Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 4, 1978/1979, 227—252

ON GENERALIZED A PRIORI ESTIMATES FOR QUASI-ELLIPTIC DIFFERENTIAL OPERATORS

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Introduction

Let P be a quasi-elliptic operator and R another operator such that the principal parts of P and R are quasi-homogeneous of the same degree, and let $B=(B_1, \ldots, B_\varkappa)$ and $Q=(Q_1, \ldots, Q_\varkappa)$ be operator vectors; here all operators are linear partial differential operators with constant coefficients.

The present paper is concerned with the validity of the generalized a priori estimate

$$(1) \qquad \|R(D)u\|^2 \leqq C\left(\|P(D)u\|^2+S(\gamma_0 B(D)u, \gamma_0 Q(D)u)+\|u\|^2\right), \qquad u\in C_0^\infty[\overline{\boldsymbol{R}}^n_+],$$

where the function $S(\cdot,\ \cdot)$ connecting the boundary values is defined by

$$S(U, V) = \int |s_\xi(F_x U, F_x V)|\, d\xi, \qquad U, V\in C_0^\infty(\boldsymbol{R}^{n-1}; \boldsymbol{C}^\varkappa),$$

with a function $\xi\mapsto s_\xi(\cdot,\ \cdot)$ from $\boldsymbol{R}^{n-1}\setminus\{0\}$ into the space of sesquilinear forms on $\boldsymbol{C}^\varkappa\times\boldsymbol{C}^\varkappa$.

In Sections 2—4 we consider the homogeneous case, i.e., the case of the principal parts, and give sufficient and necessary conditions for the estimate (1) (without the term $\|u\|^2$, of course) to be fulfilled. Among other things, we make here use of some results of our previous paper [10] in which we proved a result of V. G. Maz'ja and I. V. Gel'man [6] by the methods of M. Schechter [12].

The general non-homogeneous case is then discussed in Sections 5—8. The result is that (1) holds if and only if the principal parts satisfy the conditions given in the homogeneous case.

In the last section we shall note that for $B=Q$ the results obtained here imply a known result (see [6], [10]). As an example of applications of the results, we then consider certain types of mixed boundary value problems and give sufficient conditions for their coerciveness.

doi:10.5186/aasfm.1978-79.0420

1. Preliminaries

1.1. We first introduce some notations convenient for our purposes. Let $R^n_+=\{y=(x,t)=(x_1,\dots,x_{n-1},t)\in R^n | t>0\}$ and $\bar{R}^n_+=\{y=(x,t)\in R^n | t\geqq 0\}$. By $C_0^\infty[\bar{R}^n_+]$ we denote the space of restrictions to $\bar{R}^n_+$ of the functions of $C_0^\infty(R^n)$. The trace (restriction) operator γ_0: $C_0^\infty[\bar{R}^n_+]\to C_0^\infty(R^{n-1})$ is defined by $(\gamma_0 u)(x)==u(x,0)$. Let $\bar{z}$ be the complex conjugate of $z=\operatorname{Re} z+i\operatorname{Im} z\in C$, $|z|^2=z\bar{z}$, and let $\|\cdot\|$ stand for the norm of the space $L^2(R^n_+)$,

$$\|u\| = \int_{R^n_+} |u(y)|^2\,dy, \qquad u\in L^2(R^n_+).$$

Let $\mathscr{F}=\mathscr{F}_y$, $\mathscr{F}_x$, and $\mathscr{F}_t$ denote the Fourier transformations in R^n, R^{n-1}, and R, respectively. The partial Fourier transforms of a suitable function u of $y=(x,t)$, with respect to x and t, are defined by (cf. [4], p. 24)

$$(\mathscr{F}_x u)(\xi,t) = \hat{u}(\xi,t) = \pi_{n-1}\int e^{-i\langle x,\xi\rangle}u(x,t)\,dx$$

and

$$(\mathscr{F}_t u)(x,\zeta) = \pi_1\int e^{-it\zeta}u(x,t)\,dt,$$

respectively, where $\pi_k=(2\pi)^{-k/2}$, other notations being indicated by the definition

$$\langle y,\eta\rangle = \langle x,\xi\rangle+t\zeta = x_1\xi_1+\dots+x_{n-1}\xi_{n-1}+t\zeta$$

with the dual variable $\eta=(\xi,\zeta)$ of $y=(x,t)$.

1.2. Let $m_1,\dots,m_n$ be positive integers, $\mu=\max\{m_k | k=1,\dots,n\}$, and $q=(q',q_n)=(q_1,\dots,q_{n-1},q_n)$ with $q_k=\mu/m_k$.

We set

$$\langle\xi\rangle = \left(\sum_{k=1}^{n-1}|\xi_k|^{m_k}\right)^{1/\mu}, \qquad \xi\in R^{n-1},$$

and

$$\langle\eta\rangle = (\langle\xi\rangle^\mu+|\zeta|^{m_n})^{1/\mu}, \qquad \eta=(\xi,\zeta)\in R^n.$$

We shall consider the polynomial corresponding (via the Fourier transformation) to the linear partial differential operator

$$P(D) = P(D_x,D_t) = \sum_{\langle\alpha,q\rangle\leqq\mu} p_\alpha D^\alpha = \sum_{\langle\alpha,q\rangle\leqq\mu} p_\alpha D_x^{\alpha'}D_t^{\alpha_n}$$

with constant coefficients $p_\alpha\in C$, i.e., the polynomial

$$P(\eta) = P(\xi,\zeta) = \sum_{\langle\alpha,q\rangle\leqq\mu} p_\alpha\eta^\alpha = \sum_{\langle\alpha,q\rangle\leqq\mu} p_\alpha\xi^{\alpha'}\zeta^{\alpha_n}.$$

Here α denotes a multi-index $\alpha=(\alpha',\alpha_n)=(\alpha_1,\dots,\alpha_{n-1},\alpha_n)\in N^n$, $\eta^\alpha=\xi^{\alpha'}\zeta^{\alpha_n}=\xi_1^{\alpha_1}\dots\xi_{n-1}^{\alpha_{n-1}}\zeta^{\alpha_n}$, and $D^\alpha=D_x^{\alpha'}D_t^{\alpha_n}=D_1^{\alpha_1}\dots D_{n-1}^{\alpha_{n-1}}D_n^{\alpha_n}$ with $D_k=-i\partial/\partial x_k$ for $k<n$, $D_n=-i\partial/\partial t$.

The principal part $P^0(\eta)$ of $P(\eta)$, defined by

$$P^0(\eta) = \sum_{\langle\alpha,q\rangle=\mu} p_\alpha\eta^\alpha,$$

is q-homogeneous with $q\text{-deg}\, P^0 = \mu$; generally speaking, we say that a function h in $\mathbf{R}^n \setminus \{0\}$ is q-homogeneous of degree $r \in \mathbf{R}$, and write $q\text{-deg}\, h = r$, if

$$h(t^q \eta) = t^r h(\eta), \qquad \eta \in \mathbf{R}^n \setminus \{0\},$$

for all $t > 0$, where

$$t^q \eta = (t^{q'} \xi, t^{q_n} \zeta) = (t^{q_1} \xi_1, \dots, t^{q_{n-1}} \xi_{n-1}, t^{q_n} \zeta).$$

The q'-homogeneity in ξ is defined analogously.

The polynomial $P(\eta)$ (or the operator $P(D)$) is called quasi-elliptic of determined type $K^+ \geqq 1$ (cf. [3], [12]) if the following two conditions are satisfied:

(i) $P^0(\eta) \neq 0$ for every $\eta \in \mathbf{R}^n \setminus \{0\}$ or, equivalently (see [4], p. 103), we have with some constant $C > 0$

$$|P^0(\eta)| \geqq C \langle \eta \rangle^\mu, \qquad \eta \in \mathbf{R}^n.$$

(ii) For each $\xi \in \mathbf{R}^{n-1} \setminus \{0\}$ the equation $P^0(\xi, z) = 0$ has exactly K^+ solutions $z = \zeta(\xi)$ with $\operatorname{Im} \zeta(\xi) > 0$.

1.3. From now on let P be quasi-elliptic, and suppose that P is q-homogeneous, i.e., that $P = P^0$. For each $\xi \in \mathbf{R}^{n-1} \setminus \{0\}$ the different roots of the polynomial $P(\xi, \zeta)$ are denoted by $\zeta_\alpha(\xi)$ and their multiplicities by $k_\alpha(\xi)$, respectively.

We make the following assumption:

Hypothesis (A). *If* $\alpha \neq \beta$, *then*

$$\zeta_\alpha(\xi) \neq \zeta_\beta(\xi), \qquad \xi \in \mathbf{R}^{n-1} \setminus \{0\},$$

that is to say, $k_\alpha(\xi)$ *does not depend on* ξ.

Note that ζ_α can be assumed to be continuous in $\mathbf{R}^{n-1}_\xi$.

Remark. In [10], p. 334, the reference to Hypothesis (A) is not relevant.

Without loss of generality we may assume that there exist index sets

$$\Lambda = \{1, \dots, \lambda\}, \qquad \Lambda^+ = \{1, \dots, \lambda^+\}, \qquad \Lambda^- = \Lambda \setminus \Lambda^+$$

with some numbers $1 \leqq \lambda^+ \leqq \lambda$ such that

$$P(\xi, \zeta) = \prod_{\alpha \in \Lambda} (\zeta - \zeta_\alpha(\xi))^{k_\alpha}$$

for all $\xi \in \mathbf{R}^{n-1} \setminus \{0\}$ (this means no real restriction; see [2]) and that

$$\operatorname{Im} \zeta_\alpha(\xi) > 0 \quad \text{for } \alpha \in \Lambda^+,$$
$$\operatorname{Im} \zeta_\alpha(\xi) < 0 \quad \text{for } \alpha \in \Lambda^-.$$

Consequently, the monic polynomial

$$P_+(\xi, \zeta) = \prod_{\alpha \in \Lambda^+} (\zeta - \zeta_\alpha(\xi))^{k_\alpha}$$

in ζ is of degree $\deg_\zeta P_+(\xi, \zeta) = k_1 + \dots + k_{\lambda^+} = K^+$.

1.4. Let $R(\xi,\zeta)$ be an arbitrary q-homogeneous polynomial with q-deg $R=\mu$, and let $\xi\in\boldsymbol{R}^{n-1}\setminus\{0\}$. If $M(\xi,\zeta)$ denotes the greatest common monic divisor of the polynomials $P_+(\xi,\zeta)$ and $R(\xi,\zeta)$ (in ζ), we set

$$P'_+(\xi,\zeta)=P_+(\xi,\zeta)/M(\xi,\zeta)$$

and assume

Hypothesis (B). *The degree of* $P'_+(\xi,\zeta)$ *in* ζ *is a positive constant* $\varkappa$ *for all* $\xi\in\boldsymbol{R}^{n-1}\setminus\{0\}$.

Remark. The assumption $\varkappa>0$ is a consequence of the nature of our problem. For the case $\varkappa=0$ we refer to [6].

It can be supposed that there is an index set $\Lambda'=\{1,\dots,\lambda'\}$ with $1\leqq\lambda'\leqq\lambda^+$ such that

$$P'_+(\xi,\zeta)=\prod_{\alpha\in\Lambda'}(\zeta-\zeta_\alpha(\xi))^{\varkappa_\alpha},$$

where $\varkappa_\alpha\leqq k_\alpha$ and $\varkappa_1+\dots+\varkappa_{\lambda'}=\varkappa=\deg_\zeta P'_+(\xi,\zeta)\geqq 1$. Further, put $\varkappa_\alpha=k_\alpha$ for $\alpha\in\Lambda\setminus\Lambda'$ and $\varrho_\alpha=\{0,\dots,\varkappa_\alpha-1\}$ for $\alpha\in\Lambda$.

1.5. For every $\xi\in\boldsymbol{R}^{n-1}\setminus\{0\}$ let $s_\xi\colon \boldsymbol{C}^\varkappa\times\boldsymbol{C}^\varkappa\to\boldsymbol{C}$ be a sesquilinear form and $\boldsymbol{S}(\xi)=(S_{jk}(\xi))$ the corresponding $\varkappa\times\varkappa$-matrix. Thus

$$s_\xi(X,Y)=X\boldsymbol{S}(\xi)\bar{Y}^*=\sum S_{jk}(\xi)X_j\bar{Y}_k$$

for all $X=(X_j)$ and $Y=(Y_k)$ from $\boldsymbol{C}^\varkappa$, where the symbol * denotes the matrix transpose. Moreover, let there be set

$$S(U,V)=\int|s_\xi(\mathscr{F}_xU,\mathscr{F}_xV)|\,d\xi$$

for $U,V\in C_0^\infty(\boldsymbol{R}^{n-1};\boldsymbol{C}^\varkappa)$.

The homogeneous case

2. The problem

Theorem 2.1. *Suppose that* $P(\xi,\zeta)$ *is a* q*-homogeneous quasi-elliptic polynomial of determined type and* $R(\xi,\zeta)$ *a* q*-homogeneous polynomial such that* q-deg $P=q$-deg $R=\mu$ *and Hypotheses (A) and (B) are satisfied,* $\deg_\zeta P'_+(\xi,\zeta)=\varkappa$.

Let

$$B(\xi,\zeta)=(B_1(\xi,\zeta),\dots,B_\varkappa(\xi,\zeta))$$

and

$$Q(\xi,\zeta)=(Q_1(\xi,\zeta),\dots,Q_\varkappa(\xi,\zeta))$$

be polynomial vectors such that $B_j(\xi,\zeta)$ *and* $Q_k(\xi,\zeta)$ *are* q*-homogeneous with* q-deg $B_j=\mu_j\leqq\mu-q_n$ *and* q-deg $Q_k=\nu_k\leqq\mu-q_n$. *The elements of the matrix* $\boldsymbol{S}(\xi)=(S_{jk}(\xi))$ *given in 1.5 are assumed to be continuous and* q'*-homogeneous in* ξ *with* q'-deg $S_{jk}=2\mu-\mu_j-\nu_k-q_n$.

Then the a priori estimate

$$\|R(D)u\|^2 \leqq C(\|P(D)u\|^2 + S(\gamma_0 B(D)u, \gamma_0 Q(D)u)) \tag{2.1}$$

is valid for all $u \in C_0^\infty[\bar{R}_+^n]$ *if and only if for each* $\xi \in R^{n-1} \setminus \{0\}$ *the following conditions (I)—(III) are satisfied:*

(I) *The matrix* $S(\xi)$ *is regular.*

(II) $B(\xi, \zeta) \equiv 0$ *and* $Q(\xi, \zeta) \equiv 0$ mod $M(\xi, \zeta)$.

(III) *Both the* $B_j(\xi, \zeta)$ *and the* $Q_k(\xi, \zeta)$ *are linearly independent modulo* $P_+(\xi, \zeta)$.

Note that here, as well as in the sequel, C shall denote a generic positive constant with the dependences permitted each time and is, if necessary, identified with a subscript.

The proof of Theorem 2.1 will be given in the following two sections.

3. The sufficiency of the conditions

3.1. First of all, if

$$R'(\xi, \zeta) = R(\xi, \zeta)/M(\xi, \zeta)$$

and

$$B'(\xi, \zeta) = B(\xi, \zeta)/M(\xi, \zeta) = (B_1'(\xi, \zeta), \dots, B_\varkappa'(\xi, \zeta)),$$
$$Q'(\xi, \zeta) = Q(\xi, \zeta)/M(\xi, \zeta) = (Q_1'(\xi, \zeta), \dots, Q_\varkappa'(\xi, \zeta)),$$

we observe that it suffices to prove

Theorem 3.1. *Under the assumptions of Theorem 2.1, the coerciveness inequality*

$$\|R'(D)v\|^2 \leqq C(\|P'(D)v\|^2 + S(\gamma_0 B'(D)v, \gamma_0 Q'(D)v)) \tag{3.1}$$

holds for all $v \in C_0^\infty[\bar{R}_+^n]$ *if for each* $\xi \in R^{n-1} \setminus \{0\}$ *we have:*

(I) *The matrix* $S(\xi)$ *is regular.*

(IV) *Both the* $B_j'(\xi, \zeta)$ *and the* $Q_k'(\xi, \zeta)$ *are linearly independent modulo* $P_+'(\xi, \zeta)$.

3.2. Let $\xi \in R^{n-1} \setminus \{0\}$ and set

$$P_\alpha'(\xi, \zeta) = (\zeta - \zeta_\alpha(\xi))^{-\varkappa_\alpha} P'(\xi, \zeta) \quad \text{for } \alpha \in \Lambda' \cup \Lambda^-,$$
$$P_{+\alpha}'(\xi, \zeta) = (\zeta - \zeta_\alpha(\xi))^{-\varkappa_\alpha} P_+'(\xi, \zeta) \quad \text{for } \alpha \in \Lambda',$$
$$P_{-\alpha}(\xi, \zeta) = (\zeta - \zeta_\alpha(\xi))^{-\varkappa_\alpha} P_-(\xi, \zeta) \quad \text{for } \alpha \in \Lambda^-,$$

where $P_-(\xi, \zeta) = P(\xi, \zeta)/P_+(\xi, \zeta)$.

We first state a result obtained in [10]:

Lemma 3.2. *Suppose* $P(\xi, \zeta)$ *and* $R(\xi, \zeta)$ *are as in Theorem 2.1. Then for all* $\xi \in R^{n-1} \setminus \{0\}$

$$\int_0^\infty |R'(\xi, D_t)\hat{v}(\xi, t)|^2 dt \tag{3.2}$$
$$\leqq C\left(\int_0^\infty |P'(\xi, D_t)\hat{v}(\xi, t)|^2 dt + \sum_{\alpha \in \Lambda'} \sum_{\beta \in \varrho_\alpha} |\langle\xi\rangle^{(\varkappa_\alpha - \beta - 1/2)q_n} W_{\alpha, \varkappa_\alpha - \beta}(\xi)|^2\right)$$

for all $v \in C_0^\infty[\bar{R}_+^n]$, *where*

$$W_{\alpha k}(\xi) = \gamma_0(D_t - \zeta_\alpha(\xi))^{\varkappa_\alpha - k} P'_\alpha(\xi, D_t)\hat{v}(\xi, t), \qquad \alpha \in \Lambda' \cup \Lambda^-, \quad k = 1, \dots, \varkappa_\alpha.$$

3.3. Let $\xi \in R^{n-1} \setminus \{0\}$. There exist then polynomials $B'_{j+}(\xi, \zeta)$, $B_{j-}(\xi, \zeta)$ and $Q'_{k+}(\xi, \zeta)$, $Q_{k-}(\xi, \zeta)$ (in ζ) such that

$$\frac{B'_j(\xi, \zeta)}{P'(\xi, \zeta)} = \frac{B'_{j+}(\xi, \zeta)}{P'_+(\xi, \zeta)} + \frac{B_{j-}(\xi, \zeta)}{P_-(\xi, \zeta)} \tag{3.3}$$

and

$$\frac{Q'_k(\xi, \zeta)}{P'(\xi, \zeta)} = \frac{Q'_{k+}(\xi, \zeta)}{P'_+(\xi, \zeta)} + \frac{Q_{k-}(\xi, \zeta)}{P_-(\xi, \zeta)}. \tag{3.4}$$

Using (3.3), (3.4), and the Lagrange—Sylvester interpolation formula, we thus obtain

$$\begin{aligned}\gamma_0 B'_j(\xi, D_t)\hat{v}(\xi, t) = &\sum_{\alpha \in \Lambda'} \sum_{\beta \in \varrho_\alpha} b^+_{j\alpha\beta}(\xi) W_{\alpha, \varkappa_\alpha - \beta}(\xi) \\ &+ \sum_{\alpha \in \Lambda^-} \sum_{\beta \in \varrho_\alpha} b^-_{j\alpha\beta}(\xi) W_{\alpha, \varkappa_\alpha - \beta}(\xi)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\gamma_0 Q'_k(\xi, D_t)\hat{v}(\xi, t) = &\sum_{\gamma \in \Lambda'} \sum_{\delta \in \varrho_\gamma} q^+_{k\gamma\delta}(\xi) W_{\gamma, \varkappa_\gamma - \delta}(\xi) \\ &+ \sum_{\gamma \in \Lambda^-} \sum_{\delta \in \varrho_\gamma} q^-_{k\gamma\delta}(\xi) W_{\gamma, \varkappa_\gamma - \delta}(\xi),\end{aligned} \tag{3.6}$$

where

$$b^+_{j\alpha\beta}(\xi) = \frac{1}{\beta!}\left(\frac{\partial^\beta}{\partial\zeta^\beta}\frac{B'_{j+}(\xi, \zeta)}{P'_{+\alpha}(\xi, \zeta)}\right)\bigg|_{\zeta = \zeta_\alpha(\xi)}, \qquad \alpha \in \Lambda', \quad \beta \in \varrho_\alpha, \tag{3.7}$$

$$b^-_{j\alpha\beta}(\xi) = \frac{1}{\beta!}\left(\frac{\partial^\beta}{\partial\zeta^\beta}\frac{B_{j-}(\xi, \zeta)}{P_{-\alpha}(\xi, \zeta)}\right)\bigg|_{\zeta = \zeta_\alpha(\xi)}, \qquad \alpha \in \Lambda^-, \quad \beta \in \varrho_\alpha, \tag{3.8}$$

and, respectively,

$$q^+_{k\gamma\delta}(\xi) = \frac{1}{\delta!}\left(\frac{\partial^\delta}{\partial\zeta^\delta}\frac{Q'_{k+}(\xi, \zeta)}{P'_{+\gamma}(\xi, \zeta)}\right)\bigg|_{\zeta = \zeta_\gamma(\xi)}, \qquad \gamma \in \Lambda', \quad \delta \in \varrho_\gamma, \tag{3.9}$$

$$q^-_{k\gamma\delta}(\xi) = \frac{1}{\delta!}\left(\frac{\partial^\delta}{\partial\zeta^\delta}\frac{Q_{k-}(\xi, \zeta)}{P_{-\gamma}(\xi, \zeta)}\right)\bigg|_{\zeta = \zeta_\gamma(\xi)}, \qquad \gamma \in \Lambda^-, \quad \delta \in \varrho_\gamma. \tag{3.10}$$

Define now matrices $\boldsymbol{b}^+(\xi)$, $\boldsymbol{b}^-(\xi)$, $\boldsymbol{q}^+(\xi)$, $\boldsymbol{q}^-(\xi)$ by

$$\begin{aligned}\boldsymbol{b}^+(\xi) &= (b^+_{j\alpha\beta}(\xi)) \quad (\alpha \in \Lambda', \quad \beta \in \varrho_\alpha, \quad j = 1, \dots, \varkappa), \\ \boldsymbol{b}^-(\xi) &= (b^-_{j\alpha\beta}(\xi)) \quad (\alpha \in \Lambda^-, \quad \beta \in \varrho_\alpha, \quad j = 1, \dots, \varkappa), \\ \boldsymbol{q}^+(\xi) &= (q^+_{k\gamma\delta}(\xi)) \quad (\gamma \in \Lambda', \quad \delta \in \varrho_\gamma, \quad k = 1, \dots, \varkappa), \\ \boldsymbol{q}^-(\xi) &= (q^-_{k\gamma\delta}(\xi)) \quad (\gamma \in \Lambda^-, \quad \delta \in \varrho_\gamma, \quad k = 1, \dots, \varkappa),\end{aligned}$$

and vectors $W^+(\xi)\in \boldsymbol{C}^{\varkappa}$ and $W^-(\xi)\in \boldsymbol{C}^{K^-}$ $(K^-=\deg_\zeta P_-(\xi,\zeta))$ by

$$W^+(\xi) = \big(W_{\alpha,\varkappa_\alpha-\beta}(\xi)\big) \quad (\alpha\in\Lambda', \quad \beta\in\varrho_\alpha),$$
$$W^-(\xi) = \big(W_{\alpha,\varkappa_\alpha-\beta}(\xi)\big) \quad (\alpha\in\Lambda^-, \quad \beta\in\varrho_\alpha).$$

Then (3.5) and (3.6) can be written, taken all together, in the forms

$$(3.11) \qquad \gamma_0 B'(\xi, D)\hat{v}(\xi, t) = W^+(\xi)\boldsymbol{b}^+(\xi)+W^-(\xi)\boldsymbol{b}^-(\xi)$$

and

$$(3.12) \qquad \gamma_0 Q'(\xi, D)\hat{v}(\xi, t) = W^+(\xi)\boldsymbol{q}^+(\xi)+W^-(\xi)\boldsymbol{q}^-(\xi),$$

respectively.

3.4. It follows from (IV) that the matrices $\boldsymbol{b}^+(\xi)$ and $\boldsymbol{q}^+(\xi)$ are regular (cf. [10]). Then, by (I), the matrix $\boldsymbol{b}^+(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^+(\xi)^*}$ is regular, too. Hence

$$(3.13) \qquad \omega\boldsymbol{b}^+(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^+(\xi)^*}\,\overline{\omega^*} \neq 0 \quad \text{for} \quad \omega\in\boldsymbol{C}^{\varkappa}\setminus\{0\}.$$

In particular, let

$$\Omega = (\Omega_{\alpha\beta})_{\alpha\in\Lambda',\beta\in\varrho_\alpha}\in\boldsymbol{C}^{\varkappa}\setminus\{0\}$$

and set

$$\omega_{\alpha\beta}(\xi) = \langle\xi\rangle^{(\beta-\varkappa_\alpha+1/2)q_n}\Omega_{\alpha\beta}.$$

Then by (3.13) we have for $\omega(\xi)=\big(\omega_{\alpha\beta}(\xi)\big)$

$$(3.14) \qquad |\omega(\xi)\boldsymbol{b}^+(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^+(\xi)^*}\,\overline{\omega(\xi)^*}| > 0.$$

Now the left side of (3.14) is homogeneous in the $\Omega_{\alpha\beta}$ of degree 2 and the coefficient of the term $\Omega_{\alpha\beta}\bar{\Omega}_{\gamma\delta}$ is q'-homogeneous in ξ of degree 0. In fact, by our assumption,

$$q'\text{-deg}\, S_{jk} = 2\mu-\mu_j-\nu_k-q_n,$$

and, furthermore, it follows from (3.7) and (3.9) that (cf. [10])

$$q'\text{-deg}\, b^+_{j\alpha\beta} = \mu_j-\mu+(\varkappa_\alpha-\beta)q_n,$$
$$q'\text{-deg}\, q^+_{k\gamma\delta} = \nu_k-\mu+(\varkappa_\gamma-\delta)q_n.$$

Hence, by continuity,

$$(3.15) \qquad |\omega(\xi)\boldsymbol{b}^+(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^+(\xi)^*}\,\overline{\omega(\xi)^*}| \geqq C|\Omega|^2$$

for all $\xi\in\boldsymbol{R}^{n-1}\setminus\{0\}$ and $\Omega\in\boldsymbol{C}^{\varkappa}$. Therefore, taking

$$\Omega_{\alpha\beta} = \Omega_{\alpha\beta}(\xi) = \langle\xi\rangle^{(\varkappa_\alpha-\beta-1/2)q_n}W_{\alpha,\varkappa_\alpha-\beta}(\xi),$$

we obtain from (3.15)

$$(3.16) \qquad \sum_{\alpha\in\Lambda'}\sum_{\beta\in\varrho_\alpha}|\langle\xi\rangle^{(\varkappa_\alpha-\beta-1/2)q_n}W_{\alpha,\varkappa_\alpha-\beta}(\xi)|^2 \leqq C_1|W^+(\xi)\boldsymbol{b}^+(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^+(\xi)^*}\,\overline{W^+(\xi)^*}|.$$

3.5. On the other hand, by (3.11) and (3.12) we have

$$
\begin{aligned}
(3.17)\qquad & s_\xi\big(\gamma_0 B'(\xi, D_t)\hat{v}(\xi, t), \gamma_0 Q'(\xi, D_t)\hat{v}(\xi, t)\big) \\
&= W^+(\xi)\boldsymbol{b}^+(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^+(\xi)^*}\,\overline{W^+(\xi)^*} + W^+(\xi)\boldsymbol{b}^+(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^-(\xi)^*}\,\overline{W^-(\xi)^*} \\
&\quad + W^-(\xi)\boldsymbol{b}^-(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^+(\xi)^*}\,\overline{W^+(\xi)^*} + W^-(\xi)\boldsymbol{b}^-(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^-(\xi)^*}\,\overline{W^-(\xi)^*}.
\end{aligned}
$$

It follows from (3.8) and (3.10) that

$$
\begin{aligned}
&|W^-(\xi)\boldsymbol{b}^-(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^-(\xi)^*}\,\overline{W^-(\xi)^*}| \\
&\qquad \leqq C \sum_{\substack{\alpha\in\Lambda^- \\ \beta\in\varrho_\alpha}} \sum_{\substack{\gamma\in\Lambda' \\ \delta\in\varrho_\gamma}} \langle\xi\rangle^{(\varkappa_\alpha-\beta)q_n+(\varkappa_\gamma-\delta)q_n-q_n} |W_{\alpha,\varkappa_\alpha-\beta}(\xi)|\,|W_{\gamma,\varkappa_\gamma-\delta}(\xi)|.
\end{aligned}
$$

Since (cf. [10])

$$
|W_{\alpha k}(\xi)| \leqq C\langle\xi\rangle^{-(k-1/2)q_n}\Big(\int_0^\infty |P'(\xi, D_t)\hat{v}(\xi, t)|^2\,dt\Big)^{1/2}, \qquad \alpha\in\Lambda^-,
$$

we thus get

$$
(3.18)\qquad |W^-(\xi)\boldsymbol{b}^-(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^-(\xi)^*}\,\overline{W^-(\xi)^*}| \leqq C\int_0^\infty |P'(\xi, D_t)\hat{v}(\xi, t)|^2\,dt.
$$

Next, let $\varepsilon>0$ be arbitrary. Then

$$
\begin{aligned}
(3.19)\qquad & |W^+(\xi)\boldsymbol{b}^+(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^-(\xi)^*}\,\overline{W^-(\xi)^*}| \\
&\leqq \varepsilon C_2 \sum_{\substack{\alpha\in\Lambda' \\ \beta\in\varrho_\alpha}} |\langle\xi\rangle^{(\varkappa_\alpha-\beta-1/2)q_n} W_{\alpha,\varkappa_\alpha-\beta}(\xi)|^2 + \varepsilon^{-1} C\int_0^\infty |P'(\xi, D_t)\hat{v}(\xi, t)|^2\,dt,
\end{aligned}
$$

because

$$
|\langle\xi\rangle^{(\varkappa_\gamma-\delta-1/2)q_n} W_{\gamma,\varkappa_\gamma-\delta}(\xi)|^2 \leqq C\int_0^\infty |P'(\xi, D_t)\hat{v}(\xi, t)|^2\,dt, \qquad \gamma\in\Lambda^-.
$$

Likewise,

$$
\begin{aligned}
(3.20)\qquad & |W^-(\xi)\boldsymbol{b}^-(\xi)\boldsymbol{S}(\xi)\overline{\boldsymbol{q}^+(\xi)^*}\,\overline{W^+(\xi)^*}| \\
&\leqq \varepsilon C_3 \sum_{\substack{\alpha\in\Lambda' \\ \beta\in\varrho_\alpha}} |\langle\xi\rangle^{(\varkappa_\alpha-\beta-1/2)q_n} W_{\alpha,\varkappa_\alpha-\beta}(\xi)|^2 + \varepsilon^{-1} C\int_0^\infty |P'(\xi, D_t)\hat{v}(\xi, t)|^2\,dt.
\end{aligned}
$$

Fix now $\varepsilon>0$ such that

$$
\varepsilon \leqq \frac{1}{2} C_1^{-1}(C_2+C_3)^{-1}.
$$

By (3.18)—(3.20), we then conclude from (3.16) and (3.17) that

$$
\begin{aligned}
(3.21)\qquad & \sum_{\alpha\in\Lambda'} \sum_{\beta\in\varrho_\alpha} |\langle\xi\rangle^{(\varkappa_\alpha-\beta-1/2)q_n} W_{\alpha,\varkappa_\alpha-\beta}(\xi)|^2 \\
&\leqq C\Big(\int_0^\infty |P'(\xi, D_t)\hat{v}(\xi, t)|^2\,dt + \big|s_\xi\big(\gamma_0 B'(\xi, D_t)\hat{v}(\xi, t), \gamma_0 Q'(\xi, D_t)\hat{v}(\xi, t)\big)\big|\Big).
\end{aligned}
$$

3.6. Combining (3.2) and (3.21), we thus find

$$\int_0^\infty |R'(\xi, D_t)\hat{v}(\xi, t)|^2 dt \tag{3.22}$$

$$\leqq C\Big(\int_0^\infty |P'(\xi, D_t)\hat{v}(\xi, t)|^2 dt + \big|s_\xi(\gamma_0 B'(\xi, D_t)\hat{v}(\xi, t), \gamma_0 Q'(\xi, D_t)\hat{v}(\xi, t))\big|\Big).$$

Finally, integrate (3.22) over $\boldsymbol{R}^{n-1}$, with respect to ξ. By the Parseval formula, the result is (3.1).

This completes the proof of Theorem 3.1.

4. The necessity of the conditions

4.1. Obviously, it is sufficient to consider the case $n=1$, that is, to verify the statement below.

Theorem 4.1. *If we have*

$$\int_0^\infty |R(D_t)u|^2 dt \leqq C\Big(\int_0^\infty |P(D_t)u|^2 dt + \big|s(\gamma_0 B(D_t)u, \gamma_0 Q(D_t)u)\big|\Big) \tag{4.1}$$

for all $u \in C_0^\infty[\overline{\boldsymbol{R}}_+]$, *then:*

(I) *The matrix* $S=(S_{jk})$ *corresponding to the form* s: $\boldsymbol{C}^\varkappa \times \boldsymbol{C}^\varkappa \to \boldsymbol{C}$ *is regular.*

(II) $B(\zeta)\equiv 0$ *and* $Q(\zeta)\equiv 0$ mod $M(\zeta)$.

(III) *Both the* $B_j(\zeta)$ *and the* $Q_k(\zeta)$ *are linearly independent modulo* $P_+(\zeta)$.

The proof of Theorem 4.1 will be given in the rest of this section.

4.2. First, we may assume that there exists an integer λ^0, $0 \leqq \lambda^0 \leqq \lambda'$, such that ζ_α is a root of $M(\zeta)$ of multiplicity l_α precisely for $\lambda^0 < \alpha \leqq \lambda'$. Define $\Lambda^0 = \{1, \dots, \lambda^0\}$ for $\lambda^0 > 0$ and $\Lambda^0 = \emptyset$ for $\lambda^0 = 0$. Then

$$l_\alpha = \begin{cases} k_\alpha - \varkappa_\alpha & \text{for } \alpha \in \Lambda' \setminus \Lambda^0 \\ k_\alpha & \text{for } \alpha \in \Lambda^+ \setminus \Lambda'. \end{cases}$$

Furthermore, set

$$\varrho'_\alpha = \begin{cases} \varrho_\alpha & \text{for } \alpha \in \Lambda^0 \\ \{l_\alpha, \dots, k_\alpha - 1\} & \text{for } \alpha \in \Lambda' \setminus \Lambda^0 \end{cases}$$

and

$$\varrho''_\alpha = \{0, \dots, l_\alpha - 1\} \quad \text{for } \alpha \in \Lambda'' = \Lambda^+ \setminus \Lambda^0.$$

The general exponential solution of the equation $P_+(D_t)z=0$ has now the representation

$$z(t) = x(t) + y(t) = \sum_{\alpha \in \Lambda'} \sum_{\beta \in \varrho'_\alpha} x_{\alpha\beta} z_{\alpha\beta}(t) + \sum_{\alpha \in \Lambda''} \sum_{\beta \in \varrho''_\alpha} y_{\alpha\beta} z_{\alpha\beta}(t), \qquad x_{\alpha\beta}, y_{\alpha\beta} \in \boldsymbol{C},$$

where

$$z_{\alpha\beta}(t) = (it)^\beta e^{i\zeta_\alpha t}.$$

One immediately checks the statements

(4.2) $$R(D_t)x = 0 \quad \text{precisely for } x_{\alpha\beta} = 0,\ \alpha \in \Lambda',\ \beta \in \varrho'_\alpha,$$

and

(4.3) $$R(D_t)y = 0 \quad \text{for all } y_{\alpha\beta},\ \alpha \in \Lambda'',\ \beta \in \varrho''_\alpha.$$

Moreover, z satisfies (4.1) (cf. [1]).

4.3. Let us then suppose that

$$s(\gamma_0 B(D_t)x, \gamma_0 Q(D_t)x) = 0.$$

By the Leibniz formula, this relation proves to be equivalent to

(4.4) $$\sum_{\substack{\alpha \in \Lambda' \\ \beta \in \varrho'_\alpha}} \sum_{\substack{\gamma \in \Lambda' \\ \delta \in \varrho'_\gamma}} s\big(B^{(\beta)}(\zeta_\alpha), Q^{(\delta)}(\zeta_\gamma)\big) x_{\alpha\beta} \bar{x}_{\gamma\delta} = 0,$$

where $B^{(\beta)} = (iD_t)^\beta B$, $Q^{(\delta)} = (iD_t)^\delta Q$.

Define matrices $\boldsymbol{T}_{xx}$, $\boldsymbol{B}_x$, and $\boldsymbol{Q}_x$ by

$$\begin{aligned}
\boldsymbol{T}_{xx} &= \big(s(B^{(\beta)}(\zeta_\alpha), Q^{(\delta)}(\zeta_\gamma))\big)_{\alpha,\beta;\gamma,\delta} && (\alpha \in \Lambda',\ \beta \in \varrho'_\alpha;\ \gamma \in \Lambda',\ \delta \in \varrho'_\gamma), \\
\boldsymbol{B}_x &= \big(B^{(\beta)}(\zeta_\alpha)\big)_{\alpha,\beta} = \big(B_j^{(\beta)}(\zeta_\alpha)\big)_{\alpha,\beta;j} && (\alpha \in \Lambda',\ \beta \in \varrho'_\alpha;\ j = 1, \ldots, \varkappa), \\
\boldsymbol{Q}_x &= \big(Q^{(\delta)}(\zeta_\gamma)\big)_{\gamma,\delta} = \big(Q_k^{(\delta)}(\zeta_\gamma)\big)_{\gamma,\delta;k} && (\gamma \in \Lambda',\ \delta \in \varrho'_\gamma;\ k = 1, \ldots, \varkappa).
\end{aligned}$$

Note that they satisfy

$$\boldsymbol{T}_{xx} = \boldsymbol{B}_x \boldsymbol{S} \bar{\boldsymbol{Q}}_x^*.$$

Now (4.4) can be rewritten in the form

$$X \boldsymbol{T}_{xx} \bar{X}^* = 0$$

where $X = (x_{\alpha\beta}) \in \boldsymbol{C}^\varkappa$. By (4.1) this, however, implies $X = 0$.

Hence the matrix $\boldsymbol{T}_{xx}$ is regular and, consequently, so are $\boldsymbol{S}$, $\boldsymbol{B}_x$, and $\boldsymbol{Q}_x$.

4.4. Consider next the general solution $z = x + y$, and put

$$Y = (y_{\alpha\beta})_{\alpha \in \Lambda'', \beta \in \varrho''_\alpha} \in \boldsymbol{C}^{K''} \quad \big(K'' = \deg_\zeta M(\zeta)\big).$$

Then we have

$$s(\gamma_0 B(D_t)z, \gamma_0 Q(D_t)z) = X\boldsymbol{T}_{xx}\bar{X}^* + X\boldsymbol{T}_{xy}\bar{Y}^* + Y\boldsymbol{T}_{yx}\bar{X}^* + Y\boldsymbol{T}_{yy}\bar{Y}^*,$$

if we define

$$\boldsymbol{T}_{xy} = \boldsymbol{B}_x \boldsymbol{S} \bar{\boldsymbol{Q}}_y^*, \qquad \boldsymbol{T}_{yx} = \boldsymbol{B}_y \boldsymbol{S} \bar{\boldsymbol{Q}}_x^*, \qquad \boldsymbol{T}_{yy} = \boldsymbol{B}_y \boldsymbol{S} \bar{\boldsymbol{Q}}_y^*,$$

where

$$\begin{aligned}
\boldsymbol{B}_y &= \big(B^{(\beta)}(\zeta_\alpha)\big)_{\alpha,\beta} = \big(B_j^{(\beta)}(\zeta_\alpha)\big)_{\alpha,\beta;j} && (\alpha \in \Lambda'',\ \beta \in \varrho''_\alpha;\ j = 1, \ldots, \varkappa), \\
\boldsymbol{Q}_y &= \big(Q^{(\delta)}(\zeta_\gamma)\big)_{\gamma,\delta} = \big(Q_k^{(\delta)}(\zeta_\gamma)\big)_{\gamma,\delta;k} && (\gamma \in \Lambda'',\ \delta \in \varrho''_\gamma;\ k = 1, \ldots, \varkappa).
\end{aligned}$$

Thus the relation

$$s(\gamma_0 B(D_t)z, \gamma_0 Q(D_t)z) = 0$$

is equivalent to the equation

$$XT_{xx}\bar{X}^*+XT_{xy}\bar{Y}^*+YT_{yx}\bar{X}^*+YT_{yy}\bar{Y}^*=0. \tag{4.5}$$

Let Y now be fixed. Then X can always be chosen such that (4.5) holds. Indeed, recalling that T_{xx} is regular and noting that $T_{yx}T_{xx}^{-1}T_{xy}=T_{yy}$, one can take $X=-YT_{yx}T_{xx}^{-1}$. But then we deduce from (4.2) and (4.3) that $X=(x_{\alpha\beta})=0$. Consequently,

$$YT_{yy}\bar{Y}^*=0 \quad \text{for all} \quad Y=(y_{\alpha\beta})\in C^{K''}$$

so that $T_{yy}=0$.

Furthermore, it follows that

$$-YT_{yx}T_{xx}^{-1}=0 \quad \text{for all} \quad Y\in C^{K''}$$

and therefore

$$B_y S\bar{Q}_x^*=T_{yx}=0.$$

Hence

$$B_y=0. \tag{4.6}$$

Accordingly, we have

$$s(\gamma_0 B(D_t)z,\, \gamma_0 Q(D_t)z)=XT_{xx}\bar{X}^*+XT_{xy}\bar{Y}^*.$$

For any Y fixed, take here

$$\bar{X}^*=-T_{xx}^{-1}T_{xy}\bar{Y}^*.$$

Then (4.2) and (4.3) imply

$$-T_{xx}^{-1}T_{xy}\bar{Y}^*=0.$$

Hence

$$B_x S\bar{Q}_y^*=T_{xy}=0$$

and thus

$$Q_y=0. \tag{4.7}$$

Finally, we conclude from (4.6) and (4.7), taking the definitions of B_y and Q_y into account, that

$$B(\zeta)\equiv 0 \mod M(\zeta), \qquad Q(\zeta)\equiv 0 \mod M(\zeta),$$

i.e., condition (II) is satisfied, too.

4.5. To show (III), we first observe that (4.1) now holds for all $u\in C_0^\infty[\bar{R}_+]$ if and only if

$$\int_0^\infty |R'(D_t)v|^2\,dt \leqq C\left(\int_0^\infty |P'(D_t)v|^2\,dt+\left|s(\gamma_0 B(D_t)v,\, \gamma_0 Q(D_t)v)\right|\right) \tag{4.8}$$

for all $v\in C_0^\infty[\bar{R}_+]$. In fact, each $v\in C_0^\infty[\bar{R}_+]$ can be represented in the form $v=M(D_t)u$ with some $u\in C_0^\infty[\bar{R}_+]$ (see [1]).

Next, we shall make use of the general solution of the equation $P'_+(D_t)w=0$, given by

$$w(t) = \sum_{\alpha\in\Lambda'} \sum_{\beta\in\varrho_\alpha} w_{\alpha\beta} z_{\alpha\beta}(t), \qquad w_{\alpha\beta}\in \boldsymbol{C}.$$

Note that w also satisfies (4.8).

Consider now the relation

$$s\big(\gamma_0 B'(D_t)w,\, \gamma_0 Q'(D_t)w\big) = 0$$

or, what is the same thing, the equation

$$\sum_{\substack{\alpha\in\Lambda' \\ \beta\in\varrho_\alpha}} \sum_{\substack{\gamma\in\Lambda' \\ \delta\in\varrho_\gamma}} s\big(B'^{(\beta)}(\zeta_\alpha),\, Q'^{(\delta)}(\zeta_\gamma)\big) w_{\alpha\beta}\overline{w}_{\gamma\delta} = 0. \tag{4.9}$$

We define matrices $\boldsymbol{T}'$, $\boldsymbol{B}'$, $\boldsymbol{Q}'$ by

$$\begin{aligned}
\boldsymbol{T}' &= \big(s(B'^{(\beta)}(\zeta_\alpha),\, Q'^{(\delta)}(\zeta_\gamma))\big)_{\alpha,\beta;\gamma,\delta} && (\alpha\in\Lambda',\ \beta\in\varrho_\alpha;\ \gamma\in\Lambda',\ \delta\in\varrho_\gamma),\\
\boldsymbol{B}' &= \big(B'^{(\beta)}(\zeta_\alpha)\big)_{\alpha,\beta} = \big(B_j'^{(\beta)}(\zeta_\alpha)\big)_{\alpha,\beta;j} && (\alpha\in\Lambda',\ \beta\in\varrho_\alpha;\ j=1,\dots,\varkappa),\\
\boldsymbol{Q}' &= \big(Q'^{(\delta)}(\zeta_\gamma)\big)_{\gamma,\delta} = \big(Q_k'^{(\delta)}(\zeta_\gamma)\big)_{\gamma,\delta;k} && (\gamma\in\Lambda',\ \delta\in\varrho_\gamma;\ k=1,\dots,\varkappa),
\end{aligned}$$

and have then

$$\boldsymbol{T}' = \boldsymbol{B}'\boldsymbol{S}\overline{\boldsymbol{Q}}'^*.$$

Thus, if we set $W=(w_{\alpha\beta})\in\boldsymbol{C}^\varkappa$, (4.9) becomes

$$W\boldsymbol{T}'\overline{W}^* = 0. \tag{4.10}$$

Now suppose w satisfies (4.9). Then $R'(D_t)w=0$ and consequently $w=0$, because $R'(\zeta)$ and $P'_+(\zeta)$ are relatively prime.

Hence (4.10) can be true only for $W=(w_{\alpha\beta})=0$. The matrix $\boldsymbol{T}'$ is therefore regular. But then $\boldsymbol{B}'$ and $\boldsymbol{Q}'$ are also regular, and it follows that both the $B_j'(\zeta)$ and the $Q_k'(\zeta)$ are linearly independent modulo $P'_+(\zeta)$.

Finally, this implies (III), and thus completes the proof of Theorem 4.1.

Remark 4.2. *The constant C in (4.1) satisfies the inequality*

$$C \geqq \sup_\zeta \left|\frac{R(\zeta)}{P(\zeta)}\right|^2.$$

To see this, let $v\in C_0^\infty(\boldsymbol{R})$, and let $a\in\boldsymbol{R}$ such that $\operatorname{supp} v\cap\,]-\infty, a[\,=\emptyset$, where $\operatorname{supp} v$ denotes the support of v. If $u\in C_0^\infty(\boldsymbol{R}_+)$ is defined through $u(t)=v(t+a)$, then

$$s\big(\gamma_0 B(D_t)u,\, \gamma_0 Q(D_t)u\big) = 0.$$

By (4.1), we thus find that

$$\int_{-\infty}^{\infty} |R(D_t)v|^2\,dt \leqq C \int_{-\infty}^{\infty} |P(D_t)v|^2\,dt$$

for all $v\in C_0^\infty(\boldsymbol{R})$, from which the assertion follows, via the Fourier transformation.

The general case

5. Definitions and lemmas

5.1. We first recall the definitions of some well-known spaces of distributions (cf. [4], [13], [14], [15]).

Definition 5.1. *For $s \in \boldsymbol{R}$, the space $H^s(\boldsymbol{R}^n)$ is defined by*

$$H^s(\boldsymbol{R}^n) = \{u \in \mathscr{S}'(\boldsymbol{R}^n) \,|\, (1+\langle\eta\rangle^2)^{s/2}(\mathscr{F}u) \in L^2(\boldsymbol{R}^n_\eta)\}$$

and is provided with the norm $\|\cdot\|_{H^s(\boldsymbol{R}^n)}$,

$$\|u\|_{H^s(\boldsymbol{R}^n)} = \|(1+\langle\eta\rangle^2)^{s/2}(\mathscr{F}u)\|_{L^2(\boldsymbol{R}^n)},$$

where $\mathscr{S}'(\boldsymbol{R}^n)$ denotes the space of Schwartz's tempered distributions in $\boldsymbol{R}^n$.
Analogously,

$$H^s(\boldsymbol{R}^{n-1}) = \{v \in \mathscr{S}'(\boldsymbol{R}^{n-1}) \,|\, (1+\langle\xi\rangle^2)^{s/2}(\mathscr{F}_x v) \in L^2(\boldsymbol{R}^{n-1}_\xi)\}$$

with the norm $|\cdot|_s$,

$$|v|_s = \|v\|_{H^s(\boldsymbol{R}^{n-1})} = \|(1+\langle\xi\rangle^2)^{s/2}(\mathscr{F}_x v)\|_{L^2(\boldsymbol{R}^{n-1})}.$$

Definition 5.2. *For $s \in \boldsymbol{R}$, let $H^s(\boldsymbol{R}^n_+)$ be the space of restrictions to $\boldsymbol{R}^n_+$ of the elements of $H^s(\boldsymbol{R}^n)$, and provide it with the norm $\|\cdot\|_s$ defined by*

$$\|u\|_s = \|u\|_{H^s(\boldsymbol{R}^n_+)} = \inf\{\|U\|_{H^s(\boldsymbol{R}^n)} \,|\, U \in H^s(\boldsymbol{R}^n) \text{ with } U|_{\boldsymbol{R}^n_+} = u\}.$$

We now have (see [4], [15])

Lemma 5.3. *The spaces $C_0^\infty(\boldsymbol{R}^n)$ and $C_0^\infty[\overline{\boldsymbol{R}}^n_+]$ are dense in $H^s(\boldsymbol{R}^n)$ and $H^s(\boldsymbol{R}^n_+)$, respectively, and $H^{s_1}(\boldsymbol{R}^n) \subset H^{s_2}(\boldsymbol{R}^n)$, $H^{s_1}(\boldsymbol{R}^n_+) \subset H^{s_2}(\boldsymbol{R}^n_+)$ algebraically and topologically for $s_1 \geqq s_2$.*

For $s > q_n/2$ let l_s denote the greatest integer less than $s/q_n - 1/2$. If we set

$$(\gamma_j u)(x) = (\gamma_0 D_t^j u)(x) = (D_t^j u)(x, 0), \quad u \in C_0^\infty[\overline{\boldsymbol{R}}^n_+],$$

then we have (see [14])

Lemma 5.4. *Let $s > q_n/2$. The mapping*

$$u \mapsto (\gamma_0 u, \dots, \gamma_{l_s} u)\colon C_0^\infty[\overline{\boldsymbol{R}}^n_+] \to C_0^\infty(\boldsymbol{R}^{n-1})^{l_s+1}$$

extends by continuity to a continuous linear mapping

$$u \mapsto (\gamma_0 u, \dots, \gamma_{l_s} u)\colon H^s(\boldsymbol{R}^n_+) \to \prod_{j=0}^{l_s} H^{s-jq_n-q_n/2}(\boldsymbol{R}^{n-1}).$$

5.2. We shall often employ the following

Lemma 5.5. *Let $s_1>s_2>0$. Then for any $\varepsilon>0$ there exists a constant $C(\varepsilon)>0$ such that*

$$\|u\|_{s_2} \leqq \varepsilon\|u\|_{s_1}+C(\varepsilon)\|u\|, \quad u\in H^{s_1}(\boldsymbol{R}^n_+).$$

Proof. For each $s\geqq 0$ there is a linear extension operator E such that

$$\|Eu\|_{H^s(R^n)} \leqq C\|u\|_s$$

and

$$\|Eu\|_{L^2(R^n)} \leqq C\|u\|$$

for all $u\in C_0^\infty[\bar{\boldsymbol{R}}^n_+]$ (see [7], [9], [13]). The assertion is therefore implied by the simple lemma below.

Lemma 5.6. *If $s_1>s_2>s_3$, then for any $\varepsilon>0$ there exists a constant $C(\varepsilon)>0$ such that*

$$\|u\|_{H^{s_2}(R^n)} \leqq \varepsilon\|u\|_{H^{s_1}(R^n)}+C(\varepsilon)\|u\|_{H^{s_3}(R^n)}, \quad u\in H^{s_1}(\boldsymbol{R}^n).$$

Lemma 5.7. *Let $\alpha\in \boldsymbol{N}^n$, $k=\langle\alpha, q\rangle$, and $s\geqq k$. Then*

$$\|D^\alpha u\|_{s-k} \leqq C\|u\|_s, \quad u\in C_0^\infty[\bar{\boldsymbol{R}}^n_+].$$

Proof. The statement follows from Definitions 5.1 and 5.2 by the inequality (see [14])

$$\sum_{\langle\alpha,q\rangle\leqq k} \eta^{2\alpha} \leqq C(1+\langle\eta\rangle^2)^k.$$

Lemma 5.8. *Let $s>0$, $v>0$. For any $\varepsilon>0$ there exists a constant $C(\varepsilon)>0$ such that*

$$\sum_{\langle\alpha,q\rangle<v} \|D^\alpha u\|_s \leqq \varepsilon\|u\|_{s+v}+C(\varepsilon)\|u\|, \quad u\in C_0^\infty[\bar{\boldsymbol{R}}^n_+].$$

Proof. According to Lemma 5.7

$$\|D^\alpha u\|_s \leqq C\|u\|_{s+\langle\alpha,q\rangle}.$$

Hence, putting

$$r = \max\{\langle\alpha, q\rangle | \langle\alpha, q\rangle < v\} < v,$$

we have by Lemma 5.3

$$\sum_{\langle\alpha,q\rangle<v} \|D^\alpha u\|_s \leqq C\|u\|_{s+r}.$$

By Lemma 5.5, this yields the desired inequality.

6. Statement of the result

6.1. We consider a general polynomial of the form

$$P(\eta) = \sum_{\langle\alpha,q\rangle\leqq\mu} p_\alpha\eta^\alpha$$

with $p_\alpha\in\boldsymbol{C}$, and decompose it into the sum

$$P(\eta) = P^0(\eta)+P^{00}(\eta),$$

where $P^0(\eta)$ is the principal part of $P(\eta)$ (cf. 1.2) and

$$P^{00}(\eta) = \sum_{\langle\alpha, q\rangle < \mu} p_\alpha \eta^\alpha.$$

Let

$$R(\eta) = \sum_{\langle\alpha, q\rangle \leqq \mu} r_\alpha \eta^\alpha$$

be another polynomial with constant coefficients $r_\alpha \in \boldsymbol{C}$, and write similarly

$$R(\eta) = R^0(\eta) + R^{00}(\eta).$$

Now we make the assumptions that $P(\eta)$ is quasi-elliptic and that the q-homogeneous polynomials $P^0(\eta)$ and $R^0(\eta)$ satisfy Hypotheses (A) and (B), and put $\deg_\zeta P_+^{0\prime}(\xi, \zeta) = \varkappa$.

Let there also be given two polynomial vectors

$$B(\eta) = (B_1(\eta), \ldots, B_\varkappa(\eta))$$

and

$$Q(\eta) = (Q_1(\eta), \ldots, Q_\varkappa(\eta)),$$

where

$$B_j(\eta) = \sum_{\langle\alpha, q\rangle \leqq \mu_j} b_{j\alpha} \eta^\alpha, \quad \mu_j \leqq \mu - q_n, \quad b_{j\alpha} \in \boldsymbol{C}, \quad j = 1, \ldots, \varkappa,$$

and

$$Q_k(\eta) = \sum_{\langle\alpha, q\rangle \leqq \nu_k} q_{k\alpha} \eta^\alpha, \quad \nu_k \leqq \mu - q_n, \quad q_{k\alpha} \in \boldsymbol{C}, \quad k = 1, \ldots, \varkappa.$$

As above we also decompose

$$B(\eta) = B^0(\eta) + B^{00}(\eta)$$

and

$$Q(\eta) = Q^0(\eta) + Q^{00}(\eta).$$

Further, suppose that with each $\xi \in \boldsymbol{R}^{n-1} \setminus \{0\}$ there is associated a sesquilinear form

$$s_\xi(\cdot, \cdot): \boldsymbol{C}^\varkappa \times \boldsymbol{C}^\varkappa \to \boldsymbol{C}$$

that has the representation

$$s_\xi(\cdot, \cdot) = s_\xi^0(\cdot, \cdot) + s_\xi^{00}(\cdot, \cdot)$$

with the corresponding matrices

$$S(\xi) = (S_{jk}(\xi))_{jk},$$
$$S^0(\xi) = (S_{jk}^0(\xi))_{jk} \quad \text{(the principal part)},$$
$$S^{00}(\xi) = (S_{jk}^{00}(\xi))_{jk},$$

respectively, such that

(i) S_{jk}^0 is a q'-homogeneous continuous function in $\boldsymbol{R}^{n-1} \setminus \{0\}$ with

$$q'\text{-deg}\, S_{jk}^0 = 2\mu - \mu_j - \nu_k - q_n;$$

(ii) S_{jk}^{00} is (e.g.) such a function that there exist positive constants δ'_{jk}, δ''_{jk}, $\delta_{jk} = \delta'_{jk} + \delta''_{jk} \leqq q_n$, such that

$$|S_{jk}^{00}(\xi)| \leqq C(1 + \langle\xi\rangle)^{2\mu - \mu_j - \nu_k - q_n - \delta_{jk}}, \quad \xi \in \boldsymbol{R}^{n-1} \setminus \{0\}.$$

Finally, we define

$$S(U,V)=\int |s_\xi(\mathscr{F}_x U, \mathscr{F}_x V)|\,d\xi,$$
$$S^0(U,V)=\int |s_\xi^0(\mathscr{F}_x U, \mathscr{F}_x V)|\,d\xi,$$
$$S^{00}(U,V)=\int |s_\xi^{00}(\mathscr{F}_x U, \mathscr{F}_x V)|\,d\xi$$

for $U, V\in C_0^\infty(R^{n-1})$.

6.2. We will prove the following result.

Theorem 6.1. *Let the assumptions of 6.1 be satisfied. In order that the estimate*

$$\|R(D)u\|^2 \leqq C(\|P(D)u\|^2+S(\gamma_0 B(D)u, \gamma_0 Q(D)u)+\|u\|^2) \tag{6.1}$$

be valid for all $u\in C_0^\infty[\bar{R}_+^n]$, *it is necessary and sufficient that*

$$\|R^0(D)u\|^2 \leqq C\big(\|P^0(D)u\|^2+S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)\big), \tag{6.2}$$

for all $u\in C_0^\infty[\bar{R}_+^n]$.

Remark 6.2. *By Theorem 2.1, the inequality (6.1) is therefore fulfilled if and only if the principal parts satisfy the conditions (I)—(III) in Theorem 2.1.*

7. Proof of Theorem 6.1. Sufficiency

7.1. We first make use of (6.2) to find

$$\begin{aligned}\|R(D)u\|^2 &\leqq C\big(\|R^0(D)u\|^2+\sum_{\langle\alpha,q\rangle<\mu}\|D^\alpha u\|^2\big)\\ &\leqq C\big(\|P^0(D)u\|^2+S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)+\sum_{\langle\alpha,q\rangle<\mu}\|D^\alpha u\|^2\big)\\ &\leqq C\big(\|P(D)u\|^2+S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)+\sum_{\langle\alpha,q\rangle<\mu}\|D^\alpha u\|^2\big).\end{aligned} \tag{7.1}$$

Since

$$S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)=\int \big|s_\xi^0(\gamma_0 B^0(\xi,D_t)\hat{u}, \gamma_0 Q^0(\xi,D_t)\hat{u})\big|\,d\xi$$

and

$$\begin{aligned}&s_\xi(\gamma_0 B(\xi,D_t)\hat{u}, \gamma_0 Q(\xi,D_t)\hat{u})\\ &= s_\xi^0(\gamma_0 B^0(\xi,D_t)\hat{u}, \gamma_0 Q^0(\xi,D_t)\hat{u})+s_\xi^0(\gamma_0 B^0(\xi,D_t)\hat{u}, \gamma_0 Q^{00}(\xi,D_t)\hat{u})\\ &\quad+s_\xi^0(\gamma_0 B^{00}(\xi,D_t)\hat{u}, \gamma_0 Q^0(\xi,D_t)\hat{u})+s_\xi^0(\gamma_0 B^{00}(\xi,D_t)\hat{u}, \gamma_0 Q^{00}(\xi,D_t)\hat{u})\\ &\quad+s_\xi^{00}(\gamma_0 B(\xi,D_t)\hat{u}, \gamma_0 Q(\xi,D_t)\hat{u}),\end{aligned} \tag{7.2}$$

we have here

$$\begin{aligned}&S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)\\ &\leqq S(\gamma_0 B(D)u, \gamma_0 Q(D)u)+S^0(\gamma_0 B^0(D)u, \gamma_0 Q^{00}(D)u)\\ &\quad+S^0(\gamma_0 B^{00}(D)u, \gamma_0 Q^0(D)u)+S^0(\gamma_0 B^{00}(D)u, \gamma_0 Q^{00}(D)u)\\ &\quad+S^{00}(\gamma_0 B(D)u, \gamma_0 Q(D)u).\end{aligned} \tag{7.3}$$

7.2. Let us consider in (7.3) the term

$$S^0(\gamma_0 B^0(D)u, \gamma_0 Q^{00}(D)u) = \int \Big|\sum_{j,k} S^0_{jk}(\xi)(\gamma_0 B^0_j(\xi, D_t)\hat{u})\overline{(\gamma_0 Q^{00}_k(\xi, D_t)\hat{u})}\Big|\, d\xi.$$

If $\nu_k > 0$ ($Q_k = Q^0_k$ for $\nu_k = 0$), then it follows from the q'-homogeneity of S^0_{jk} and Lemmas 5.4 and 5.7 that

$$\begin{aligned}(7.4)\qquad &\int |S^0_{jk}(\xi)(\gamma_0 B^0_j(\xi, D_t)\hat{u})\overline{(\gamma_0 Q^{00}_k(\xi, D_t)\hat{u})}|\, d\xi \\ &\leqq C\Big(\int |\langle\xi\rangle^{\mu-\mu_j-q_n/2}\gamma_0 B^0_j(\xi, D_t)\hat{u}|^2 d\xi\Big)^{1/2}\Big(\int |\langle\xi\rangle^{\mu-\nu_k-q_n/2}\gamma_0 Q^{00}_k(\xi, D_t)\hat{u}|^2 d\xi\Big)^{1/2} \\ &\leqq C|\gamma_0 B^0_j(D)u|_{\mu-\mu_j-q_n/2}|\gamma_0 Q^{00}_k(D)u|_{\mu-\nu_k-q_n/2} \\ &\leqq C\|u\|_\mu \sum_{\langle\alpha, q\rangle<\nu_k} \|D^\alpha u\|_{\mu-\nu_k}.\end{aligned}$$

Employing a known inequality (see [14]) and the extension operator (cf. the proof of Lemma 5.5 and [9]), we easily see that

$$(7.5)\qquad \|u\|_\mu \leqq C_1 \sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\| + C_2\|u\|, \qquad u \in C_0^\infty[\bar{R}^n_+].$$

Now let $\varepsilon > 0$ be arbitrary. Then, since we have by (7.5) and Lemma 5.8

$$\sum_{\langle\alpha, q\rangle<\nu_k} \|D^\alpha u\|_{\mu-\nu_k} \leqq \varepsilon C_3 \sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\| + C_4(\varepsilon)\|u\|,$$

it follows from (7.4), again by (7.5), that

$$\begin{aligned}&\int |S^0_{jk}(\xi)(\gamma_0 B^0_j(\xi, D_t)\hat{u})\overline{(\gamma_0 Q^{00}_k(\xi, D_t)\hat{u})}|\, d\xi \\ &\leqq \varepsilon C_5 \sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\|^2 + C_6(\varepsilon)\|u\|^2.\end{aligned}$$

Hence

$$(7.6)\qquad S^0(\gamma_0 B^0(D)u, \gamma_0 Q^{00}(D)u) \leqq \varepsilon C_7 \sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\|^2 + C_8(\varepsilon)\|u\|^2.$$

Likewise,

$$(7.7)\qquad S^0(\gamma_0 B^{00}(D)u, \gamma_0 Q^0(D)u) \leqq \varepsilon C_9 \sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\|^2 + C_{10}(\varepsilon)\|u\|^2$$

and

$$(7.8)\qquad S^0(\gamma_0 B^{00}(D)u, \gamma_0 Q^{00}(D)u) \leqq \varepsilon C_{11} \sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\|^2 + C_{12}(\varepsilon)\|u\|^2.$$

Consider then the last term of (7.3), that is, the term

$$S^{00}(\gamma_0 B(D)u, \gamma_0 Q(D)u) = \int \Big|\sum_{j,k} S^{00}_{jk}(\xi)(\gamma_0 B_j(\xi, D_t)\hat{u})\overline{(\gamma_0 Q_k(\xi, D_t)\hat{u})}\Big|\, d\xi.$$

Due to condition (ii), we have by Lemmas 5.3, 5.4, 5.7

$$\begin{aligned}&\int |S^{00}_{jk}(\xi)(\gamma_0 B_j(\xi, D_t)\hat{u})\overline{(\gamma_0 Q_k(\xi, D_t)\hat{u})}|\, d\xi \\ &\leqq C \sum_{\langle\alpha, q\rangle\leqq\mu_j} \|D^\alpha u\|_{\mu-\mu_j-\delta'_{jk}} \sum_{\langle\alpha, q\rangle\leqq\nu_k} \|D^\alpha u\|_{\mu-\nu_k-\delta''_{jk}} \\ &\leqq C\|u\|_{\mu-\delta'_{jk}}\|u\|_{\mu-\delta''_{jk}}.\end{aligned}$$

It therefore follows, by use of Lemma 5.5, that

$$S^{00}(\gamma_0 B(D)u, \gamma_0 Q(D)u) \leqq \varepsilon C_{13} \sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\|^2 + C_{14}(\varepsilon)\|u\|^2. \tag{7.9}$$

7.3. From (7.3), (7.6)—(7.9) we thus obtain, applying Lemma 5.8 and (7.5),

$$\begin{aligned} &S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u) + \sum_{\langle\alpha, q\rangle<\mu} \|D^\alpha u\|^2 \\ &\leqq S(\gamma_0 B(D)u, \gamma_0 Q(D)u) + \varepsilon C_{15} \sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\|^2 + C_{16}(\varepsilon)\|u\|^2. \end{aligned} \tag{7.10}$$

Next, note that according to Sections 2—4

$$\|D^\alpha u\|^2 \leqq C(\|P^0(D)u\|^2 + S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u))$$

for every $\alpha \in N^n$ with $\langle\alpha, q\rangle = \mu$, and consequently

$$\sum_{\langle\alpha, q\rangle=\mu} \|D^\alpha u\|^2 \leqq C_{17}(\|P(D)u\|^2 + S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u) + \sum_{\langle\alpha, q\rangle<\mu} \|D^\alpha u\|^2).$$

Hence, if we fix ε such that

$$\varepsilon \leqq \frac{1}{2} C_{15}^{-1} C_{17}^{-1},$$

it follows from (7.10) that

$$\begin{aligned} &S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u) + \sum_{\langle\alpha, q\rangle<\mu} \|D^\alpha u\|^2 \\ &\leqq \|P(D)u\|^2 + 2S(\gamma_0 B(D)u, \gamma_0 Q(D)u) + C\|u\|^2. \end{aligned}$$

Finally, combining this with (7.1), we find

$$\|R(D)u\|^2 \leqq C(\|P(D)u\|^2 + S(\gamma_0 B(D)u, \gamma_0 Q(D)u) + \|u\|^2).$$

8. Proof of Theorem 6.1. Necessity

8.1. Assuming that (6.1) is valid we first have

$$\begin{aligned} \|R^0(D)u\|^2 &\leqq C(\|R(D)u\|^2 + \sum_{\langle\alpha, q\rangle<\mu} \|D^\alpha u\|^2) \\ &\leqq C(\|P(D)u\|^2 + S(\gamma_0 B(D)u, \gamma_0 Q(D)u) + \sum_{\langle\alpha, q\rangle<\mu} \|D^\alpha u\|^2) \\ &\leqq C(\|P^0(D)u\|^2 + S(\gamma_0 B(D)u, \gamma_0 Q(D)u) + \sum_{\langle\alpha, q\rangle<\mu} \|D^\alpha u\|^2). \end{aligned}$$

By (i), (ii), and (7.2), we therefore obtain the inequality

$$\begin{aligned} \|R^0(D)u\|^2 &\leqq C_1(\|P^0(D)u\|^2 + S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)) \\ &+ C_2 \sum_{j,k} \Big(\sum_{\langle\alpha, q\rangle=\mu_j} |\gamma_0 D^\alpha u|_{\mu-\mu_j-q_n/2} \sum_{\langle\beta, q\rangle<\nu_k} |\gamma_0 D^\beta u|_{\mu-\nu_k-q_n/2} \\ &\quad + \sum_{\langle\alpha, q\rangle<\mu_j} |\gamma_0 D^\alpha u|_{\mu-\mu_j-q_n/2} \sum_{\langle\beta, q\rangle=\nu_k} |\gamma_0 D^\beta u|_{\mu-\nu_k-q_n/2} \\ &\quad + \sum_{\langle\alpha, q\rangle<\mu_j} |\gamma_0 D^\alpha u|_{\mu-\mu_j-q_n/2} \sum_{\langle\beta, q\rangle<\nu_k} |\gamma_0 D^\beta u|_{\mu-\nu_k-q_n/2} \\ &\quad + \sum_{\langle\alpha, q\rangle\leqq\mu_j} |\gamma_0 D^\alpha u|_{\mu-\mu_j-q_n/2-\delta'_{jk}} \sum_{\langle\beta, q\rangle\leqq\nu_k} |\gamma_0 D^\beta u|_{\mu-\nu_k-q_n/2-\delta''_{jk}} \Big) \\ &+ C_3 \sum_{\langle\alpha, q\rangle<\mu} \|D^\alpha u\|^2 \\ &= C_1 I_1 + C_2 I_2 + C_3 I_3. \end{aligned} \tag{8.1}$$

8.2. Let $\xi \in R^{n-1} \setminus \{0\}$ be fixed. Suppose $f \in C_0^\infty(R^{n-1})$, $g \in C_0^\infty[\bar{R}_+]$, and $h>0$, and define a function $u \in C_0^\infty[\bar{R}_+^n]$ by

$$u(x, t) = \left(\frac{1}{h}\right)^{(n-1)/2} f\left(\frac{x}{h}\right) e^{i\langle x, \xi\rangle} g_\xi(t),$$

where $g_\xi(t) = g(\langle \xi \rangle^{q_n} t)$.

The Leibniz formula yields

$$R^0(D)u = \sum_{\langle \alpha', q' \rangle \leqq \mu} \frac{1}{\alpha'!} \left(\frac{1}{h}\right)^{(n-1)/2+|\alpha'|} (D_x^{\alpha'} f)\left(\frac{x}{h}\right) R^{0(\alpha')}(\xi, D_t) g_\xi(t) e^{i\langle x, \xi\rangle},$$

where $\alpha'! = \alpha_1! \dots \alpha_{n-1}!$ and $|\alpha'| = \alpha_1 + \dots + \alpha_{n-1}$. Hence

$$\text{(8.2)} \quad \|R^0(D)u\|^2 = \int_0^\infty \left(\int \left| \sum_{\langle \alpha', q' \rangle \leqq \mu} \frac{1}{\alpha'!} \left(\frac{1}{h}\right)^{|\alpha'|} (D_x^{\alpha'} f)(x) R^{0(\alpha')}(\xi, D_t) g_\xi(t) \right|^2 dx \right) dt.$$

An application of the dominated convergence theorem shows now that the right side of (8.2) tends to

$$\int_0^\infty |R^0(\xi, D_t) g_\xi(t)|^2 dt \int |f(x)|^2 dx$$

as $h \to \infty$.

Similarly, if $h \to \infty$, we find

$$\|P^0(D)u\|^2 \to \int_0^\infty |P^0(\xi, D_t) g_\xi(t)|^2 dt \int |f(x)|^2 dx.$$

Consider next the term

$$S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)$$
$$= \int \Big| \sum_{j,k} S_{jk}^0(\theta) (\gamma_0 B_j^0(\theta, D_t) \hat{u}(\theta, t)) \overline{(\gamma_0 Q_k^0(\theta, D_t) \hat{u}(\theta, t))} \Big| d\theta.$$

Since

$$\text{(8.3)} \qquad \hat{u}(\theta, t) = g_\xi(t) h^{(n-1)/2} (\mathscr{F}_x f)(h(\theta - \xi)),$$

we obtain, substituting $\tau = h(\theta - \xi)$,

$$S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)$$
$$= \int \left| \sum_{j,k} S_{jk}^0\left(\xi + \frac{\tau}{h}\right) \left(\gamma_0 B_j^0\left(\xi + \frac{\tau}{h}, D_t\right) g_\xi(t)\right) \overline{\left(\gamma_0 Q_k^0\left(\xi + \frac{\tau}{h}, D_t\right) g_\xi(t)\right)} \right| |(\mathscr{F}_x f)(\tau)|^2 d\tau,$$

where the right side tends to

$$|s_\xi^0(\gamma_0 B^0(\xi, D_t) g_\xi(t), \gamma_0 Q^0(\xi, D_t) g_\xi(t))| \int |f(x)|^2 dx$$

as $h \to \infty$.

Using (8.3), we have, if $h \to \infty$,

$$|\gamma_0 D^\alpha u|_s^2 \to (1 + \langle \xi \rangle^2)^s \xi^{2\alpha'} |\gamma_0 D_t^{\alpha_n} g_\xi|^2 \int |f(x)|^2 dx,$$

and consequently

$$I_2 \to \sum_{j,k} \Big(\sum_{\substack{\langle \alpha, q\rangle = \mu_j \\ \langle \beta, q\rangle < \nu_k}} + \sum_{\substack{\langle \alpha, q\rangle < \mu_j \\ \langle \beta, q\rangle = \nu_k}} + \sum_{\substack{\langle \alpha, q\rangle < \mu_j \\ \langle \beta, q\rangle < \nu_k}} \Big) (1+\langle \xi \rangle^2)^{(2\mu-\mu_j-\nu_k-q_n)/2}$$

$$\times |\xi^{\alpha'+\beta'}| \, |\gamma_0 D_t^{\alpha_n} g_\xi| \, |\gamma_0 D_t^{\beta_n} g_\xi| \int |f(x)|^2 \, dx$$

$$+ \sum_{j,k} \sum_{\substack{\langle \alpha, q\rangle \leqq \mu_j \\ \langle \beta, q\rangle \leqq \nu_k}} (1+\langle \xi \rangle^2)^{(2\mu-\mu_j-\nu_k-q_n-\delta_{jk})/2}$$

$$\times |\xi^{\alpha'+\beta'}| \, |\gamma_0 D_t^{\alpha_n} g_\xi| \, |\gamma_0 D_t^{\beta_n} g_\xi| \int |f(x)|^2 \, dx .$$

Finally, we see, again by (8.3), that

$$I_3 \to \sum_{\langle \alpha, q\rangle < \mu} \xi^{2\alpha'} \int_0^\infty |D_t^{\alpha_n} g_\xi(t)|^2 \, dt \int |f(x)|^2 \, dx$$

when $h \to \infty$.

8.3. By (8.1) we have thus established the inequality

$$\int_0^\infty |R^0(\xi, D_t) g_\xi(t)|^2 \, dt \tag{8.4}$$

$$\leqq C \Big(\int_0^\infty |P^0(\xi, D_t) g_\xi(t)|^2 \, dt + \big| s_\xi^0 \big(\gamma_0 B^0(\xi, D_t) g_\xi, \gamma_0 Q^0(\xi, D_t) g_\xi\big)\big|$$

$$+ \sum_{j,k} \Big(\sum_{\substack{\langle \alpha, q\rangle = \mu_j \\ \langle \beta, q\rangle < \nu_k}} + \sum_{\substack{\langle \alpha, q\rangle < \mu_j \\ \langle \beta, q\rangle = \nu_k}} + \sum_{\substack{\langle \alpha, q\rangle < \mu_j \\ \langle \beta, q\rangle < \nu_k}} \Big) (1+\langle \xi \rangle^2)^{(2\mu-\mu_j-\nu_k-q_n)/2}$$

$$\times |\xi^{\alpha'+\beta'}| \, |\gamma_0 D_t^{\alpha_n} g_\xi| \, |\gamma_0 D_t^{\beta_n} g_\xi|$$

$$+ \sum_{j,k} \sum_{\substack{\langle \alpha, q\rangle \leqq \mu_j \\ \langle \beta, q\rangle \leqq \nu_k}} (1+\langle \xi \rangle^2)^{(2\mu-\mu_j-\nu_k-q_n-\delta_{jk})/2} |\xi^{\alpha'+\beta'}| \, |\gamma_0 D_t^{\alpha_n} g_\xi| \, |\gamma_0 D_t^{\beta_n} g_\xi|$$

$$+ \sum_{\langle \alpha, q\rangle < \mu} \xi^{2\alpha'} \int_0^\infty |D_t^{\alpha_n} g_\xi(t)|^2 \, dt \Big).$$

Substitute here

$$\xi = \langle \xi \rangle^{q'} \theta, \qquad \tau = \langle \xi \rangle^{q_n} t, \tag{8.5}$$

and divide by $\langle \xi \rangle^{2\mu - q_n}$. Then (8.4) becomes

$$\int_0^\infty |R^0(\theta, D_\tau) g(\tau)|^2 \, d\tau$$

$$\leqq C \Big(\int_0^\infty |P^0(\theta, D_\tau) g(\tau)|^2 \, d\tau + \big| s_\theta^0 \big(\gamma_0 B^0(\theta, D_\tau) g(\tau), \gamma_0 Q^0(\theta, D_\tau) g(\tau)\big)\big|$$

$$+ \sum_{j,k} \Big(\sum_{\substack{\langle \alpha, q\rangle = \mu_j \\ \langle \beta, q\rangle < \nu_k}} + \sum_{\substack{\langle \alpha, q\rangle < \mu_j \\ \langle \beta, q\rangle = \nu_k}} + \sum_{\substack{\langle \alpha, q\rangle < \mu_j \\ \langle \beta, q\rangle < \nu_k}} \Big) \langle \xi \rangle^{\langle \alpha, q\rangle - \mu_j + \langle \beta, q\rangle - \nu_k}$$

$$\times (\langle \xi \rangle^{-2} + \langle \theta \rangle^2)^{(2\mu-\mu_j-\nu_k-q_n)/2} |\theta^{\alpha'+\beta'}| \, |\gamma_0 D_\tau^{\alpha_n} g(\tau)| \, |\gamma_0 D_\tau^{\beta_n} g(\tau)|$$

$$+ \sum_{j,k} \sum_{\substack{\langle \alpha, q\rangle \leqq \mu_j \\ \langle \beta, q\rangle \leqq \nu_k}} \langle \xi \rangle^{\langle \alpha, q\rangle - \mu_j + \langle \beta, q\rangle - \nu_k - \delta_{jk}}$$

$$\times (\langle \xi \rangle^{-2} + \langle \theta \rangle^2)^{(2\mu-\mu_j-\nu_k-q_n-\delta_{jk})/2} |\theta^{\alpha'+\beta'}| \, |\gamma_0 D_\tau^{\alpha_n} g(\tau)| \, |\gamma_0 D_\tau^{\beta_n} g(\tau)|$$

$$+ \sum_{\langle \alpha, q\rangle < \mu} \langle \xi \rangle^{2(\langle \alpha, q\rangle - \mu)} \theta^{2\alpha'} \int_0^\infty |D_\tau^{\alpha_n} g(\tau)|^2 \, d\tau \Big).$$

Now, letting $\langle\xi\rangle$ tend to infinity, we find

$$\int_0^\infty |R^0(\theta, D_\tau)g(\tau)|^2 d\tau \leqq C\Big(\int_0^\infty |P^0(\theta, D_\tau)g(\tau)|^2 d\tau + \big|s_\theta^0(\gamma_0 B^0(\theta, D_\tau)g(\tau), \gamma_0 Q^0(\theta, D_\tau)g(\tau))\big|\Big)$$

and hence by means of (8.5)

$$\int_0^\infty |R^0(\xi, D_t)g_\xi(t)|^2 dt \leqq C\Big(\int_0^\infty |P^0(\xi, D_t)g_\xi(t)|^2 dt + \big|s_\xi^0(\gamma_0 B^0(\xi, D_t)g_\xi(t), \gamma_0 Q^0(\xi, D_t)g_\xi(t))\big|\Big).$$

This implies that

$$(8.6) \qquad \int_0^\infty |R^0(\xi, D_t)v(t)|^2 dt \leqq C\Big(\int_0^\infty |P^0(\xi, D_t)v(t)|^2 dt + \big|s_\xi^0(\gamma_0 B^0(\xi, D_t)v, \gamma_0 Q^0(\xi, D_t)v)\big|\Big)$$

for all $\xi \in \boldsymbol{R}^{n-1}\diagdown\{0\}$ and all $v \in C_0^\infty[\bar{\boldsymbol{R}}_+]$.

8.4. To complete the proof, let $u \in C_0^\infty[\bar{\boldsymbol{R}}_+^n]$ and $\xi \in \boldsymbol{R}^{n-1}\diagdown\{0\}$. Then the function $v \in C_0^\infty[\bar{\boldsymbol{R}}_+]$ defined by

$$v(t) = (\mathscr{F}_x u)(\xi, t)$$

satisfies (8.6), that is, we have

$$\int_0^\infty |R^0(\xi, D_t)(\mathscr{F}_x u)(\xi, t)|^2 dt \leqq C\Big(\int_0^\infty |P^0(\xi, D_t)(\mathscr{F}_x u)(\xi, t)|^2 dt + \big|s_\xi^0(\gamma_0 B^0(\xi, D_t)(\mathscr{F}_x u), \gamma_0 Q^0(\xi, D_t)(\mathscr{F}_x u))\big|\Big).$$

Integrating the above inequality over $\boldsymbol{R}_\xi^{n-1}$, we obtain

$$\int_0^\infty \Big(\int |(\mathscr{F}_x(R^0(D)u))(\xi, t)|^2 d\xi\Big) dt \leqq C\Big(\int_0^\infty \Big(\int |(\mathscr{F}_x(P^0(D)u))(\xi, t)|^2 d\xi\Big) dt + S^0(\gamma_0 B^0(D)u, \gamma_0 Q^0(D)u)\Big)$$

and consequently (6.2).

Thus we have proved Theorem 6.1.

9. Concluding remarks

9.1. The case $B=Q$. If we have $B=Q$ and choose the matrix $S(\xi)=(S_{jk}(\xi))$ such that

$$S_{jj}(\xi) = \langle\xi\rangle^{2(\mu-\mu_j-q_n/2)}+1$$

and

$$S_{jk}(\xi) = 0 \quad \text{for } j \neq k,$$

then we obtain as a result (see [6], [10]; cf. also [5]):

Theorem 9.1. *Under the assumptions of 6.1, the estimate*

$$\|R(D)u\|^2 \leqq C\Big(\|P(D)u\|^2+\sum_{j=1}^{\varkappa}|\gamma_0 B_j(D)u|^2_{\mu-\mu_j-q_n/2}+\|u\|^2\Big)$$

is valid for all $u\in C_0^\infty[\bar{R}^n_+]$ *if and only if for each* $\xi\in R^{n-1}\setminus\{0\}$ *the following conditions are satisfied:*

(i) $B^0(\xi,\zeta)\equiv 0 \bmod M^0(\xi,\zeta)$.

(ii) *The polynomials* $B_j^0(\xi,\zeta)$, $j=1,\ldots,\varkappa$, *are linearly independent modulo* $P_+^0(\xi,\zeta)$.

Indeed, by Remark 6.2, the statement follows from the fact that the matrix $S^0(\xi)$ is now regular, and from the trivial inequality ($s\geqq 0$)

$$C_1(1+\langle\xi\rangle^2)^s \leqq 1+\langle\xi\rangle^{2s} \leqq C_2(1+\langle\xi\rangle^2)^s.$$

9.2. Now we will show that the above results can be used to find sufficient conditions for the validity of coerciveness inequalities for some types of mixed boundary problems, too (cf. [11]; see also [8]).

A. We first introduce some notations. For $s\geqq 0$, let $[\cdot]_s$ be the functional on $C_0^\infty(R^{n-1})$ defined by

$$[v]_s = \|\langle\xi\rangle^s(\mathcal{F}_x v)\|_{L^2(R^{n-1}_\xi)}, \qquad v\in C_0^\infty(R^{n-1}).$$

We set

$$C_{00}^\infty(R^{n-1}) = \{v\in C_0^\infty(R^{n-1})\,|\,\operatorname{supp} v\cap R^{n-2}\times\{0\} = \emptyset\}$$

and

$$C_{00}^\infty[\bar{R}^n_+] = \{u\in C_0^\infty[\bar{R}^n_+]\,|\,\gamma_0 u\in C_{00}^\infty(R^{n-1})\}.$$

If the extension of $w\in C_0^\infty(R^{n-1}_\pm)$ by 0 to R^{n-1} is denoted by $w_\pm$, then $w_\pm\in C_{00}^\infty(R^{n-1})$, and we define

$$[w]_{+,s} = [w_+]_s, \qquad [w]_{-,s} = [w_-]_s.$$

Note that every $v\in C_{00}^\infty(R^{n-1})$ can be represented in the form

$$v = v_+ + v_- \quad \text{with } v_\pm = (v|_{R^{n-1}_\pm})_\pm\in C_{00}^\infty(R^{n-1}).$$

B. Let there be given polynomials P and R and polynomial vectors B and Q as in 6.1 such that their principal parts satisfy (II) and (III) in Theorem 2.1. Furthermore, suppose that for each $\xi\in \boldsymbol{R}^{n-1}\setminus\{0\}$ there is a sesquilinear form s_ξ: $\boldsymbol{C}^\varkappa\times\boldsymbol{C}^\varkappa\to\boldsymbol{C}$, the corresponding matrix $S(\xi)$ being as in Theorem 2.1, such that (I) and (e.g.) the following two conditions are satisfied:

(i) Each $S_{jk}(\xi)$ is a polynomial in ξ.

(ii) There exists a positive constant C_0 such that, for every $\xi\in\boldsymbol{R}^{n-1}\setminus\{0\}$,

$$|s_\xi(B(\xi, U), Q(\xi, U))| \leqq C_0 \operatorname{Re} s_\xi(B(\xi, U), Q(\xi, U)) \tag{9.1}$$

for all $U=(U_0, \dots, U_{r-1})\in \boldsymbol{C}^r$ (with sufficiently large $r\leqq \mu/q_n$), where we have used the notation

$$B_j(\xi, U) = \sum_{\langle\alpha, q\rangle\leqq\mu_j} b_{j\alpha}\xi^{\alpha'}U_{\alpha_n}, \quad \text{etc.}$$

Under the preceding assumptions, we shall verify the estimate

$$\|R(D)u\| \leqq C\Big(\|P(D)u\| + \sum_{j=1}^{\varkappa}[\gamma_0 B_j(D)u]_{+,\mu-\mu_j-q_n/2} + \sum_{k=1}^{\varkappa}[\gamma_0 Q_k(D)u]_{-,\mu-\nu_k-q_n/2} + \|u\|\Big) \tag{9.2}$$

for all $u\in C_{00}^\infty[\overline{\boldsymbol{R}}_+^n]$.

Note that the above conditions may certainly be weakened; for instance, on the right-hand side of (9.1) one can replace s_ξ by any suitable sesquilinear form.

C. Accordingly to Theorem 6.1, we first have

$$\|R(D)u\|^2 \leqq C\Big(\|P(D)u\|^2 + \int |s_\xi(\mathscr{F}_x(\gamma_0 B(D)u), \mathscr{F}_x(\gamma_0 Q(D)u))|\, d\xi + \|u\|^2\Big) \tag{9.3}$$

for all $u\in C_0^\infty[\overline{\boldsymbol{R}}_+^n]$. If $v, w\in C_{00}^\infty(\boldsymbol{R}^{n-1})$, we find by (i)

$$\int S_{jk}(\xi)(\mathscr{F}_x v_+)(\xi)\overline{(\mathscr{F}_x w_-)(\xi)}\, d\xi = 0.$$

Therefore, if $u\in C_{00}^\infty[\overline{\boldsymbol{R}}_+^n]$, it follows from (ii) that

$$\begin{aligned}&\int |s_\xi(\mathscr{F}_x(\gamma_0 B(D)u), \mathscr{F}_x(\gamma_0 Q(D)u))|\, d\xi \\ &\leqq C_0 \operatorname{Re}\Big(\int s_\xi(\mathscr{F}_x(\gamma_0 B(D)u)_+, \mathscr{F}_x(\gamma_0 Q(D)u))\, d\xi \\ &\qquad + \int s_\xi(\mathscr{F}_x(\gamma_0 B(D)u), \mathscr{F}_x(\gamma_0 Q(D)u)_-)\, d\xi\Big).\end{aligned}$$

By Hölder's inequality, we obtain

$$\begin{aligned}&\int |S_{jk}(\xi)\mathscr{F}_x(\gamma_0 B_j(D)u)_+ \overline{\mathscr{F}_x(\gamma_0 Q_k(D)u)}|\, d\xi \\ &\leqq C[\gamma_0 B_j(D)u]_{+,\mu-\mu_j-q_n/2}[\gamma_0 Q_k(D)u]_{\mu-\nu_k-q_n/2},\end{aligned}$$

and analogously

$$\begin{aligned}&\int |S_{jk}(\xi)\mathscr{F}_x(\gamma_0 B_j(D)u)\overline{\mathscr{F}_x(\gamma_0 Q_k(D)u)_-}|\, d\xi \\ &\leqq C[\gamma_0 B_j(D)u]_{\mu-\mu_j-q_n/2}[\gamma_0 Q_k(D)u]_{-,\mu-\nu_k-q_n/2}.\end{aligned}$$

Now let $\varepsilon>0$ be arbitrary. Then we conclude from above, by Lemma 5.7, that

$$\begin{aligned}(9.4)\qquad &\int \left|s_\xi(\mathcal{F}_x(\gamma_0 B(D)u),\ \mathcal{F}_x(\gamma_0 Q(D)u))\right| d\xi \\ &\leqq \varepsilon C_1\|u\|_\mu^2+\varepsilon^{-1}C_2\left(\sum_{j=1}^{\varkappa}[\gamma_0 B_j(D)u]^2_{+,\mu-\mu_j-q_n/2}+\sum_{k=1}^{\varkappa}[\gamma_0 Q_k(D)u]^2_{-,\mu-\nu_k-q_n/2}\right).\end{aligned}$$

D. By (7.5) we have

$$(9.5)\qquad \|u\|_\mu^2 \leqq C_3 \sum_{\langle\alpha,q\rangle=\mu}\|D^\alpha u\|^2+C_4\|u\|^2,$$

and, on the other hand, Theorem 6.1 yields

$$\begin{aligned}(9.6)\qquad &\sum_{\langle\alpha,q\rangle=\mu}\|D^\alpha u\|^2 \\ &\leqq C_5\left(\|P(D)u\|^2+\int \left|s_\xi(\mathcal{F}_x(\gamma_0 B(D)u),\ \mathcal{F}_x(\gamma_0 Q(D)u))\right| d\xi+\|u\|^2\right).\end{aligned}$$

Choose now $\varepsilon>0$ such that

$$\varepsilon \leqq \frac{1}{2}(C_1C_3C_5)^{-1}.$$

Then it follows from (9.4)—(9.6) that

$$\begin{aligned}&\int \left|s_\xi(\mathcal{F}_x(\gamma_0 B(D)u),\ \mathcal{F}_x(\gamma_0 Q(D)u))\right| d\xi \\ &\leqq C\Bigg(\|P(D)u\|^2+\sum_{j=1}^{\varkappa}[\gamma_0 B_j(D)u]^2_{+,\mu-\mu_j-q_n/2} \\ &\qquad +\sum_{k=1}^{\varkappa}[\gamma_0 Q_k(D)u]^2_{-,\mu-\nu_k-q_n/2}+\|u\|^2\Bigg).\end{aligned}$$

Finally, combining this with (9.3), we obtain

$$\begin{aligned}\|R(D)u\|^2 \leqq C\Bigg(&\|P(D)u\|^2+\sum_{j=1}^{\varkappa}[\gamma_0 B_j(D)u]^2_{+,\mu-\mu_j-q_n/2} \\ &+\sum_{k=1}^{\varkappa}[\gamma_0 Q_k(D)u]^2_{-,\mu-\nu_k-q_n/2}+\|u\|^2\Bigg)\end{aligned}$$

for all $u\in C_{00}^\infty[\bar{R}_+^n]$, which at once implies the desired estimate (9.2).

E. For an example of this type of boundary problem, let $n=2$, $(m_1, m_2)=(8, 4)$, and consider the polynomial

$$P(\xi,\zeta) = \xi^8+\zeta^4.$$

Then $\mu=8$, $q=(q_1, q_2)=(1, 2)$, and $\varkappa=2$. Let $R(\xi,\zeta)$ be a q-homogeneous monomial with q-deg $R=8$, and take

$$B_1(\xi,\zeta)=1,\qquad B_2(\xi,\zeta)=\zeta$$

and

$$Q_1(\xi,\zeta)=\xi^2,\qquad Q_2(\xi,\zeta)=\xi^2\zeta;$$

consequently $\mu_1=0$, $\mu_2=2$, $\nu_1=2$, $\nu_2=4$.

Next, define

$$S(\xi) = \begin{bmatrix} 2\xi^{12} & i\xi^{10} \\ -i\xi^{10} & 2\xi^{8} \end{bmatrix}.$$

If $\xi \neq 0$, then $S(\xi)$ is regular and

$$s_\xi(\mathscr{F}_x(\gamma_0 B(D)u), \mathscr{F}_x(\gamma_0 Q(D)u)) = U(\xi)T(\xi)\overline{U(\xi)}^*,$$

where

$$U(\xi) = (\hat{u}(\xi, 0), (D_t\hat{u})(\xi, 0))$$

and

$$T(\xi) = \xi^2 S(\xi).$$

Now the form that corresponds to the matrix $T(\xi)$ is hermitian and positive definite. Thus we have even $s_\xi = \operatorname{Re} s_\xi$.

Therefore, (ii) is satisfied, too.

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Received 11 November 1978