Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 5, 1980, 3—12

EXPLICIT UPPER BOUNDS FOR THE SOLUTIONS OF SOME DIOPHANTINE EQUATIONS

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1. Introduction

Let L be an algebraic number field of degree $l\geqq 1$ with a ring of integers $\mathbf{Z}_L$. Let $F(\mathbf{x})=F(x_1, \ldots, x_m)\in \mathbf{Z}_L[x_1, \ldots, x_m]$ be a decomposable form of degree $n\geqq 3$ in $m\geqq 2$ variables. We may suppose without loss of generality that the coefficient of x_1^n is not zero (see e.g. [2] and [11]). Suppose that in the factorization

$$F(\mathbf{x}) = a_0 L_1(\mathbf{x}) \ldots L_n(\mathbf{x}), \quad 0 \neq a_0 \in \mathbf{Z}_L,$$

of F the system $\mathscr{L}$ of the linear forms $L_j(\mathbf{x})=x_1+\alpha_{2j}x_2+\ldots+\alpha_{mj}x_m, j=1, \ldots, n,$ is connected (i.e. for any distinct i, j with $1\leqq i, j\leqq n$ there is a sequence $L_i=L_{i_1}, \ldots, L_{i_v}=L_j$ in $\mathscr{L}$ such that for each u with $1\leqq u\leqq v-1$ $L_{i_u}, L_{i_{u+1}}$ have a linear combination with non-zero algebraic coefficients which belongs to $\mathscr{L}$; cf. [13] or [11]) and that there is no $0\neq\mathbf{x}\in L^m$ for which $L_j(\mathbf{x})=0$, $j=1, \ldots, n$ [1]. Let $\beta, \pi_1, \ldots, \pi_s$ be fixed non-zero algebraic integers in L and let d be a positive integer. Assume that $\pi_1, \cdots, \pi_s$ are not units. In [11] we obtained as a consequence of our main result that the diophantine equation

$$F(\mathbf{x}) = \beta\pi_1^{z_1}\ldots\pi_s^{z_s} \tag{1}$$

has only finitely many solutions in $\mathbf{x}\in\mathbf{Z}_L^m$, $z_1, \ldots, z_s\in\mathbf{Z}$ with $N_{L/Q}((x_1, \ldots, x_m))\leqq d$ and $z_1, \ldots, z_s\geqq 0$ and that all these solutions can be effectively determined.

In the proof of our theorem in [11] there occurred various constants which were said to be effectively computable, but which in fact were not explicitly calculated. The purpose of the present paper is to derive appropriate values for these constants, and thereby to obtain an explicit upper bound for the sizes of the solutions of (1). Our main result generalizes a recent theorem of Kotov and Sprindžuk [17] concerning the Thue—Mahler equation. Further, it generalizes some results of Győry [6], [7], Győry and Papp [13], [14], [15] and Trelina [25] concerning norm form, discriminant form and index form equations, respectively.

[1] If in particular $m=2$, then every binary form $F\in\mathbf{Z}_L[x, y]$ which has at least three pairwise nonproportional linear forms in its factorization satisfies these conditions.

doi:10.5186/aasfm.1980.0508

2. Results

Let L, $F(\mathbf{x})=F(x_1, \dots, x_m)$, $\beta, \pi_1, \dots, \pi_s$ and d be defined as above. Denote by G the splitting field of $F(\mathbf{x})$ over L and let $[G:\mathbf{Q}]=g$, $[G:L]=f$. Let R_G and h_G (resp. R_L and h_L) be the regulator and the class number of G (resp. of L). Let r denote the number of fundamental units in G. Put $|N_{L/Q}(\beta)|\leqq b$ and[2] $\overline{|a_0\alpha_{ij}|}\leqq A$ (with $\alpha_{1j}=1$ for $j=1, \dots, n$). Suppose, for convenience, that in (1) $(\pi_i)=\mathfrak{p}_i^{h_L}$, $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ being distinct prime ideals in L with norms $N(\mathfrak{p}_i)=p_i^{f_i}$, $i=1, \dots, s$. Here $p_1, \dots, p_s$ denote rational primes not exceeding P. With the above notations we have the following

Theorem 1. *Let L, $F(x_1, \dots, x_m)$, $\beta, \pi_1, \dots, \pi_s$ and d be as above. Then for every solution of the equation* (1) *in $x_1, \dots, x_m\in\mathbf{Z}_L$, $z_1, \dots, z_s\in\mathbf{Z}$ with $N_{L/Q}((x_1, \dots, x_m))\leqq d$, $z_1, \dots, z_s\geqq 0$ there exists a unit ε in L such that*

$$\max\{\overline{|\varepsilon x_1|}, \dots, \overline{|\varepsilon x_m|}, (p_1^{f_1 z_1}\dots p_s^{f_s z_s})^{h_L/ln}\}\leqq d^{1/l}T, \tag{2}$$

where

$$T=\exp\{n^2 C h_L R_L^* P^g (\log P)^5 R_G \log^3(R_G^* h_G)(R_G+h_G\log P)^{sf+2}\cdot$$
$$\cdot(R_G+sh_G\log P+n\log A+\log b)\},$$

$$C=\big(25(r+sf+3)g\big)^{22r+13sf+2srf+42},\quad R_G^*=\max(R_G, e)\quad \textit{and}\quad R_L^*=\max(R_L, e).$$

Suppose $\max_{1\leqq i\leqq s}\overline{|\pi_i|}\leqq\mathcal{P}(\geqq e)$. From Theorem 1 we can easily deduce the following theorem.

Theorem 2. *Let L, $F(x_1, \dots, x_m)$, $\beta, \pi_1, \dots, \pi_s$ and d be defined as above. Then all solutions of the equation* (1) *in $x_1, \dots, x_m\in\mathbf{Z}_L$, $z_1, \dots, z_s\in\mathbf{Z}$ with $N((x_1, \dots, x_m))\leqq d$, $z_1, \dots, z_s\geqq 0$ satisfy*

$$\max\{\overline{|x_1|}, \dots, \overline{|x_m|}\}\leqq\overline{|\beta|}^{1/n}(d^{1/l}T)^{l\left(\frac{s}{h_L}\log\mathcal{P}+1\right)}, \tag{3}$$
$$p_1^{f_1 z_1}\dots p_s^{f_s z_s}\leqq(d^{1/l}T)^{ln/h_L}.$$

Theorem 2 can be regarded as a $\mathfrak{p}$-adic analogue of our Theorem 1 in [13].

Our theorems have several consequences. In this paper we restrict ourselves to some applications concerning diophantine equations. Further applications will be given in a separate paper.

We shall state the consequences of Theorem 2 only, but from Theorem 1 one can deduce similar corollaries.

Let $L, \beta, \pi_1, \dots, \pi_s$ and d be as above and let $f(x, y)\in\mathbf{Z}_L[x, y]$ be a binary form of degree $n\geqq 3$. There is an extensive literature on the Thue—Mahler equation

$$f(x, y)=\beta\pi_1^{z_1}\dots\pi_s^{z_s} \tag{4}$$

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[2] $\overline{|\gamma|}$ denotes the maximum absolute value of the conjugates of an algebraic number γ.

and its applications; we refer the reader to the papers of Coates [3], [4], [5], Sprindžuk [22], [23], [24], Kotov [16] and Kotov and Sprindžuk [17] and thence to the literature mentioned there. The best known upper bounds for the solutions of (4) are due to Kotov and Sprindžuk [17]. Their bounds are especially good in terms of s. However, in [17] it is assumed that $n \geqq 5$ and the constants corresponding to our constant C are not explicitly calculated in terms of l and n.

Corollary 1. *Let $f(x, y) \in \mathbf{Z}_L[x, y]$ be a binary form of degree $n \geqq 3$ with splitting field G over L. Suppose that $f(1, 0) \neq 0$ and that $f(x, 1)$ has at least three distinct zeros. Then all solutions of the equation* (4) *in $x, y \in \mathbf{Z}_L$, $z_1, \ldots, z_s \in \mathbf{Z}$ with $N((x, y)) \leqq d$, $z_1, \ldots, z_s \geqq 0$ satisfy*

$$(5) \qquad \max\{\overline{|x|}, \overline{|y|}\} \leqq \overline{|\beta|}^{1/n} (d^{1/l} T^*)^{l\left(\frac{s}{h_L}\log \mathcal{P}+1\right)},$$

$$p_1^{f_1 z_1} \ldots p_s^{f_s z_s} \leqq (d^{1/l} T^*)^{ln/h_L},$$

where[3]

$$T^* = \exp\{n^2 C h_L R_L^* P^g (\log P)^5 R_G \log^3 (R_G^* h_G) \cdot$$

$$\cdot (R_G + h_G \log P)^{sf+2} (R_G + s h_G \log P + n \log(2\overline{|f|}) + \log b)\}$$

with the C defined above.

In terms of R_G, h_G and P our upper bound in (5) is better than that of Kotov and Sprindžuk [17].

Let K be an extension of degree $n \geqq 3$ of L and let G be the smallest normal extension of L containing K. Let $a_0 N_{K/L}(x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m) \in \mathbf{Z}_L[x_1, \ldots, x_m]$ be a norm form in $m \geqq 2$ variables such that $K = L(\alpha_2, \cdots, \alpha_m)$, $[L(\alpha_i):L] = n_i \geqq 3$, $i = 2, \ldots, m$, and $n_2 \ldots n_m = n$. Consider the norm form equation

$$(6) \qquad a_0 N_{K/L}(x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m) = \beta \pi_1^{z_1} \ldots \pi_s^{z_s}$$

with the $\beta, \pi_1, \ldots, \pi_s$ introduced above. Put $\max(\overline{|a_0|}, \overline{|a_0 \alpha_2|}, \ldots, \overline{|a_0 \alpha_m|}) \leqq A$. When $s=0$, that is, when no $\pi_1, \ldots, \pi_s$ are specified, Papp and I [13], [15] obtained explicit bounds for the sizes of the solutions of (6). Those bounds depend on A, β and certain parameters of L and K. As a consequence of Theorem 2 we obtain the following $\mathfrak{p}$-adic generalization of these results.

Corollary 2. *Under the above assumptions all solutions of the equation* (6) *in $x_1, \ldots, x_m \in \mathbf{Z}_L$, $z_1, \ldots, z_s \in \mathbf{Z}$ with $N((x_1, \ldots, x_m)) \leqq d$, $z_1, \ldots, z_s \geqq 0$ satisfy* (3).

Our bound established here depends on G instead of K. In the case $L = \mathbf{Q}$ our Corollary 2 makes effective, for a wide class of norm forms, a recent theorem of Schlickewei [21] on norm form equations.

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[3] As usual, $\overline{|f|}$ denotes the maximum absolute value of the conjugates of the coefficients of the polynomial f.

Let $L, K, G, \beta, \pi_1, \ldots, \pi_s$ be as above. Let $\alpha_1, \ldots, \alpha_m$ be $m \geqq 2$ algebraic integers in K such that $K = L(\alpha_1, \ldots, \alpha_m)$. Suppose that $1, \alpha_1, \ldots, \alpha_m$ are linearly independent over L and that $\max_{1 \leqq i \leqq m} \overline{|\alpha_i|} \leqq A$. Generalizing my results obtained in the case $L = \mathbf{Q}$, $s = 0$ ([6], [7]), in [14] Papp and I established upper bounds for the solutions of the discriminant form equation

$$\mathrm{Discr}_{K/L}(\alpha_1 x_1 + \ldots + \alpha_m x_m) = \beta \pi_1^{z_1} \ldots \pi_s^{z_s}. \tag{7}$$

Corollary 3. *Under the above assumptions all solutions of the equation* (7) *in* $x_1, \ldots, x_m \in \mathbf{Z}_L$, $z_1, \ldots, z_s \in \mathbf{Z}$ *with* $N((x_1, \ldots, x_m)) \leqq d$, $z_1, \ldots, z_s \geqq 0$ *satisfy* (3) *with* n *replaced by* n^3.

It is difficult to compare the estimates of our Corollaries 3 and 4 with those obtained in [14] because the bounds derived in [14] depend on certain parameters of L and K, but not on G. Further, in [14] the constants corresponding the C defined above are not explicitly given [4)] in terms of l and n. Corollaries 3 and 4 give better bounds in b than those obtained in [14]. Since in [14] we derived our estimates in a more direct way (from my results [8], [9] concerning algebraic integers of given discriminant), the estimates of [14] are much better in terms of A.

Let again $L, K, G, \beta, \pi_1, \ldots, \pi_s$ be defined as above. Consider an order $\mathcal{O}$ of the field extension K/L (i.e. a subring of $\mathbf{Z}_K$ containing $\mathbf{Z}_L$ that has the full dimension n as a $\mathbf{Z}_L$-module) and suppose that $\mathcal{O}$ has a relative integral basis of the form $1, \alpha_1, \ldots, \alpha_{n-1}$ over L. (Such an integral basis exists for a number of orders of K/L; see e.g. [1], [18] and [14]). It is easy to see (cf. [14]) that

$$\mathrm{Discr}_{K/L}(\alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1}) = [\mathrm{Ind}_{K/L}(\alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1})]^2 D_{K/L}(1, \alpha_1, \ldots, \alpha_{n-1}),$$

where the decomposable form $\mathrm{Ind}_{K/L}(\alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1}) \in \mathbf{Z}_L[x_1, \ldots, x_{n-1}]$ is called the index form of the basis $1, \alpha_1, \ldots, \alpha_{n-1}$ of $\mathcal{O}$ over L. Denote by $D_{K/L}(\mathcal{O})$ the principal ideal generated by $D_{K/L}(1, \alpha_1, \ldots, \alpha_{n-1})$ and suppose $\max_{1 \leqq i \leqq n-1} \overline{|\alpha_i|} \leqq A$.

Consider the index form equation

$$\mathrm{Ind}_{K/L}(\alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1}) = \beta \pi_1^{z_1} \ldots \pi_s^{z_s}. \tag{8}$$

In the case $L = \mathbf{Q}$, $s = 0$ the author [6], [7] obtained an effectively computable bound for the solutions $x_1, \ldots, x_{n-1}$ of (8). Later Trelina [25] generalized this result of the author to arbitrary $s \geqq 1$. Independently of Trelina, Papp and I [14] established an upper bound for the solutions $x_1, \ldots, x_m \in \mathbf{Z}_L$, $z_1, \ldots, z_s \geqq 0$ in the general case when $s \geqq 1$ and L is an arbitrary but fixed number field. In [25] and [14] the constants corresponding to our constant C are effectively computable but are not explicitly calculated in terms of l and n.

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4) In the special case $s = 0$ our constants in [14] and [13] are explicitly computed.

Corollary 4. *All solutions of the equation* (8) *in* $x_1, \ldots, x_{n-1} \in \mathbf{Z}_L$, $z_1, \ldots, z_s \in \mathbf{Z}$ *with* $N((x_1, \ldots, x_{n-1})) \leqq d$, $z_1, \ldots, z_s \geqq 0$ *satisfy*

$$\text{(9)} \qquad \max\{\overline{|x_1|}, \ldots, \overline{|x_{n-1}|}\} \leqq \overline{|D_{K/L}(1, \alpha_1, \ldots, \alpha_{n-1})\beta^2|}^{1/n} (d^{1/l} T^{**})^{l\left(\frac{s}{h_L}\log \mathcal{P}+1\right)},$$

$$p_1^{f_1 z_1} \ldots p_s^{f_s z_s} \leqq (d^{1/l} T^{**})^{ln/h_L},$$

where

$$T^{**} = \exp\{n^6 C h_L R_L^* P^g (\log P)^5 R_G \log^3 (R_G^* h_G) \cdot$$

$$\cdot (R_G + h_G \log P)^{sf+2} (R_G + s h_G \log P + n^3 \log A + \log(b^2 N(D_{K/L}(\mathcal{O}))))\}$$

with the C defined above.

In the special case $L = \mathbf{Q}$ (9) is better in terms of b, P and s than the bound obtained by Trelina [25]. On the other hand, her bound is better in A than (9) (because she followed a similar argument to that applied earlier in [6]; see also [14]). In [25] the bound depends on the maximal prime factor and the number of distinct prime factors of $D_{K/Q}(\mathcal{O})$, but not on $D_{K/Q}(\mathcal{O})$. We could easily get a similar result for the equation (8) by taking the prime ideal factorization of $D_{K/L}(1, \alpha_1, \ldots, \alpha_{n-1})$ in (26) and applying our Corollary 3.

It is evident that our Corollaries 3 and 4 have applications to algebraic integers with given discriminant and given index, respectively. However, in case of algebraic numbers the more direct deductions used in [8] and [9] yield better estimates.

3. Proofs

To prove our theorems we need some lemmas. We keep the notations of Section 2. We suppose that there are r_1 real and $2r_2$ complex conjugate fields to G and that they are chosen in the usual manner: if θ is in G then $\theta^{(i)}$ is real for $1 \leqq i \leqq r_1$ and $\theta^{(i+r_2)} = \overline{\theta^{(i)}}$ for $r_1 + 1 \leqq i \leqq r_1 + r_2$.

Lemma 1. *Let* α *be a non-zero element in* G *with* $|N_{G/Q}(\alpha)| = M$ *and let* v *be a positive integer. There exists a unit* ε *in* G *such that*

$$\text{(10)} \qquad \left|\log |M^{-1/g} (\alpha \varepsilon^v)^{(i)}|\right| \leqq \frac{c_1^* r v}{2} R_G, \quad i = 1, \ldots, g,$$

where $c_1^* = (6rg^2/\log g)^r$ *or* $c_1^* = 1$ *according as* $r \geqq 1$ *or* $r = 0$.

Proof. This is Lemma 3 in [10].

Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_t$ be distinct prime ideals in G lying above rational primes not exceeding P, and let $\mu_1, \ldots, \mu_t$ be algebraic integers in G such that $(\mu_i) = \mathfrak{P}_i^{h_G}$, $i = 1, \ldots, t$. Let $\beta_j = \varkappa_j \mu_1^{u_{1j}} \ldots \mu_t^{u_{tj}}$, where $0 \neq \varkappa_j \in \mathbf{Z}_G$ with $|N_{G/Q}(\varkappa_j)| \leqq N$, and let $u_{1j}, \ldots, u_{tj}$ be non-negative rational integers, $j = 1, 2, 3$. Suppose $\lambda_1, \lambda_2, \lambda_3$ are non-zero algebraic integers in G satisfying $\max_j \overline{|\lambda_j|} \leqq H$.

Lemma 2. *If*

$$\lambda_1\beta_1+\lambda_2\beta_2+\lambda_3\beta_3 = 0 \tag{11}$$

then

$$\beta_j = \sigma\delta_j, \tag{12}$$

where $\sigma=\eta\mu_1^{a_1}\dots\mu_t^{a_t}$ *with some unit* $\eta\in G$ *and non-negative rational integers* $a_1, \dots, a_t$ *and* $\delta_j\in \mathbf{Z}_G$ *such that*

$$\max_{1\leqq j\leqq 3} \overline{|\delta_j|} \leqq \exp\{c_2^* P^g (\log P)^3 R_G \log^3 (R_G^* h_G)(R_G+h_G\log P)^{t+2}\cdot \cdot (R_G+th_G\log P+\log(HN)\}, \tag{13}$$

where $c_2^*=(25(r+t+3)g)^{20r+13t+2rt+40}$.

Proof. This is a special case of our Lemma 6 in [12]. The proof of this lemma is based on some explicit estimates of van der Poorten [19] and van der Poorten and Loxton [20].

Proof of Theorem 1. We follow the proof of our theorem established in [11]. It will be assumed that the reader is familiar with the contents of [11], and only a minimal amount of the discussion of that paper will be repeated here.

Writing $\alpha'_{ij}=a_0\alpha_{ij}$, we have $\alpha'_{ij}\in \mathbf{Z}_G$ for each i and j. We shall prove our theorem for the equation

$$f(\mathbf{x}) = a_0^{n-1}F(\mathbf{x}) = \prod_{j=1}^{n} L'_j(\mathbf{x}) = a_0^{n-1}\beta\pi_1^{z_1}\dots\pi_s^{z_s}, \tag{14}$$

where $L'_j(\mathbf{x})=\alpha'_{1j}x_1+\dots+\alpha'_{mj}x_m$.

Let $x_1, \dots, x_m$, $z_1, \dots, z_s$ be any solution of (14) with $x_1, \dots, x_m\in \mathbf{Z}_L$, $N((x_1, \dots, x_m))\leqq d$ and $z_1, \dots, z_s\geqq 0$. Put

$$\beta_j = \alpha'_{1j}x_1+\dots+\alpha'_{mj}x_m, \quad j = 1, \dots, n. \tag{15}$$

Let $\mathfrak{P}_1, \dots, \mathfrak{P}_t$ be all distinct prime ideals in G lying above $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. From (14) we get

$$(\beta_j) = \mathfrak{A}_j\mathfrak{P}_1^{U_{1j}}\dots\mathfrak{P}_t^{U_{tj}}, \quad j = 1, \dots, n, \tag{16}$$

where the $\mathfrak{A}_j$ are integral ideals in G such that $\mathfrak{A}_1\dots\mathfrak{A}_n|(a_0^{n-1}\beta)$ and the U_{kj} are non-negative rational integers. The definitions and notations given in [11] remain unchanged, except that we now have $(\chi_j)=\mathfrak{A}_j\mathfrak{P}_1^{r_{1j}}\dots\mathfrak{P}_t^{r_{tj}}$,

$$|N_{G/Q}(\chi_j)| \leqq A^{(n-1)g} b^f p^{tgh_G} \tag{17}$$

and, by Lemma 1,

$$\overline{|\mu_k|} \leqq c_3^* P^{h_G}, \quad \overline{|\chi_j|} \leqq c_3^* A^{n-1} b^{f/g} P^{th_G} \tag{18}$$

for $k=1, \dots, t$ and $j=1, \dots, n$, where $c_3^*=\exp\{(c_1^* r/2)R_G\}$. In [11] we may choose $c_9=2A^2$ and we may apply Lemma 2 to (17) of [11]. Then we get

$$\beta_q = \sigma\delta_q, \quad \sigma = \varepsilon_3\mu_1^{a_1}\dots\mu_t^{a_t}, \tag{19}$$

where ε_3 is a unit in G, $a_1, \ldots, a_t \in \boldsymbol{Z}$ with $a_k \geqq 0$ and $\delta_q \in \boldsymbol{Z}_G$ with

$$(20)\qquad \overline{|\delta_q|} \leqq \exp\{2g c_2^* P^g (\log P)^3 R_G \log^3 (R_G^* h_G)(R_G + h_G \log P)^{t+2} \cdot$$
$$\cdot (R_G + t h_G \log P + n \log A + \log b)\} = T_1, \quad q = 1, 2, 3,$$

in place of (27) of [11].

Continuing the argument of [11], a suitable choice for c_{36} is $ng/h_G \log 2$. Further, in view of the above estimates (18), (20) and of (32) of [11] we obtain (34) of [11], that is,

$$(21)\qquad \overline{|\tau_j|} \leqq \prod_{k=1}^{t} \overline{|\mu_k|}^{b_k^*} \overline{|\varphi_j/\psi_j|} \leqq \exp\{c_{37} s \log P \log T_1\} = T_2$$

with $c_{37} = c_1^* rnfg R_G$.

In place of (35) of [11] we get now from (14)

$$(22)\qquad (a_0^{n-1}\beta)\mathfrak{p}_1^{v_1} \ldots \mathfrak{p}_s^{v_s} = (\beta_1 \ldots \beta_n) = ((\vartheta \mu_1^{d_1} \ldots \mu_t^{d_t})^n \tau_1 \ldots \tau_n),$$

where $v_k = z_k h_L$. Formula (21) implies

$$\operatorname{ord}_{\mathfrak{P}} \left(\prod_{j=1}^{n} \tau_j \right) \leqq ng \log T_2$$

for each prime ideal of G lying above $\mathfrak{p}_k$. Similarly,

$$\operatorname{ord}_{\mathfrak{P}} (a_0^{n-1}\beta) \leqq (n-1) g \log A + f \log b.$$

The argument of [11] applies if we replace in (36) and (37) of [11] $\min\left(v_k e_k - \operatorname{ord}_{\mathfrak{P}}\left(\prod_{j=1}^n \tau_j\right), v_k e_k\right)$ by

$$\min \left(v_k e_k + \operatorname{ord}_{\mathfrak{P}}(a_0^{n-1}\mathfrak{P}) - \operatorname{ord}_{\mathfrak{P}} \left(\prod_{j=1}^{n} \tau_j \right), v_k e_k \right).$$

Then we have (38) and (39) of [11] with $c_{39} = h_L ng$ and $c_{40} = 2c_{39}$. Let now $\mathfrak{p}_1^{h_L y_1} \ldots \mathfrak{p}_s^{h_L y_s} = (\pi_1^{y_1} \ldots \pi_s^{y_s}) = (\varkappa)$, where $\varkappa \in \boldsymbol{Z}_L$, and choose ξ as in [11]. Then (39) of [11] implies that a suitable value for c_{41} is $c_{40} fg = 2h_L fg^2 n$.

Lemma 1 together with (34) and (41) of [11] imply that an appropriate choice for c_{42} is $2nc_{41} R_L^*/g$, and it follows immediately that suitable values for c_{43} and c_{44} occurring in (43) and (45) of [11] are given by $(3/2)c_{42}$ and $2c_{42}$, respectively.

To estimate $\overline{|v|}$ and $\overline{|v_i|}$ we can use Hadamard's inequality. Since $m \leqq nf$, it is easily seen that appropriate choices for c_{45} and c_{46} are $(nf)^{nf/2} A^{nf}$ and $(nf)^{nf/2} A^{nf-1}$, respectively. Consequently, c_{47} can be taken as $(c_{45})^{g/f} d = (nf)^{ng/2} A^{ng} d$. In view of Lemma 1 c_{48} can obviously be taken as

$$(c_{47})^{1/l} \exp\{(c_1^* r/2) R_L\} = (nf)^{ng/2l} A^{ng/l} d^{1/l} \exp\{(c_1^* r/2) R_L\}.$$

Finally, in (52) of [11] a suitable choice for c_{49} is $c_{48} c_{46} c_{45}^{l-1}$, which is less than

$$(nf)^{2ng} A^{2ng} d^{1/l} \exp\left\{\frac{c_1^* r}{2} R_L\right\}.$$

So, by (52) of [11] we have

$$\max_{1\leqq i\leqq m}\overline{|x_i'|} < c_{49}T_3 = c_{49}\exp\{8nfgh_L R_L^* s\log P\log T_2\} \tag{23}$$

$$= c_{49}\exp\{8rc_1^*(fgn)^2 h_L R_L^* s^2(\log P)^2\log T_1\}$$

$$= d^{1/l}\exp\{16r(sfn)^2 g^3 c_1^* c_2^* h_L R_L^* P^g(\log P)^5 R_G\log^3(R_G^* h_G)\cdot$$

$$\cdot(R_G+h_G\log P)^{sf+2}(R_G+sh_G\log P+n\log A+\log b)\}.$$

Since $16r(sf)^2g^3c_1^*c_2^* < (25(r+sf+3)g)^{22r+13sf+2rsf+42}$, (23) provides the desired upper bound for $\max_{1\leqq i\leqq m}\overline{|x_i'|}$.

Finally, it follows from (44), (45) and (50) of [11] that

$$(p_1^{f_1z_1}\dots p_s^{f_sz_s})^{fh_L} \leqq |N_{G/Q}(a_0^{n-1}\beta\pi_1^{z_1}\dots\pi_s^{z_s})| = |N_{G/Q}(\beta_1\dots\beta_n)|$$

$$= |N_{G/Q}(\varkappa)|^n|N_{G/Q}(\xi_1\dots\xi_n)| \leqq c_{47}^{nf}T_3^{ng},$$

and this completes the proof of Theorem 1.

Proof of Theorem 2. Let $x_1, \dots, x_m, z_1, \dots, z_s$ be an arbitrary but fixed solution of (1) with the given properties. By Theorem 1 there exists a unit ε for which (2) holds. Further, from (1) we get

$$F(\varepsilon x_1, \dots, \varepsilon x_m) = \varepsilon^n\beta\pi_1^{z_1}\dots\pi_s^{z_s}. \tag{24}$$

But it follows from (2) that

$$\overline{|a_0^{n-1}\beta\pi_1^{z_1}\dots\pi_s^{z_s}|} \leqq A^{n-1}\overline{|\beta|}\prod_{i=1}^{s}\overline{|\pi_i|}^{z_i} \leqq A^{n-1}\overline{|\beta|}(d^{1/l}T)^{\frac{nls}{h_L}\log\mathscr{P}} = T_4,$$

where T denotes the expression occurring in (2). On the other hand, by (14) we have

$$\overline{|a_0^{n-1}F(\varepsilon x_1, \dots, \varepsilon x_m)|} \leqq (mAd^{1/l}T)^n \leqq (nfAd^{1/l}T)^n = T_5.$$

Thus we obtain from (24)

$$\overline{|\varepsilon^{-n}|} = \overline{|\varepsilon^{-1}|}^n \leqq T_4\cdot T_5^{l-1} \leqq \overline{|\beta|}(nf)^{n(l-1)}A^{nl-1}(d^{1/l}T)^{\frac{nls}{h_L}\log\mathscr{P}+n(l-1)},$$

whence

$$\overline{|\varepsilon^{-1}|} \leqq \overline{|\beta|}^{1/n}(nf)^{l-1}A^l(d^{1/l}T)^{\frac{ls}{h_L}\log\mathscr{P}+(l-1)}. \tag{25}$$

Finally (2) and (25) imply (3).

Proof of Corollary 1. In what follows it will again be supposed that the reader is familiar with the proofs of the corollaries occurring in [11]. By assumption there are at least three pairwise nonproportional linear factors in the factorization

$$f(x, y) = a_0(x+\alpha_1y)\dots(x+\alpha_ny).$$

So, in order to apply our Theorem 2 it suffices to give an upper bound for $\max(\overline{|a_0|}, \overline{|a_0\alpha_1|}, \dots, \overline{|a_0\alpha_n|})$. But it is known that $\overline{|a_0\alpha_i|} < \overline{|a_0|} + \overline{|f|} \leqq 2\overline{|f|}$; hence (5) follows from (3).

Proof of Corollary 2. The argument of the proof of Corollary 4 of [11] shows that the equation (6) satisfies all the conditions of our Theorem 2. This proves the required assertion.

Proof of Corollary 3. As is known, there exists $\alpha=\alpha_1+a_2\alpha_2+\ldots+a_m\alpha_m$ with $D_{K/L}(\alpha)\neq 0$, $a_i\in\mathbf{Z}_L$, $\overline{|a_i|}\leqq n^4$, $i=2,\ldots,m$. Let $x_1=x_1'$, $x_i=a_ix_1'+x_i'$, $i=2,\ldots,m$. Then (7) gives $\mathrm{Discr}_{K/L}(\alpha x_1'+\alpha_2x_2'+\ldots+\alpha_mx_m')=\beta\pi_1^{z_1}\ldots\pi_s^{z_s}$. By applying Theorem 2 to this equation our statement follows (cf. [11]).

Proof of Corollary 4. Every solution of (8) satisfies

$$\mathrm{Discr}_{K/L}(\alpha_1x_1+\ldots+\alpha_{n-1}x_{n-1}) = D_{K/L}(1,\alpha_1,\ldots,\alpha_{n-1})\beta^2\pi_1^{2z_1}\ldots\pi_s^{2z_s}. \tag{26}$$

Now Corollary 3 applies and (9) follows immediately.

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Received 26 January 1979