A NON-NORMAL FUNCTION WHOSE DERIVATIVE HAS 
FINITE AREA INTEGRAL OF ORDER $0<p<2$

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1. Introduction

Let $f$ be a function holomorphic in $D = \{ |z| < 1 \}$, and let

$$I_p(f) = \iint_D |f'(z)|^p \, dx \, dy \quad (z = x + iy, \ 0 < p < \infty).$$

It is known that if $I_2(f) = \infty$, then $f$ is normal in $D$ in the sense of O. Lehto and K. I. Virtanen [4], or equivalently, $\sup_{z \in D} (1 - |z|) |f'(z)|/(1 + |f(z)|^2) < \infty$. We shall show that there exists a non-normal $f$ such that $I_p(f) < \infty$ for each $p, \ 0 < p < 2$.

H. Allen and C. Belna [1, Theorem 1] proved that there exists a non-normal $f$ such that $I_1(f) < \infty$. Our example therefore fills up the remaining gap between 1 and 2. Note that if $I_p(f) < \infty$, then $I_q(f) < \infty$ for all $q, \ 0 < q < p$.

Theorem. Let the zeros $\{z_n\}$ of the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{\bar{z}_n z^n}$$

satisfy the inequality

$$1 - |z_{n+1}| \equiv \beta (1 - |z_n|) < \beta, \quad n \equiv 1,$$

where $\beta, \ 0 < \beta < 1$, is a constant. Assume that 1 is the limit of a subsequence of $\{z_n\}$. Then the function

$$f(z) = B(z) \log \frac{1}{1 - z} \quad (\log 1 = 0)$$

is not normal in $D$, yet it satisfies $I_p(f) < \infty$ for all $p, \ 0 < p < 2$.

A typical example of $\{z_n\}$ is $\{1 - \beta^n\}$.

doi:10.5186/aasfm.1978-79.0431
2. Proof of Theorem

Lemma 1. Consider $B$ of (1.1) with (1.2). Assume that a holomorphic function $g$ in $D$ satisfies
\[ \lim_{j \to \infty} g(z_{n_j}) = \infty \]
for a certain subsequence $\{z_{n_j}\}$ of $\{z_n\}$. Then the product $Bg$ is not normal in $D$.

Since
\[ |B'(z_n)| = \frac{1}{1 - |z_n|^2} \prod_{k=1, k \neq n}^{\infty} \frac{|z_n - z_k|}{|1 - \overline{z_k}z_n|}, \]
it follows from the proof of [5, Theorem 2] that
\[ (1 - |z_n|) |B'(z_n)| \equiv \frac{1}{2} \prod_{k=1, k \neq n}^{\infty} \frac{|z_n - z_k|}{|1 - \overline{z_k}z_n|} \equiv \frac{1}{2} \left( \prod_{k=1}^{\infty} \left( \frac{1 - \beta^k}{1 + \beta^k} \right)^2 \right) = a > 0. \]
Since
\[ (1 - |z_{n_j}|)^{-1/2} |g(z_{n_j})| = 1 - |z_{n_j}| \|B'(z_{n_j})g(z_{n_j})\| \equiv a ||g(z_{n_j})|| \to \infty \]
as $j \to \infty$, the function $Bg$ is not normal.

Lemma 2. Assume that $p > 1/2$ and $1/2 < \gamma < 1$. Then, for each $r$, $1/(2\gamma) \leq r < 1$, and for each real constant $\tau$, the following estimate
\[ J(\gamma, r, p) = \int_0^{2\pi} |1 - \gamma r e^{i(0 - \tau)}|^{-2p} d\theta \leq c_p (1 - \gamma r)^{-2p+1} \]
holds, where
\[ c_p = \int_{-\infty}^{+\infty} [1 + 2\pi^{-3/2}]^{-p} dt < \infty. \]
Since $1/2 \equiv \gamma r - 1 < 1$, it follows from the known estimate [2, p. 66, line 4 from above] that
\[ J(\gamma, r, p) = \int_{-\pi}^{\pi} \frac{d\theta}{(1 - 2\gamma r \cos \theta + \gamma^2 r^2)^p} \equiv c_p (1 - \gamma r)^{-2p+1}. \]

Proof of Theorem. Set $g(z) = \log [1/(1-z)]$ in $D$. It follows from Lemma 1 that $f = Bg$ is not normal.

It suffices to prove that $I_p(f) < \infty$ for each $p$, $1 < p < 2$. From $f' = B'g + Bg'$ it follows that
\[ |f'|^p \leq 2^p (|B'g|^p + |Bg'|^p) \]
(see [2, p. 2]). Since
\[ \int_D |B(z)g'(z)|^p dx dy \equiv \int_D |g'(z)|^p dx dy = \int_D |1-z|^p dx dy < \infty, \]
A non-normal function whose derivative has finite area integral of order $0 < p < 2$

because $p < 2$, we have only to prove that

$$P \equiv \int_{3/4}^{1} \int_{|z| < 1} |B'(z) g(z)|^p \, dx \, dy < \infty.$$  

From $M(r, g) = \max_{|z| = r} |g(z)| \equiv -\log (1 - r) + (\pi/2)$, $0 < r < 1$, we conclude that, for each $s > 0$,

$$\int_0^1 M(r, g)^s \, dr < \infty.$$  

Let $N$ be a natural number such that $1/(2|z_n|) < 3/4$ for all $n > N$. Since

$$|B'(z)| = \left| \sum_{n=1}^{\infty} \frac{|z_n|^2 - 1}{z_n} \frac{1}{\sum_{k=1}^{\infty} |z_k|} \frac{z_k - z}{1 - \overline{z} \, z} \right| \leq 2 \sum_{n=1}^{\infty} n^{-1} \cdot n \frac{1 - |z_n|}{|1 - \overline{z} \, z|^2},$$  

it follows from the Hölder inequality with $p^{-1} + q^{-1} = 1$ that

$$|B'(z)|^p \leq 2^p \left( \sum_{n=1}^{\infty} n^{-q} \right)^{p/q} \sum_{n=1}^{\infty} n^{p-2} \left( \frac{1 - |z_n|^2}{|1 - \overline{z} \, z|^2} \right)^p \leq K_1 \left[ \sum_{n=1}^{N} n^{p} (1 - |z_n|^2)^{-p} + \sum_{n=N+1}^{\infty} n^p \left( \frac{1 - |z_n|^2}{|1 - \overline{z} \, z|^2} \right)^p \right] \leq K_2 + K_1 \sum_{n=N+1}^{\infty} n^p (1 - |z_n|^p) \left( 1 - \overline{z} \, z \right)^{-2p}, \quad z \in D.$$  

Here and hereafter, $K_j$, $j = 1, \ldots, 8$, are positive constants. It now follows from Lemma 2 that, for each $r$, $3/4 < r < 1$,

(2.4) \quad \int_0^{2\pi} |B'(re^{i\theta}) g(re^{i\theta})|^p \, d\theta \leq M(r, g)^p \left[ K_3 + K_1 \sum_{n=N+1}^{\infty} n^p (1 - |z_n|) \, J(|z_n|, r, p) \right] \leq K_3 M(r, g)^p + K_4 \sum_{n=N+1}^{\infty} n^p (1 - |z_n|) \, M(r, g)^p (1 - |z_n|^r)^{-2p+1} \leq Q(r).$$

Therefore, on considering (2.3), one obtains the bound

(2.5) \quad P \equiv \int_{3/4}^{1} Q(r) \, dr \leq K_5 + K_4 \sum_{n=N+1}^{\infty} n^p (1 - |z_n|) \, \int_0^1 M(r, g)^p (1 - |z_n|^r)^{-2p+1} \, dr.$$

Choose $\lambda > 1$ and $\mu > 1$ such that

$$\mu < (p - 1)^{-1} \quad \text{and} \quad \lambda^{-1} + \mu^{-1} = 1.$$
It then follows from the Hölder inequality, together with (2.3), that the last integral in (2.5) is not greater than

\[ \left[ \int_0^1 M(r, g)^2 \, dr \right]^{1/2} \left[ \int_0^1 (1 - |z_n| r)^{\beta p (2p + 1)} \, dr \right]^{1/\beta} \]

where

\[ t = -2p + 1 + \mu^{-1} < 0. \]

It then follows from (2.5) that

\[ P \equiv K_5 + K_7 \sum_{n=N+1}^{\infty} n^p (1 - |z_n|)^{1+p} \equiv K_5 + K_8 \sum_{n=1}^{\infty} n^2 \beta^p n \beta^{p+1} < \infty \]

because

\[ 1 - |z_n| \leq \beta^{n-1} (1 - |z_1|), \quad n \equiv 1, \]

and \( t + p > 0. \)

3. Concluding remarks

Since the function \( g(z) = \log \left[ 1/(1-z) \right] \) is a member of

\[ H \equiv \bigcap_{0 < \lambda < \infty} H^\lambda, \]

where \( H^\lambda \) is the Hardy class (see [7, p. 58]), it follows from \( |f| \leq |g| \), that \( f \) of Theorem also belongs to \( H \). On the other hand, if \( h \) is holomorphic in \( D \) such that \( I_2(h) < \infty \), then \( h \in H \) (see [2, p. 106]). It might be of interest to note that this follows from [3, Theorem] because the area of the image \( h(D) \) (not the Riemannian image \( F \)) of \( D \) by \( h \) is less than that of \( F \), being \( I_2(h) \).

It is obvious that \( f \) in Theorem is not Bloch [6], that is,

\[ \sup_{z \in D} (1 - |z|) |f'(z)| = \infty. \]

However, \( f \) is "near Bloch" in the sense that

\[ \sup_{z \in D} (1 - |z|^\alpha) |f'(z)| < \infty \]

for every \( \alpha > 1 \). In fact, for each \( h \in H \), and for each \( \alpha > 1 \), (3.1) is valid if \( f \) is replaced by \( h \). For the proof we set \( p = (\alpha - 1)^{-1} \). Since \( h \in H^p \), it follows from the inequality [2, Lemma, p. 36] (see also [2, p. 144, line 2 from above]) that

\[ \sup_{z \in D} (1 - |z|)^{\alpha - 1} |h(z)| < \infty. \]

This, together with [2, Theorem 5.5 in the case \( p = \infty \), p. 80], shows that (3.1) is true for \( h \).

Consider next the weighted integral \( I_2(h) \),

\[ A_2(h) = \iint_D (1 - |z|)^{\alpha} |h'(z)|^2 \, dx \, dy \quad (\alpha \equiv 0) \]
A non-normal function whose derivative has finite area integral of order \(0 < p < 2\) of arbitrary \(h\) holomorphic in \(D\). Obviously, \(I_\alpha(h) = A_\alpha(h)\), and \(A_\alpha(h) \equiv A_\beta(h)\) if \(\alpha \equiv \beta\). We shall show that our function \(f\) in Theorem also satisfies \(A_\alpha(f) < \infty\) for all \(\alpha > 0\).

For the proof we may assume that \(0 < \alpha < 1\). Since

\[
\iint_D (1 - |z|)^p |B(z)g(z)|^2 dx \, dy \equiv \iint_D |1 - z|^{2p} dx \, dy < \infty,
\]

it suffices to show that

\[
P_1 \equiv \int_{\alpha < |z| < 1} (1 - |z|)^p |B'(z)g(z)|^2 \, dx \, dy < \infty.
\]

We follow the proof of (2.2) up to the estimate (2.4), where, in the present case, we may set \(p = 2\). Then,

\[
\int_0^{2\pi} |B'(re^{i\theta})g(re^{i\theta})|^2 \, d\theta \equiv C_1 M(r, g)^2 + C_2 \sum_{n=N+1}^{\infty} n^2 (1 - |z_n|)^2 M(r, g)^2 (1 - |z_n|)^{-3}
\]

\[
\equiv Q_1(r), \quad 3/4 < r < 1.
\]

Here and hereafter, \(C_j, j=1, \ldots, 5\), are positive constants. On considering

\[(1-r)^2 M(r, g)^2 \leq M(r, g)^2\]

and

\[(1-r)^2 (1-|z_n|r)^{-3} \leq (1-|z_n|r)^{-3+\xi},\]

one obtains from (2.3) the following estimate:

(3.2) \[
P_1 \equiv \int_{3/4}^1 (1-r)^2 Q_1(r) \, dr
\]

\[
\equiv C_3 + C_2 \sum_{n=N+1}^{\infty} n^2 (1-|z_n|)^2 \int_0^1 M(r, g)^2 (1-|z_n|r)^{-3+\xi} \, dr.
\]

With the choice \(1 < \mu < (1-\alpha)^{-1}, \mu^{-1} + \mu^{-1} = 1\), the last integral in (3.2) is not greater than

\[
\left[ \int_0^1 M(r, g)^2 \, dr \right]^{1/\mu} \left[ \int_0^1 (1-|z_n|r)^{-3+\xi} \, dr \right]^{1/\mu} \equiv C_4 (1-|z_n|)^{-3+\xi+\mu^{-1}}.
\]

It follows that

\[
P_1 \equiv C_3 + C_5 \sum_{n=1}^{\infty} n^2 (1-|z_n|)^{-1+\xi+\mu^{-1}} < \infty,
\]

because \(-1+\alpha+\mu^{-1} > 0\).
References


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Received 22 January 1979