ON TWO-DIMENSIONAL QUASICONFORMAL GROUPS

PEKKA TUKIA

A quasiconformal group is a group $G$ of homeomorphisms of some open set $U$ in the $n$-ball $S^n$ such that each element of $G$ is $K$-quasiconformal for some fixed $K \geq 1$. If we wish to specify $K$ and $U$, we say that $G$ is a $K$-quasiconformal group acting in $U$ (or of $U$).

One can obtain quasiconformal groups as follows. Let $V \subset S^n$ be open and let $G$ be a group of conformal homeomorphisms of $V$. (If $n \geq 3$ and $V$ is connected, $G$ is a group of Möbius transformations.) Let $f: V \to U$ be quasiconformal. Then $fGf^{-1}$ is a quasiconformal group. It has been proposed by F. Gehring that all quasiconformal groups are of this form. We offer here a proof of this conjecture for quasiconformal groups acting in open subsets of the Riemann sphere.

Kuusalo [3, Theorem 3 p. 21] has proved the following theorem, which is related to ours. Let $S$ be a quasiconformal 2-manifold. Then $S$ has a conformal structure which is compatible with the quasiconformal structure of $S$. Quasiconformal groups have been considered also in Gehring—Palka [2]. The theory of quasiconformal mappings we need can be found in Lehto—Virtanen [4].

I wish to thank Kari Hag who read the manuscript and made many valuable remarks. In particular, she informed me of Maskit’s work [5, 6], and the Corollary was suggested by her.

Theorem. Let $G$ be a $K$-quasiconformal group acting in an open subset $U \subset S^2$. Then there is a $K$-quasiconformal homeomorphism $h: U \to V \subset S^2$ such that $hGh^{-1}$ is a group of conformal self-maps of $V$ and that $h$ is the restriction of a $K$-quasiconformal homeomorphism $f$ of $S^2$ whose complex dilatation $\mu_f$ vanishes a.e. outside $U$.

A consequence of the above theorem is that if the fundamental group of a plane domain $U$ is non-cyclic and $G$ is a quasiconformal group of $U$, $G$ is discrete, i.e. there are no sequences $f_n \in G \setminus \{id\}$, $n \geq 0$, such that $\lim_{n \to \infty} f_n = \text{id}$. This follows since the group of conformal self-maps of a plane domain with non-cyclic fundamental group is discrete. This is in turn a consequence of the theory of Fuchsian groups, since in this case the universal cover of $U$ is the open unit disk and the limit set of the cover translation group contains more than two points.

If $V \subset S^2$ is open and connected, there is a conformal homeomorphism $g: V \to V' \subset S^2$ such that $gGg^{-1}$ is a Möbius transformation of $V'$ whenever $\alpha$ is

a conformal self-map of $V$, cf. Maskit [5, 6]; if $V$ is simply connected, this follows by Riemann's mapping theorem. Therefore we have the following

**Corollary.** Let $G$ and $U$ be as in the above theorem and assume that $U$ is connected. Then there is a $K$-quasiconformal homeomorphism $h': U \to U' \subset S^2$ such that $h'Gh'^{-1}$ is a group of Möbius transformations of $U'$.

Note that in general we cannot extend $h'$ to a quasiconformal homeomorphism of $S^2$.

**Proof of the theorem.** We find the conditions for $\mu$ guaranteeing that if a quasi-conformal map $f$ of $S^2$ has the complex dilatation $\mu$, then $f$ is a solution of our problem. Clearly, we must set

(1) \[ \mu|S^2\setminus U = 0. \]

The conformality of $f \circ g \circ (f^{-1}|f(U))$ for $g \in G$ is equivalent to the validity of

(2) \[ \mu_f(x) = \mu_{fg}(x) \]

for almost all $x \in U$ when $g \in G$ is fixed. Computing $\mu_{fg}$ in terms of $\mu_f$ and $\mu_g$ we must have

(3) \[ \mu_f(x) = \frac{\mu_g(x) + \mu_f(g(x))e^{-2i \arg s_2(x)}}{1 + \mu_g(x)\mu_f(g(x))e^{-2i \arg s_2(x)}} \]

a.e. in $U$. If we can find a measurable function $\mu: S^2 \to \mathbb{C}$ with $\|\mu\|_\infty \leq (K-1)/(K+1)$ that satisfies (1) and (3) for $\mu = \mu_f$ a.e. in $U$ if $g \in G$ is given, then any homeomorphic solution $f$ of the equation

(4) \[ \mu_f = \mu \quad \text{a.e. in } S^2 \]

is also a solution of our problem.

We write (3) in the form

(5) \[ \mu_f(x) = T_g(x)(\mu_f(g(x))) = \mu_{fg}(x), \]

where

\[ T_g(x)(z) = \frac{\mu_g(x) + e^{-2i \arg s_2(x)}z}{1 + \mu_g(x)e^{-2i \arg s_2(x)}z} = \frac{a + bz}{b + az} \]

with $a = g_z(x), b = g_z(x)$. Thus, whenever defined (i.e. a.e. in $U$ for fixed $g$), $T_g(x)$ is a conformal self-map of the open unit disk $D$ and is an isometry in the hyperbolic metric.

Consider the sets $M_x = \{\mu_g(x): g \in G\}, x \in U$. By (5) we have

\[ T_g(x)(M_{g(x)}) = \{T_g(x)(\mu_g(g(x))): g' \in G\} = \{\mu_{g'g}(x): g' \in G\} = \{\mu_{g'}(x): g' \in G\} = M_x \]

for almost all $x \in U$ and every $g \in G$ if $G$ is countable. Let us assume that there is a map $X \mapsto P(X) \in D$ that assigns a point to every non-empty subset $X \subset D$ which is bounded in the hyperbolic metric in such a way that $P(g(X)) = g(P(X))$ for
every isometry $g$ of $D$. Then, for countable $G$, the map $\mu(x)=P(M_x)$ satisfies (3) (with $\mu_f=\mu$) a.e. in $U$ for all $g \in G$.

Now we construct such a map $P$. Let $X \subset D$ be bounded, $X \neq \emptyset$. Then there is a unique closed hyperbolic disk $D(x, r)$ with center $x$ and radius $r \geq 0$ with the properties

(i) $D(x, r) \supset X$, and
(ii) if $D(y, r') \supset X$, $y \neq x$, then $r' \geq r$.

To see the existence of $D(x, r)$ we can reason as follows. In any case there is a smallest $r \geq 0$ such that if $r' \geq r$, there is $y \in D$ with $D(y, r') \supset X$. Next it is easy to see that there is at least one $x \in D$ such that $D(x, r) \supset X$. Assume that there is another point $y \in D$ with $D(y, r) \supset X$. Let $w$ be one of the two points of $\partial D(x, r) \cap \partial D(y, r)$ and let $z$ be the orthogonal projection (in hyperbolic geometry) of $w$ onto the hyperbolic line through $x$ and $y$. Consider the hyperbolic triangle with vertices $x$, $z$ and $w$. It has a right angle at $z$ and therefore it is geometrically evident that $d(x, w)=r'=d(z, w)$. This follows also from the relation $r'=\cosh d(x, z)\cosh d(z, w)$ (cf. e.g. Coxeter [1]). But then, if $r'=d(z, w)\leq r'$ and $D(z, r') \supset D(x, r) \cap D(y, r) \supset X$. This proves the uniqueness of $x$. Therefore, if we let $P(X)$ be the center of the smallest closed hyperbolic disk containing $X$, we have a well-defined map $P$. Clearly, $P(g(X))=g(P(X))$ for any isometry $g$ of $D$. It has also the following property.

(A) If $X \subset D(y, s)$, $X \neq \emptyset$, then $P(X) \subset D(y, s)$.

To see the validity of (A), note first that $r \leq s$ if $r$ is the radius of the smallest disk containing $X$. Then, if $d(y, P(X)) \geq s$, we can reason as above and find $D(z, r') \supset X$ with $r' \leq r$. Therefore $d(y, P(X)) \leq s$.

Now we assume for a moment that $G$ is countable, $G=\{g_0, g_1, \ldots\}$. We define a map $\mu$ by setting

$$
\mu|S^2 \setminus U = 0, \quad \text{and} \quad \mu(x) = P(M_x) \quad \text{if} \quad x \in U,
$$

which defines $\mu$ a.e. in $S^2$. Since $M_x \subset D(0, r)$, where $r=d(0, (K-1)/(K+1))$, for almost all $x \in U$, $\|\mu\|_\infty \leq (K-1)/(K+1)$ by (A). We have already observed that $\mu$ satisfies (3) (with $\mu_f=\mu$) a.e. in $U$ for all $g \in G$. It is also measurable. To see this, let

$$
\mu_n(x) = P\{\mu_{g_i}(x): i \leq n\}
$$

if $x \in U$ and $n \geq 0$. Then $\mu_n$ is a.e. defined and it is certainly measurable. Since $\mu(x) = \lim_{n \to \infty} \mu_n(x)$ a.e. in $U$, also $\mu$ is measurable. Therefore, if $G$ is countable, there is a map $f$ satisfying the conditions of the theorem.

If $G$ is not countable, there is always a countable subgroup $G' \subset G$ which is dense in the topology of uniform convergence in compact sets. This follows from the separability of the set of all continuous maps $U \to S^2$ in this topology (they can be approximated by PL maps). Then if $f$ satisfies the conditions of the theorem with respect to $G'$, it satisfies them also with respect to $G$. 

On two-dimensional quasiconformal groups 75
Remarks. An earlier version of this note proved the preceding theorem under the assumptions that $G$ was discrete and that the limit set of $G$ had zero measure. Unable to find a solution of (3) with $\mu=\mu_f$ for general $G$, I had to make these assumptions on which one can find a measurable fundamental set $S$ for $G$, and set $\mu|S=0$, determining $\mu$ completely. After it was written, F. W. Gehring called my attention to the paper [7] by Sullivan. This paper contains a sketch of the proof of the above theorem. Sullivan's proof differs from ours in the definition of $\mu$; we have defined $\mu(x) = P(M_x)$ whereas Sullivan sets $\mu(x) = B(M_x)$, where $B(X)$ is the barycenter of the convex hull of $X$, both in hyperbolic geometry. Since Sullivan gives only the barest outline of the proof and since he makes also some unnecessary assumptions (e.g. $G$ was assumed to be discrete), the publication of this little note is perhaps justified.

The use of the map $P$ seems also to have some slight advantages over the use of the map $B$. No doubt one can take barycenters also in hyperbolic geometry, but to prove the existence of $B(X)$ is non-trivial, whereas this proof is very simple for $P(X)$. Secondly, $X \mapsto P(X)$ is continuous but $X \mapsto B(X)$ is not. That is, if $\varepsilon > 0$ is given, there is $\delta = \delta(\varepsilon) > 0$ such that if $X, Y \subseteq D$ are non-empty and bounded and if

$$ g(X, Y) = \sup \{d(x, Y), d(X, y) : x \in X, y \in Y\} < \delta, $$

then $d(P(X), P(Y)) < \varepsilon$; cf. (8) below. To see the discontinuity of $B$, let $A = \{0, 1\}$ and $A_n = \{0, 1, 1+i/n\} \subseteq C, n=0$. Then if we take the barycenter of the convex hull in the euclidean geometry of $C$, we have $B(A) = 1/2$ but $\lim_{n \to \infty} B(A_n) = 2/3$.

Appendix 1. It is easy to derive an estimate for $d(P(X), P(Y))$ in terms of $g(X, Y)$, and since we will need it in a future paper, we do it here. A consequence of this estimate is that if the family $\{\mu_g : g \in G\}$ is equicontinuous, $\mu_f|U$ is continuous. Let $X, Y \subseteq D$ be non-empty and bounded. Let $d = g(X, Y), x = P(X), y = P(Y)$, and let $D(x, r) \supset X$ and $D(y, r') \supset Y$ be the smallest disks containing $X$ and $Y$, respectively. Then $D(x, r+d) \supset Y$, implying $r' \leq r+d$. Similarly, $D(y, r'+d) \supset X$, and therefore $D(y, r+2d) \supset X$. We consider the disks $D(x, r) \supset X$ and $D(y, r+2d) \supset X$. If $2d \leq d(x, y) \leq 2r+2d$, $\partial D(x, r) \cap \partial D(y, r+2d) \neq \emptyset$. We assume now that $d(x, y) > 2d$. Since $D(x, r+d) \supset Y$, by (A) $y \in D(x, r+d)$, implying $d(x, y) \leq r+d$. Therefore there is a point $w \in \partial D(x, r) \cap \partial D(y, r+2d)$. Consider the hyperbolic triangle $T$ with vertices $x, y$ and $w$. Let $z$ be the orthogonal projection (in hyperbolic geometry) of $w$ onto the hyperbolic line through $x$ and $y$. If $z \in T, z \neq x$, then $r'' = d(z, w) < r$, and $D(z, r'') \supset D(x, r) \cap D(y, r+2d) \supset X$, contradicting the definition of $r$. Therefore $z \notin T \setminus \{x\}$, i.e., $\varnothing \neq \pi/2$ when $\varnothing$ is the angle of $T$ at $x$. Now, keep $r$ and $r+2d$ fixed and decrease $\varnothing$ from $\pi$ to $\pi/2$. Then $d(x, y)$ increases from $2d$ to a value $d'$ with $\cosh (r+2d) = \cosh r \cosh d'$. This is geometrically evident and follows also from the relation $\cosh (r+2d) = \cosh r \cosh (d(x, y) - \sinh r \sinh d(x, y) \cos \varnothing)$; cf. [1]. It follows

$$ \cosh d(x, y) \leq \cosh (r+2d)/\cosh r. $$
This is also valid if \( d(x, y) \geq 2d \), which case we have excluded from the above discussion.

We have \( \cosh (r + 2d)/\cosh r = e^{2d(1 + e^{-2(r + 2d)}/(1 + e^{-2r})} \leq e^{2d} \). Thus, substituting back into (6) \( d = \varrho(X, Y), x = P(X) \) and \( y = P(Y) \), we get

\[
(7) \quad d(P(X), P(Y)) \leq \arccosh e^{2r(X, Y)} = \log \left( e^{2r(X, Y)} + (e^{2r(X, Y)} - 1)^{1/2} \right) < 2\rho(X, Y) + \log 2.
\]

If \( \rho(X, Y) \) is small, we get a Hölder-type inequality. Let \( c = e^{2r(X, Y)} - 1 \). Then \( \log (e^{2r(X, Y)} + (e^{2r(X, Y)} - 1)^{1/2}) = \log (1 + c + (2c + c^2)^{1/2}) = \log (1 + c^{1/2}(c^{1/2} + (2 + c)^{1/2})) \leq c^{1/2}(c^{1/2} + (2 + c)^{1/2}) \). If \( \rho(X, Y) \geq R \), \( c \leq 2e^{2R} \rho(X, Y) \). We have then by (7)

\[
(8) \quad d(P(X), P(Y)) \leq C(R)\rho(X, Y)^{1/2} \quad \text{if} \quad \rho(X, Y) \leq R,
\]

where \( C(R) = 2e^R((\text{Re}^{2R})^{1/2} + (1 + \text{Re}^{2R})^{1/2}) \).

Note that \( P(X) \) exists and that (6), (7), (8) and (A) are valid also if \( X \) and \( Y \) are non-empty bounded subsets of the \( n \)-dimensional hyperbolic space.

Appendix 2. (Added December 1979.) It is possible to give a sharper estimate for the dilatation of the map \( f \) of the preceding Theorem. In fact,

\( f \) is \( K' \)-quasiconformal where \( K' = (1 + 1/K + \sqrt{K - 1}/\sqrt{K})/\sqrt{2} \geq \min(K^{1/2}, \sqrt{2K}) \).

This is an immediate consequence of the following lemma. Note that always \( 0 = \mu_{id}(z) \in M_\Sigma \) and that \( d(0, (K - 1)/(K + 1)) = \log K \) when the hyperbolic metric of \( D \) is given by \( 2|dz|/(1 - |z|^2) \) in which the formulae of hyperbolic trigonometry are valid.

Lemma. Let \( X \subset D(0, r), r \geq 0, \) and assume that \( 0 \in X \). Then the center of the smallest hyperbolic disk containing \( X \) satisfies

\[
d(0, P(X)) \leq \beta(r) = \arccosh (\cosh r)^{1/2}.
\]

We have the following relations for the function \( \beta: r/2 < \beta(r) = r/\sqrt{2} \) if \( r > 0, \beta(r) < r/2 + \log \sqrt{2} \) and \( \lim_{r \to \infty} (\beta(r) - r/2) = \log \sqrt{2}. \)

Proof. Let \( D(x, \varrho) \) be the smallest hyperbolic disk containing \( X \). We can assume that \( x \in R, x \geq 0. \) It also suffices to consider the case \( d(0, x) > r/2; \) by (A) always \( x \in D(0, r). \) Then

\[
r/2 < d(0, x) \equiv \varrho \equiv r,
\]

since \( 0 \in D(x, \varrho) \) and in any case \( \varrho \leq r. \) Thus \( \partial D(x, \varrho) \cap \partial D(0, r) \) consists of two points; let \( z \) be one of them. Let \( w \) be the orthogonal projection (in hyperbolic geometry) of \( z \) onto \( R \cap D (= \text{the hyperbolic line joining 0 and x}). \) We consider the following three cases

\[
(\alpha) \quad w \equiv 0; \quad (\beta) \quad 0 < w < x; \quad (\gamma) \quad w \equiv x.
\]

Let \( T \) be the hyperbolic triangle with vertices \( 0, x \) and \( z. \) In case \( (\alpha) \) the angle of \( T \) at \( 0 \equiv \pi/2. \) Therefore [1, eq. 12.94] \( \cosh \varrho = \cosh d(x, z) \equiv \cosh d(0, x) \cosh d(0, z) = \cosh d(0, x) \cosh r. \) This implies \( \varrho > r \) which is impossible by (9). Thus \( (\alpha) \) is impos-
sible. Case (β) cannot occur either, since now \( D(w, d(w, z)) \supset D(0, r) \cap D(x, \varrho) \supset X \). This is impossible since the triangle with vertices \( x, z \) and \( w \) has a right angle at \( w \) and thus \( d(w, z) < d(x, z) = \varrho \).

Thus, if \( d(0, x) > r/2 \), (γ) is the only possibility. Now the angle of \( T \) at \( x \equiv \pi/2 \). This, together with (9), implies \( \cosh r = \cosh d(0, z) \equiv \cosh d(0, x) \cosh d(x, z) = \cosh d(0, x) \cosh \varrho \equiv \cosh^2 d(0, x) \). Thus \( d(0, x) \equiv ar \cosh (\cosh r)^{1/2} \), proving the inequality for \( d(0, P(X)) = d(0, x) \).

We then examine the properties of \( \beta(r) \). Differentiating \( \beta(r) \), we get

\[
\beta'(r) = \frac{\cosh r/2}{\sqrt{2} \cosh r} = \sqrt{\frac{e^r + e^{-r} + 2}{4(e^r + e^{-r})}} \in (1/2, 1/\sqrt{2})
\]

if \( r > 0 \), proving the first inequalities for \( \beta \). We get the next, since

\[
\begin{align*}
\beta(r) &= \log \left( \sqrt{(e^r + e^{-r})/2} + \sqrt{(e^r + e^{-r})/2 - 1} \right) \\
&= \log \left( \sqrt{e^r + e^{-r} + e^{i\pi/2} - e^{-r/2}} \right) - \log \sqrt{2} \\
&< \log \left( e^{i\pi/2} + e^{-r/2} + e^{i\pi/2} - e^{-r/2} \right) - \log \sqrt{2} = r/2 + \log \sqrt{2}.
\end{align*}
\]

Finally, the above expression for \( \beta(r) \) gives immediately \( \lim_{r \to \infty} (\beta(r) - r/2) = \log \sqrt{2} \).

We remark that the function \( \beta \) is best possible in the above lemma. In fact, let \( T \) be the triangle with vertices \( 0, x \) and \( y \) where \( x, y \in \partial D(0, r) \) and \( T \) has equal angles at \( x \) and \( y \). Choose these angles in such a way that if \( t \) is the orthogonal projection of \( 0 \) onto the opposite side, we have \( d(t, 0) = d(t, x) = d(t, y) \). Then \( d(0, t) = d(0, P(T)) = \beta(r) \).

References


University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

Received 18 April 1979

Revision received 15 August 1979