_m \). Observe that \( l_{\zeta_j} - l_{\zeta_m} \in H(\hat{C} - \{\zeta_j, \zeta_m\}) \) \((j = 1, \ldots, m)\) and vanishes at \( \infty \); here \( \hat{C} = C \cup \{\infty\} \), the extended complex plane. Consider the function

\[
h(z) = u(z) - \sum_{j=1}^{m-1} t_j (l_{\zeta_j}(z) - l_{\zeta_m}(z)).
\]

By (6) and the above remark, we see that \( h \in H(\hat{C} - \zeta_m) \), and

\[
h(z) = \text{Re} \left( \left( \sum_{j=1}^{m} t_j \right) l_{\zeta_m}(z) + \sum_{n=0}^{\infty} a_{mn}(z - \zeta_m)^n \right)
\]
in a certain \( A_0(\zeta_m, \eta_m) \). We denote by \( C \) the boundary of the disk \( A(\zeta_m, \sigma_m/2) : |z - \zeta_m| < \sigma_m/2 \). Then \( h \) is harmonic on \( \hat{C} - \partial A(\zeta_m, \sigma_m/2) \) and the Gauss theorem assures the vanishing of the flux of \( h \) across \( C \). On the other hand this flux is the sum of

\[
\left( \sum_{j=1}^{m} t_j \right) \oint_C d\zeta_m(z) = 2\pi \sum_{j=1}^{m} t_j
\]
and \( \oint_C d(\text{Re} \sum_{n=0}^{\infty} a_{mn}(z - \zeta_m)^n) = 0 \). Therefore \( \sum_{j=1}^{m} t_j = 0 \) and \( h \in H(\hat{C}) \). Since \( h(\infty) = 0 \), we conclude that \( h = 0 \) on \( C \), i.e.

\[
u(z) = h_t(z) = \text{Re} \left( \left( \sum_{j=1}^{m} \zeta_j t_j \right) z^{-1} + \sum_{n=2}^{\infty} a_n z^{-n} \right)
\]

with \( \alpha_n = n^{-1} \sum_{j=1}^{m} \varepsilon_j^n t_j \). Since \( u \in H_2(\mathbb{R}) \), (6) implies that \( \sum_{j=1}^{m} \zeta_j t_j = 0 \). By taking the real and imaginary parts we deduce \( At = 0 \), that is \( t \in \mathcal{S} \). Thus \( t \mapsto h_t : S \to H_2(\mathbb{R}) \) is a linear bijection and we have established (9).
3. Suppose $\zeta$ consists of at least four points. Then, since rank $A(\zeta)\leq 3$ in any case, we conclude by (9) that $\dim H_2(R)\geq 4 - 3 = 1$. Next suppose $\zeta$ consists of three points. Then $\dim H_2(R) = 3 - \text{rank } A(\zeta)$. Although rank $A(\zeta)\geq 1$, the equality here cannot occur since otherwise we would have $\zeta_1 = \zeta_2 = \zeta_3$ in $\zeta$. Therefore rank $A = 2$ or 3. In the latter case, $\dim H_2(R) = 0$. In the former case, $\dim H_2(R) = 1$.

The relation rank $A = 2$ is, in the present situation, equivalent to $\text{det } A = 0$, which in turn is equivalent to $\zeta_1$, $\zeta_2$, and $\zeta_3$ in $\zeta$ being collinear. If $\zeta$ consists of two points $\zeta_1$, $\zeta_2$, then rank $A(\zeta) = 2$ since $\zeta_1 \neq \zeta_2$. Therefore $\dim H_2(R) = 2 - \text{rank } A(\zeta)$. If $\zeta$ consists of one point, then rank $A = 1$ and again $\dim H_2(R) = 0$. The case $\zeta = \emptyset$ (empty) may be treated directly. In this case, in view of (6), any $u \in H_2(R)$ must belong to $u \in H(\hat{C})$ and $u(\infty) = 0$. Therefore $u \equiv 0$, that is $\dim H_2(R) = \dim H_2(\hat{C}) = 0$.

Note that $R \subset R'$ implies $H_2(R') \subset H_2(R)$. We have proved:

Theorem. The space $H_2(R)$ is degenerate, i.e. $H_2(R) = \{0\}$, if and only if $C - R$ consists of at most two points or three noncollinear points.

4. In our earlier paper [3] we considered the degenerate class $O_{H_2}$ of Riemannian manifolds $M$ for which $H_2(M) = \{0\}$. We also considered the class $O_{SH_2}$ of those $M$ which contain a subregion $N \in O_{H_2}$ with an exterior point. For plane regions $R$ we have thus determined the classes $O_{H_2}$ and $O_{SH_2}$ as follows: $R \in O_{H_2}$ if and only if $C - R$ contains at most two points or three noncollinear points; $O_{SH_2} = \emptyset$.

5. Consider the set $\zeta = \{\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_m\}$, $m \geq 3$, with the noncollinear points $\zeta_1 = 1 + i$, $\zeta_2 = 1$, $\zeta_3 = 0$. We have rank $A = 3$ and

$$\dim H_2(R_{\zeta}) = m - 3.$$  

The set $R^{(k)} = R_{\zeta}$ with $m = 3 + k$ for $k = 0, 1, 2, \ldots$ satisfies

$$\dim H_2(R^{(k)}) = k$$  

for $k = 0, 1, 2, \ldots, \aleph$. Here $\aleph$ is the countably infinite cardinal number and $R^{(\aleph)} = R_{\zeta} = C - \zeta$ with $\zeta = \{1 + i, 1, 0, \zeta_4, \zeta_5, \ldots\}$ closed and all points in $\zeta$ distinct. We still have to prove (10) for $K = \aleph$. Let $\zeta(m) = \{1 + i, 1, 0, \zeta_4, \zeta_5, \ldots, \zeta_m\}$ ($m = 4, 5, \ldots$). Observe that area integrals over $R_{\zeta}$ are identical with those over any $R_{\zeta(m)}$ and a fortiori $u \to u|_{R_{\zeta}}$ is an isometric injection: $H_2(R_{\zeta(m)}) \to H_2(R_{\zeta})$. Therefore $\dim H_2(R_{\zeta}) \equiv \dim H_2(R_{\zeta(m)}) = m - 3$ for every $m = 4, 5, \ldots$. On the other hand, since $L_2(R)$ is separable for any subregion (and actually for any measurable subset) $R$ of $C$, $\dim H_2(R) \equiv \aleph$ as a closed subspace of $L_2(R)$ with $\dim L_2(R) = \aleph$. We thus deduce (10) also for $k = \aleph$. In summary:

Theorem. The dimension of $H_2(R)$ for any plane region $R$ is at most countably infinite and there actually exists a plane region $R$ such that the dimension of $H_2(R)$ is an arbitrarily preassigned countable cardinal number.
6. Let \( R \) be a plane region and \( \zeta \in R \). Then \( H_2(R) \) is a closed subspace of \( H_2(R - \zeta) \). We denote by \( H_2(R)_{\zeta}^\perp \) the orthogonal complement of \( H_2(R) \) in \( H_2(R - \zeta) \):

\[
H_2(R - \zeta) = H_2(R) \oplus H_2(R)_{\zeta}^\perp.
\]

It may happen that \( H_2(R)_{\zeta}^\perp = \{0\} \). In this case we say that \( (R, \zeta) \) is a noneffective pair. Otherwise we assert that \( \dim H_2(R)_{\zeta}^\perp = 1 \). In fact, suppose \( u_1, u_2 \in H_2(R)_{\zeta}^\perp \). In view of (5)

\[
u_j(z) = \text{Re} \left( -c_j \log (z - \zeta) + \sum_{n=0}^{\infty} a_n(z - \zeta)^n \right)
\]

in a certain \( \Delta_0(\zeta, \varrho) \), where \( c_j \in \mathbb{R} \) with \( c_j \neq 0 \) \((j = 1, 2)\). Therefore \( c_2 u_1 - c_1 u_2 \in H_2(R) \) and we have \( (u_j, c_2 u_1 - c_1 u_2) = 0 \) \((j = 1, 2)\). From these it follows that \( \|c_2 u_1 - c_1 u_2\|^2_R = 0 \), i.e. \( u_1 \) and \( u_2 \) are linearly dependent. We say that \( (R, \zeta) \) is an effective pair if it is noneffective. For an effective pair \( (R, \zeta) \) we have seen that \( H_2(R)_{\zeta}^\perp \) has a single generator:

\[
H_2(R)_{\zeta}^\perp = RH_R(\zeta, \zeta)
\]

where the generator \( H_R(\zeta, \zeta) \) is so normalized that

\[
H_R(z, \zeta) = \text{Re} \left( -\log (z - \zeta) + \sum_{n=0}^{\infty} a_n(z - \zeta)^n \right).
\]

7. We next discuss the effectiveness of a point \( \zeta \) in a given plane region \( R \). There are three cases:

Case 1. \( (R, \zeta) \) is a noneffective pair for every \( \zeta \in R \).

Case 2. \( (R, \zeta) \) is an effective pair for some \( \zeta \in R \), noneffective for some other \( \zeta \in R \).

Case 3. \( (R, \zeta) \) is an effective pair for every \( \zeta \in R \).

We call \( R \) a weak, unstable, or strong region according as case 1, 2, 3 occurs. We remark that for \( R \subset R' \), if \( R' \) is strong so is \( R \).

Once more we consider \( R_\zeta = C - \zeta \) with \( \zeta \) the set \( \{\zeta_1, \ldots, \zeta_m\} \) of distinct points in \( C \). Let \( \zeta \in R_\zeta \) be an arbitrary point and let \( \zeta' = \zeta \cup \{\zeta\} \). Similarly let \( A(\zeta) \) and \( A(\zeta') \) be the matrices (8) associated with \( \zeta \) and \( \zeta' \). Then by (9) we deduce

\[
\dim H_2(R_\zeta)_{\zeta'}^\perp = 1 - (\text{rank } A(\zeta') - \text{rank } A(\zeta)),
\]

and conclude that \( (R_\zeta, \zeta) \) is effective if and only if

\[
\text{rank } A(\zeta) = \text{rank } A(\zeta').
\]

Suppose \( m \geq 3 \). Recall that \( \text{rank } A(\zeta) \geq 2 \). From this we can easily see that (15) is valid for every choice of \( \zeta \in R_\zeta \) if and only if \( \text{rank } A(\zeta) = 3 \), i.e. there are three noncollinear points in \( \zeta \). If \( m \geq 1 \), then \( \zeta' \) contains at most two points and therefore \( H_2(R_\zeta - \zeta) = H_2(R_\zeta') = H_2(R_\zeta) = \{0\} \) and a fortiori \( \dim H_2(R)_{\zeta}^\perp = 0 \). In
the remaining case $\zeta$ contains at least two points and $\zeta$ is collinear. Let $l$ be the
line on which every point of $\zeta$ lies. In this case $\text{rank } A(\zeta) = 2$. If $\zeta \notin l$, then
$\text{rank } A(\zeta') = 2$ and $(R_\zeta, \zeta)$ is effective. If $\zeta \in l$, then $\text{rank } A(\zeta') = 3$ and
$(R_\zeta, \zeta)$ is noneffective.

We summarize our observations thus far:

**Theorem.** A plane region $R$ is strong if and only if $C - R$ contains three non-
collinear points, $R$ is weak if and only if $C - R$ contains at most one point, and $R$
is unstable if and only if $C - R$ contains at least two points and is a proper subset
of a line.

8. In nos. 8—11 we will concentrate on strong regions $R$. We recall that they
are characterized by the existence of three noncollinear points $\zeta_1, \zeta_2, \zeta_3$, in $C - R$.
The region $R_0 = C - \{\zeta_1, \zeta_2, \zeta_3\}$ is of course strong and by virtue of $R_0 \supset R$ we
have $H_2(R_0) \subset H_2(R)$. For each $\zeta \in R$ we consider the function

$$h(z, \zeta) = \left\{ \sum_{j=1}^{3} t_j(\zeta) l_j(z) \right\} + l(\zeta(z))$$

where the $t_j = t_j(\zeta) \in R \ (j = 1, 2, 3)$ will be later so chosen that $h(\cdot, \zeta) \in H_2(R_0 - \zeta) \subset
H_2(R - \zeta)$. Let the Laurent expansion of $h(\cdot, \zeta)$ about $\infty$ be

$$h(z, \zeta) = \text{Re} \left\{ - \left( \left( \sum_{j=1}^{3} t_j \right) + 1 \right) \log z + \left( \left( \sum_{j=1}^{3} \zeta_j t_j \right) + \zeta \right) z^{-1} + \sum_{n=2}^{\infty} \zeta_n z^{-n} \right\}.$$  

In order that $h(\cdot, \zeta) \in H_2(R_0 - \zeta)$ it is necessary and sufficient that the $t_j = t_j(\zeta)$
$(j = 1, 2, 3)$ satisfy the equation

$$\begin{vmatrix}
1 & 1 & 1 \\
\text{Re } \zeta_1 & \text{Re } \zeta_2 & \text{Re } \zeta_3 \\
\text{Im } \zeta_1 & \text{Im } \zeta_2 & \text{Im } \zeta_3
\end{vmatrix} = - \begin{vmatrix}
1 \\
\text{Re } \zeta \\
\text{Im } \zeta
\end{vmatrix}.$$  

The determinant of the first matrix is nonzero since $\zeta_1, \zeta_2, \zeta_3$ are not collinear. Therefore
by the Cramer formula the solution vector with components $t_j = t_j(\zeta)$ $(j = 1, 2, 3)$
in $R$ satisfying (17) is uniquely determined, and continuous (in fact harmonic) with
respect to $\zeta$. The function $h(\cdot, \zeta)$ in (16) with the $t_j = t_j(\zeta)$ $(j = 1, 2, 3)$ thus determined
by (17) belongs to $H_2(R_0 - \zeta) \subset H_2(R - \zeta)$ and we also have $h(z, \cdot) \in H(R - \zeta)$.

9. We next study the continuity of the mapping $\zeta \to h(z, \zeta) : R \to L_2(R)$. We can write

$$h(z, \zeta) = \text{Re} \left\{ \sum_{n=2}^{\infty} n^{-1} \zeta_n(\zeta) z^{-n} \right\}$$

in some $A_0(\infty, \sigma)$ with $\zeta_n(\zeta) = \sum_{j=1}^{3} t_j(\zeta) \zeta_j^n + \zeta^n \ (n = 2, 3, \ldots)$. Suppose $\zeta, \zeta' \in A(0, \sigma)$:
$|z| < \sigma \ (j = 1, 2, 3)$. Then the continuity of $\zeta \to t_j(\zeta)$ implies the existence of a constant
$K_\sigma > 0$ such that $|\zeta_n(\zeta)| \leq K_\sigma \sigma^n \ (n = 2, 3, \ldots)$, and we obtain $|h(z, \zeta)| \leq
K_\sigma \sum_{n=2}^{\infty} n^{-1} \sigma^n |z|^{-n}$ on $A_0(\infty, \sigma)$ with $\sigma > \sigma$. Therefore

$$|h(\cdot, \zeta)|_{A_0(\infty, \sigma)} \leq \sqrt{\pi/2} K_\sigma \sigma^n / (\sigma - \sigma).$$
Let \( \zeta, \zeta' \in R \) be contained in \( A(0, \sigma), \sigma > \sigma \), and \( A = A(0, \sigma) - A(0, \sigma) \). Then
\[
\|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{A}^{2} \equiv \|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{A}^{2},
\]
which is dominated by the sum of
\[
\|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{A}^{2}, \|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{A}^{2}, \text{ and } \|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{A}^{2}.
\]
The first of these three terms is dominated by
\[
\left( \sum_{j=1}^{3} \left| t_{j}(\zeta) - t_{j}(\zeta') \right| \right)^{2} + \left| l_{\zeta} - l_{\zeta'} \right|_{A}^{2},
\]
the second by
\[
\pi(\sigma^{2} - \sigma^{2})(\sup_{z \in A} |h(z, \zeta) - h(z, \zeta')|)^{2},
\]
and the last by
\[
\left( \|h(\cdot, \zeta)\|_{A}^{2} + \|h(\cdot, \zeta')\|_{A}^{2} \right)^{2},
\]
which is not greater than
\[
2\pi K\sigma^{4}/(\sigma - \sigma)^{2} \quad \text{by (18).}
\]
We conclude that
\[
\lim_{\zeta \to \zeta'} \sup_{z \in A} |h(\cdot, \zeta) - h(\cdot, \zeta')|_{A} = \sqrt{2\pi K\sigma^{4}/(\sigma - \sigma)}.
\]
On letting \( \sigma \to +\infty \) we obtain
\[
\lim_{\zeta \to \zeta'} \|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{A} = 0.
\]

10. Let \( u_{\zeta}(z) = h(z, \zeta) - H\_2(z, \zeta) \). Clearly \( u_{\zeta} \in H\_2(R) \) and thus the projection of \( h(\cdot, \zeta) \in H\_2(R-\zeta) \) on \( H\_2(R) = H\_2(R, \zeta) \) is \( H\_2(R, \zeta) \). Observe that
\[
\|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{R}^{2} = \|H\_R(\cdot, \zeta) - H\_R(\cdot, \zeta')\|_{R}^{2} + \|u_{\zeta} - u_{\zeta'}\|_{R}^{2}.
\]
We have obtained the following

\textbf{Theorem.} On a strong region \( R \)
\[
\lim_{\zeta \to \zeta'} \|H\_R(\cdot, \zeta) - H\_R(\cdot, \zeta')\|_{R} = 0.
\]

11. Let \( R' \) and \( R'' \) be subregions of a strong region \( R \), hence strong regions. If \( R' \subset R'' \), then \( H\_R(\cdot, \zeta) \) belongs to \( H\_R(R') \) and is orthogonal to \( H\_R(R', \zeta) \) over \( R' \). Therefore
\[
\|H\_R(\cdot, \zeta) - H\_R(\cdot, \zeta')\|_{R'}^{2} = \|H\_R(\cdot, \zeta)\|_{R'}^{2} - \|H\_R(\cdot, \zeta')\|_{R'}^{2}.
\]
We denote by \( \{\Omega\} \) the family of regular subregions \( \Omega \) of \( R \), a directed set by inclusion, and set \( H\_\Omega(z, \zeta) = 0 \) for \( z \in R - \Omega \). In view of (20), \( \{\|H\_\Omega(\cdot, \zeta)\|_{R}\}_{\Omega \in \{\Omega\}} \) is an increasing net and \( \{H\_\Omega(\cdot, \zeta)\}_{\Omega \in \{\Omega\}} \) is a Cauchy net in \( L\_2(R) \). It is easy to check that the limit is \( H\_R(\cdot, \zeta) \) and
\[
\|H\_R(\cdot, \zeta) - H\_\Omega(\cdot, \zeta)\|_{R}^{2} = \|H\_R(\cdot, \zeta)\|_{R}^{2} - \|H\_\Omega(\cdot, \zeta)\|_{R}^{2}.
\]
On any compact subset \( E \) of \( R \), both \( \|H\_\Omega(\cdot, \zeta)\|_{R}^{2} \) (\( \Omega \supset E \)) and \( \|H\_R(\cdot, \zeta)\|_{R}^{2} \) are continuous (see (19)). Thus the Dini theorem implies:

\textbf{Theorem.} If \( R \) is strong, then for any compact subset \( E \) of \( R \)
\[
\lim_{\Omega \to R} \sup_{\zeta \in E} \|H\_R(\cdot, \zeta) - H\_\Omega(\cdot, \zeta)\|_{R} = 0.
\]
12. We remark that the biharmonic Green's function of the disk \( D(0, \varrho) \) is (cf. e.g. Garabedian [2])

\[
\beta_{D(0, \varrho)}(z, \zeta) = \frac{1}{8\pi} \left[ |z - \zeta|^2 \log \left| \frac{\varrho(z - \zeta)}{\varrho^2 - \varrho^2 z} \right| + \frac{1}{2\varrho^2} (|z|^2 - \varrho^2)(|\zeta|^2 - \varrho^2) \right]
\]

on \( D(0, \varrho) \times D(0, \varrho) \). Hence, clearly \( C \in O_\beta \). Similarly, the biharmonic Green's function of the punctured disk \( D_0(0, \varrho) \) is (cf. [4], [5])

\[
\beta_{D_0(0, \varrho)}(z, \zeta) = \beta_{D(0, \varrho)}(z, \zeta) - |6\pi \varrho^{-2} \beta_{D(0, \varrho)}(z, 0) \beta_{D(0, \varrho)}(\zeta, 0)|
\]

on \( D_0(0, \varrho) \times D_0(0, \varrho) \). Therefore \( C - \{0\} \in O_\beta \).

In view of the above, relation (4), and the theorem in no. 7, the assertions 1° and 3° can easily be deduced. By nos. 9 and 10 the first three assertions in 2° are clear. Since \( \beta_{\Omega}(\cdot, \zeta) \) is biharmonic on \( \Omega - \zeta \) and uniformly convergent to \( \beta_R(\cdot, \zeta) \) on every compact subset of \( R - \zeta \), \( \beta_R(\cdot, \zeta) \) is also biharmonic on \( R - \zeta \).

References


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