ON THE LENGTH OF ASYMPTOTIC PATHS OF ENTIRE FUNCTIONS OF ORDER ZERO

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Suppose that \( f \) is an entire function of finite order and that \( P \) is a locally rectifiable path on which \( f(z) \to \infty \). Let \( l(r) \) be the length of \( P \) in \( |z| < r \). We shall consider the following question of Erdős (Hayman [1, Problem 2.41]): If \( f \) has zero order, or more generally finite order, can a path \( P \) be found for which \( l(r) = O(r) \) \( (r \to \infty) \)? Such a path \( P \) exists if

\[
\log M(r, f) = O((\log r)^2).
\]

In fact, Hayman [2] has proved that if \( f \) satisfies (A) we may choose a ray through the origin for \( P \). We shall show that (A) is the best possible growth condition under which there exists a path \( P \) satisfying \( l(r) = O(r) \).

**Theorem.** Given any increasing function \( \varphi(r) \) such that \( \varphi(r) \to \infty \) as \( r \to \infty \), there exists an entire function \( f \) such that

\[
\log M(r, f) = O(\varphi(r)(\log r)^2),
\]

and if \( P \) is any locally rectifiable curve on which \( f(z) \to \infty \) then

\[
l_0 = \exp \left\{ \frac{\arg z}{4\pi n} \right\},
\]

\( 0 \leq \arg z < \infty \), and let \( f_n \) be an entire function such that \( f_n(z) \to 0 \) on \( \Gamma_n \). We set

\[
g_n(z) = z^{-m}f_n(z),
\]

where \( m \) is chosen such that \( g_n(0) \neq 0 \) (if \( f_n(0) \neq 0 \) then \( m = 0 \)) and

\[
h_n(z) = \frac{g_n(z)}{g_n(0)},
\]

where \( \varrho_n \geq 0 \) is chosen such that \( |h_n(z)| < 1/8 \) on the path

\[
\gamma_n: |z| = \exp \left\{ \frac{\arg z}{4\pi n} \right\},
\]

**Proof.** Let \( \Gamma_n \) be the path

\[
|z| = \exp \left\{ \frac{\arg z}{4\pi n} \right\},
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\gamma_n: |z| = \exp \left\{ \frac{\arg z}{4\pi n} \right\},
\]

\(0 \leq \arg z \leq 4\pi n\). Since \(h_n\) is an entire function we see by means of the Taylor-series that there exists a polynomial \(P_n\) such that \(P_n(0) = 1\) and \(|P_n(z)| < 1/4\) on \(\gamma_n\). Let

\[ P_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{a_n,k}\right). \]

We get the desired function writing

\[ f(z) = \prod_{n=1}^{\infty} \left(P_n(z|r_n)\right)^{s_n}, \]

where \(r_1 = e, s_1 = 1, r_{n+1} > r_n^4\), and for \(n \geq 2\),

(i) \[ \varphi(\sqrt[r_n]{r}) > 8t_n \sum_{k=1}^{n-1} s_k t_k, \]

and

(ii) \[ \frac{s_n}{\log r_n} = 2 \sum_{k=1}^{n-1} s_k t_k < \frac{\log r_n}{t_n}. \]

We denote \(b_n = \min \{a_{n,k} : k = 1, 2, \ldots, t_n\}\). Assume that \(b_n \sqrt[r_n]{r} > 2\). Then we get for \(|z| \leq \sqrt[r_n]{r}\)

\[ \log |P_n(z|r_n)^{s_n}| \leq \left|s_n t_n \log \left(1 - \frac{\sqrt[r_n]{r}}{r_n b_n}\right)\right| \leq s_n t_n \frac{2}{b_n \sqrt[r_n]{r}}. \]

It follows from (ii) that

\[ \log |P_n(z|r_n)^{s_n}| \leq \frac{2(\log r_n)^2}{b_n \sqrt[r_n]{r}} \to 0 \]
as \(r_n \to \infty\). Therefore we may assume that \(r_n \to \infty\) as \(n \to \infty\) so rapidly that

(iii) \[ 1/2 < \left|\prod_{k=n+1}^{\infty} (P_k(z/r_k))^{s_k}\right| < 2 \]
on \(|z| \leq \sqrt[r_n+1]{r}\).

Let \(\sqrt[r_n]{r} \leq |z| \leq \sqrt[r_n+1]{r}\). Then it follows from (iii) that

\[ \log |f(z)| = \sum_{k=1}^{n} \log \left|(P_k(z/r_k))^{s_k}\right| + \log \left|\prod_{k=n+1}^{\infty} (P_k(z/r_k))^{s_k}\right| \leq \sum_{k=1}^{n} s_k t_k \log |z| + \log 2. \]

If \(\sqrt[r_n]{r} \leq r \leq \sqrt[r_n+1]{r}\) we see now from (ii) that

\[ \log M(r, f) \leq 2s_n t_n \log r \leq 4t_n \left(\sum_{k=1}^{n-1} s_k t_k\right) (\log r_n) \log r. \]

Since \(r_n \leq r^2\) we get

\[ \log M(r, f) \leq 8t_n \left(\sum_{k=1}^{n-1} s_k t_k\right) (\log r)^2 \]
and it follows from (i) that

(iv) \[ \log M(r, f) \leq \varphi(r) (\log r)^2 \]
if \(\sqrt[r_n]{r} \leq r \leq \sqrt[r_n+1]{r}\). Therefore \(f\) satisfies the condition (1).
We denote by $\beta_n$ the path

$$|z| = r_n \exp \left\{ \frac{\arg z}{4\pi n} \right\},$$

$0 \leq \arg z \leq 4\pi n$. Let $z \in \beta_n$. Then

$$\log |f(z)| \equiv \sum_{k=1}^{n-1} s_k t_k \log |z| + s_n \log (1/4),$$

and because $|z| \equiv er_n$, we see from (ii) that

$$\log |f(z)| \equiv 2 \left( \sum_{k=1}^{n-1} s_k t_k \right) \log r_n - s_n = 0.$$ 

This implies that

(v) $|f(z)| \equiv 1$

on $\beta_n$.

Let $P$ be any rectifiable path on which $f(z) \to \infty$. It follows from (v) that $P$ does not intersect the path $\beta_n$ if $n$ is large enough. Therefore $l(er_n) > 2n\pi r_n$ for all large values of $n$ and we get

$$\lim_{r \to \infty} \sup_{r \in \beta_n} \frac{l(r)}{r} = \infty.$$ 

The theorem is proved.

Remark. After this paper had been written, I was told that the same result was proved by A. A. Goldberg and A. E. Eremenko: On the asymptotic paths of entire functions of finite order (in Russian). Mat. Sb. 109 (151) No 4, 1979, 555—581.

References


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