ON THE BOUNDARY BEHAVIOR OF LOCALLY 
K-QUASICONFORMAL MAPPINGS IN SPACE

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1. Introduction

Let $B^n$, $n \geq 2$, be the $n$-dimensional unit ball in $R^n$, let $b \in \partial B^n$, and let $f: B^n \to G'$ be a quasiconformal mapping. Suppose that $b \in E_\varepsilon$ for all $\varepsilon > 0$, where $E_\varepsilon = \{x \in B^n : |f(x)| < \varepsilon\}$. This condition means that 0 belongs to the cluster set $C(f, b)$ of $f$ at $b$. Write $\delta_\varepsilon = \text{cap dens} (E_\varepsilon, b)$, where $\text{cap dens}$ refers to the lower (conformal) capacity density (for definitions, cf. Section 2). The conformal capacity density has been studied e.g. in [6] and [1]. In [20, 5.5] we proved that $f$ has angular limit 0 at $b$ if $\delta_\varepsilon (\log (1/\varepsilon))^{n-1} \to \infty$ when $\varepsilon \to 0$, i.e. if the numbers $\delta_\varepsilon$ do not tend "too" rapidly to 0. An alternative proof was presented in [21]. As it was shown in [20, Section 5], this result is a quasiconformal counterpart of a theorem of J. L. Doob [2, Theorem 4] about bounded analytic functions.

The purpose of the present paper is to prove related theorems for locally $K$-quasiconformal mappings. The main result, proved in Section 3, reads as follows. Let $f: B^n \to R^n$ be locally $K$-quasiconformal, let $b \in \partial B^n$, and let $D$ be an open cone in $B^n$ with vertex $b$. Write $\delta_\varepsilon = \text{cap dens} (D \cap f^{-1} B^n(\varepsilon), b)$. If $n \geq 3$ and $C(f, b) \subset \partial f B^n$ and $\delta_\varepsilon (\log (1/\varepsilon))^{n-1} \to \infty$ when $\varepsilon \to 0$, then $f$ has angular limit 0 at $b$. The proof of this theorem is based on the method used in [21, 4.12] and on an injectivity theorem of Martio, Rickman, and Väisälä [10, 2.3], which yields an upper bound for the maximal multiplicity of a locally $K$-quasiconformal mapping of $B^n$ in a non-tangential domain, provided that the dimension $n \geq 3$. Instead of a cone with a fixed angle, like $D$ above, one may consider in the definition of $\delta_\varepsilon$ cones with the central angle increasing towards $\pi/2$ in a tempered way as $\varepsilon \to 0$. For details we refer the reader to Theorem 3.1.

In Section 4 we consider the situation of the above result if the condition $C(f, b) \subset \partial f B^n$ is removed. Employing now a different method we prove the following theorem. Let $f: B^n \to R^n$ be locally $K$-quasiconformal, let $b \in \partial B^n$, and let $D$ be an open cone in $B^n$ with vertex $b$. Suppose that $E \subset D$ and $\text{cap dens} (E, b) > 0$. If $n \geq 3$ and $f(x)$ tends to 0 when $x$ approaches $b$ through the set $E$, then $f$ has angular limit 0 at $b$. By an example we show that this result is, in a sense, the best possible.
Finally, in Section 5, we consider a subclass of quasiregular mappings of \( B^n, n \geq 2 \), characterized by the property that the maximal multiplicity is uniformly bounded in each hyperbolic ball with a fixed radius (cf. (5.1)). From the injectivity theorem [10, 2.3] it follows that locally \( K \)-quasiconformal mappings of \( B^n, n \geq 3 \), have the same property. It is pointed out that the results in Sections 3 and 4 hold for mappings in this larger class as well. A normality criterion, related to a problem of W. K. Hayman, is given for functions in the mentioned class.

2. Preliminary results

The notation and terminology will be, in general, as in [20], [21], and [8]. For definitions and basic properties of quasiconformal and quasiregular mappings we refer the reader to Väisälä's book [19] and to the papers of Martio, Rickman, and Väisälä [8], [9], [10]. A mapping \( f: G \to R^n \) is locally \( K \)-quasiconformal if there exists a number \( K \in [1, \infty) \) such that \( f \) is \( K \)-quasiconformal in a neighborhood of each point of \( G \). Here \( G \subseteq R^n \) is a domain. A sense-preserving mapping is locally \( K \)-quasiconformal if and only if it is a \( K \)-quasiregular local homeomorphism (cf. [8, p. 14]).

2.1. Notation. If \( x \in R^n, n \geq 2, \) and \( r > 0 \), then \( B^n(x, r) = \{ y \in R^n: |x - y| < r \} \), \( S^{n-1}(x, r) = \partial B^n(x, r) \), \( B^n(r) = B^n(0, r) \), \( S^{n-1}(r) = S^{n-1}(0, r) \), \( B^n = B^n(1) \), and \( S^{n-1} = S^{n-1}(1) \). For \( x \in R^n \) and \( r > s > 0 \) we write \( R(x, r, s) = B^n(x, r) \setminus B^n(x, s) \) and \( R(r, s) = R(0, r, s) \). The standard unit coordinate vectors are \( e_1, \ldots, e_n \).

2.2. Path families and their modulus. A path is a continuous nonconstant mapping \( \gamma: \Delta \to A, A \subseteq R^n \), where \( \Delta \) is an interval on the real axis. The point set \( \gamma \Delta \) will be denoted by \( |\gamma| \). Given \( E, F, \) and \( G \) in \( R^n \), we let \( \Delta(E, F; G) \) be the family of all paths \( \gamma: [0, 1] \to G \) with \( \gamma(0) \in E \) and \( \gamma(1) \in F \) (cf. [19, p. 21]). For the definition and basic properties of the \((n-)\)modulus \( M(\Gamma) \) of a path family \( \Gamma \) we refer the reader to Väisälä’s book [19, Section 6]. If \( u \in R^n \) and \( t > r > 0 \) and \( \Gamma \) is a path family such that \( |\gamma| \) intersects both boundary components of \( R(u, t, r) \) for each \( \gamma \in \Gamma \), then the following estimate holds ([19, 7.5]):

\[
M(\Gamma) \leq o_{n-1} \left( \log \frac{t}{r} \right)^{1-n}.
\]

Here \( o_{n-1} \) is the surface area of \( S^{n-1} \). For \( E \subseteq R^n, x \in R^n, \) and \( t > r > 0 \) we abbreviate

\[
M_t(E, r, x) = M(\Delta(S^{n-1}(x, t), B^n(x, r) \cap E; R^n)),
\]

\[
M(E, r, x) = M_{2r}(E, r, x).
\]
The lower and upper capacity densities of $E$ at $x$ are defined by (cf. [20] and Martio—Sarvas [11])

$$\text{cap dens}(E, x) = \liminf_{r \to 0} M(E, r, x),$$

$$\text{cap dens}(E, x) = \limsup_{r \to 0} M(E, r, x).$$

If $E$ is compact, this definition is equivalent to the one employed in [11], which is based on the use of $n$-capacities of condensers (cf. Ziemer [22]). Some sufficient conditions for $\text{cap dens}(E, x) > 0$ were given in [20, Section 2]. See also Martio [6, 3.1]. From a result of Wallin it follows that there are sets $E$ with $\text{cap dens}(E, 0) = 0$ which have Hausdorff dimension zero [20, 2.5 (3)]. For $t > s > r > 0$

$$M_t(E, r, x) \leq M_s(E, r, x) \equiv \left(\frac{\log(t/r)}{\log(s/r)}\right)^{n-1} M_s(E, r, x).$$

One can prove (2.4) by making use of a radial quasiconformal mapping which is identity in $B'(r)$ and maps $R(s, r)$ onto $R(t, r)$ (cf. [11, 2.7]). Using (2.4) we prove the following lemma.

2.5. Lemma. $\text{cap dens}(E, 0) = \liminf_{r \to 0} M(E \cap B^n(r), r, 0)$.

Proof. Denote by $a$ and $b$ the left and right hand sides of the equality, respectively. Obviously $a \geq b \geq 0$. Hence it suffices to prove $a \equiv b$ and we may assume that $a > 0$. Choose $a' \in (0, a)$ and $r_0 \in (0, 1)$ in such a way that $M(E, r, 0) \geq a'$ for all $r \in (0, r_0)$. Fix $r' \in (0, r_0)$. For all $k = 2, 3, ...$ we get by (2.4)

$$M(E \cap B^n(r), r, 0) \equiv M_{a'}(E \cap B^n(r), r(1 - 1/k), 0)$$

$$= M_{a'}(E, r(1 - 1/k), 0) \geq d_k^{1-n} M(E, r(1 - 1/k), 0) \geq d_k^{1-n} a',$$

where $d_k = \log(2 / (1 - 1/k)) \log 2$. Since $d_k \to 1$, this implies $M(E \cap B^n(r), r, 0) \geq a'$. Hence $b \geq a'$. Letting $a' \to a$ yields the desired conclusion.

The next lemma was proved by Näkki [15] (cf. also Martio, Rickman, and Väisälä [10, 3.11]). It will be called here, as in [15], the comparison principle for the modulus. Throughout the paper we let $c_n$ denote the positive constant in [19, 10.9], depending only on $n$.

2.6. Lemma. Let $F_1$, $F_2$, and $F_3$ be three sets in $\mathbb{R}^n$ and let $\Gamma_{ij} = \Delta(F_i, F_j; \mathbb{R}^n)$, $1 \leq i, j \leq 3$. If there exist $x \in \mathbb{R}^n$ and $0 < a < b$ such that $F_1, F_2 \subset B^n(x, a)$ and $F_3 \subset \mathbb{R}^n \setminus B^n(x, b)$, then

$$M(\Gamma_{12}) \geq 3^{-n} \min\left\{M(\Gamma_{13}), M(\Gamma_{23}), c_n \log \frac{b}{a}\right\}.$$
Proof. From the choice of \( \lambda \) it follows by (2.3) that
\[
M(\overline{B}^n(s/\lambda), s, 0) \leq \frac{t}{6},
\]
and hence the proof follows from Lemma 2.6 and the estimates
\[
M(F_j, s, 0) \equiv \delta_j - t/6 \leq 3^n s t/6, \quad j = 1, 2.
\]

2.8. The hyperbolic metric. The hyperbolic metric \( \varrho \) in \( B^n \) is defined by the element of length \( d_\varrho = |dx|/(1-|x|^2) \). If \( a \) and \( b \) are points in \( B^n \), then \( \varrho(a, b) \) denotes the geodesic distance between \( a \) and \( b \) corresponding to this element of length. For \( b \in B^n \) and \( M \in (0, \infty) \) we let \( D(b, M) \) denote the hyperbolic ball \( \{ x \in B^n : \varrho(b, x) < M \} \). Let \( r_b = \min \{ |z-b| : z \in \partial D(b, M) \} \). By integrating we get
\[
(2.9)
\]
\[
r_b = \frac{(1-|b|^2) \tanh M}{1 + |b| \tanh M}.
\]
This implies that \( B^n(b, \tanh M(1-|b|)) \subset D(b, M) \).

In what follows we shall need some properties of normal mappings. We recall that a mapping \( f : B^n \to \mathbb{R}^n \) is said to be normal if for each sequence \( (h_k) \) of conformal self-mappings of \( B^n \) there is a subsequence of \( (f \circ h_k^{-1}) \) converging uniformly on compact subsets of \( B^n \) (or briefly c-uniformly) towards a limit mapping \( g : B^n \to \mathbb{R}^n \) (cf. [19, p. 68], [20, Section 3]). The cluster set of \( f \) at \( b \in \partial B^n \) is the set \( C(f, b) \) of all points \( b' \in \mathbb{R}^n \) for which there exists a sequence \( (x_k) \) in \( B^n \) with \( x_k \to b \) and \( f(x_k) \to b' \). The next lemma makes use of some ideas of Bagemihl's and Seidel's [9], p. 5].

2.10. Lemma. Let \( f : B^n \to \mathbb{R}^n \) be a quasiregular mapping, let \( (b_k) \) be a sequence in \( B^n \) with \( b_k \to b \in \partial B^n \) and \( f(b_k) \to \alpha \), and let \( M \in (0, \infty) \) and \( E = \cup D(b_k, M) \). Suppose that \( \alpha \in \partial f B^n \). If \( f \) is normal or if \( C(f, b) \subset \partial f B^n \), then \( f(x) \to \alpha \) as \( x \to b \) through the set \( E \).

Proof. It is well known that \( C(f, b) \) is a non-empty compact connected set (cf. [19, 17.1, 17.5 (1)]). If \( C(f, b) \) consists of one point, there is nothing to prove. Otherwise \( C(f, b) \) is a non-degenerate continuum. If \( C(f, b) \subset \partial f B^n \), it follows that \( \text{cap} (\mathbb{R}^n \setminus f B^n) = 0 \) in the terminology of [9], and hence \( f \) is normal by [9, 3.17]. Hence \( f \) is quasiregular and normal, and it follows from [17, p. 497] that the condition in [20, 6.3] is satisfied. The proof follows from [20, 6.3].

The assumption \( \alpha \in \partial f B^n \) in the above lemma can be replaced by the requirement that \( f^{-1}(\alpha) \) be finite (cf. [1, p. 5], [20, 6.4]). By considering the behavior of the function \( f : B^3 \to B^3, f(z) = \exp ((z + 1)/(z - 1)), \) near \( z = 1 \) we see that this assumption cannot be dropped. The next example shows that corresponding functions exist when the dimension \( n = 3 \).

2.11. Example. We shall slightly modify the example constructed by Martio and Srebro in [12, 4.1]. Let \( g : R^3_+ \to T \) be the locally \( K \)-quasiconformal automorphic mapping constructed in [12, 4.1], and let \( h : B^3 \to R^3_+ \) be the Möbius
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transformation \( h(x) = 2(x + e_1)|x + e_1|^\gamma - e_1 \). Here \( T \) is an open solid torus and the mapping \( f = g \circ h : B^3 \to T \) has a continuous extension \( \tilde{f} : B^3 \setminus \{e_1, -e_1\} \to T \) with \( \tilde{f}(\partial B^3 \setminus \{e_1, -e_1\}) \subset \partial T \). By the construction of \( g, f \) maps the segment \( \{e_1 t : -1 < t < 1\} \) onto a closed curve \( \partial S \subset T \). Fix \( \alpha, \beta \in \mathbb{S}, \alpha \neq \beta \). By the construction of \( \gamma \) there are increasing sequences \((s_k), (t_k)\) in \((0, 1)\) with \( s_k < t_k < s_k+1 \) for all \( k \) such that \( \lim s_k = 1, \lim t_k = -1 \) and such that \( g(s_k e_1, t_k e_1) = M \) for all \( k \) and some \( M \in (0, \infty) \). Thus the conclusion of Lemma 2.10 does not hold for this function \( f \). Hence the assumption \( \alpha \in \partial B^n \) cannot be dropped.

Let \( f : B^n \to R^n \) be a mapping, \( y \in R^n \), and \( D \subset R^n \). Then we denote by \( N(y, f, D) \) the number of the points in \( f^{-1}(y) \cap D \). The maximal multiplicity of \( f \) in \( D \) is

\[
N(f, D) = \sup \{N(y, f, D) : y \in R^n\}.
\]

The next lemma follows from [10, 2.3].

2.12. Lemma. Let \( n \geq 3 \) and \( K \geq 1 \). Then there is a constant \( \psi(n, K) \in (0, 1) \) such that if \( f : B^n \to R^n \) is a locally \( K \)-quasiconformal mapping, then \( f \) is injective in \( B^n(\psi(n, K)) \). Moreover, for every \( r \in (0, 1) \) there is a number \( c(n, K, r) \in [1, \infty) \) depending on \( n, K, \) and \( r \), and a number \( b(n) \) depending only on \( n \) such that

\[
N(f, B^n(r)) \leq c(n, K, r) \leq \left( \frac{b(n)}{\psi(n, K)(1-r)} \right)^n.
\]

Proof. The first part of the lemma was proved by Martio, Rickman, and Väisälä [10, 2.3]. From the first part it follows that \( f \) is injective in \( B'_x = B^n(x, \psi(n, K)t) \), \( x \in B^n \), when \( 0 < t \leq 1 \). One may define \( c(n, K, r) \) to be the smallest number of balls \( B'_x \) needed to cover \( B^n(r) \). The estimate for \( c(n, K, r) \) follows from known properties of coverings by families of balls (cf. [14, Lemma 3] and [4, p. 197, Lemma 3.2]).

Note that Lemma 2.12 is false for \( n = 2 \) (cf. [10, 2.11]).

2.13. Remark. One can improve the upper bound for \( c(n, K, r) \) by making use of ideas presented in [7, 5.27]. In this way one obtains an estimate of the type

\[
c(n, K, r) \leq A(1-r)^{-n} \log (2/(1-r)),
\]

where \( A > 0 \) depends only on \( n \) and \( K \), but we shall not need such an estimate here.

Using Lemma 2.12 we shall now prove an upper bound for the maximal multiplicity of a locally \( K \)-quasiconformal mapping in a non-tangential domain of a particular shape. For \( b \in \partial B^n \) and \( \varphi \in (0, \pi/2) \) we let \( K(b, \varphi) \) denote the cone \( \{z \in R^n : (b|b-z| > |b-z| \cos \varphi) \} \). Here \( (x|y) \) is the inner product \( \sum_{i=1}^{n} x_i y_i \).

2.14. Lemma. If \( n \geq 3, K \geq 1, \) and \( \varphi \in (0, \pi/2) \), then there are constants \( a(n, K) > 0 \) and \( d(n, K, \varphi) > 0 \), depending only on the numbers indicated, with the following properties. Let \( f : B^n \to R^n \) be a locally \( K \)-quasiconformal mapping, \( b \in \partial B^n \),
Let \( t \in (0, \cos \phi) \), and let \( A^\phi_\lambda(t) = K(b, \varphi) \cap R(b, t, t/\lambda) \) for \( \lambda > 1 \). Then for \( \lambda \geq 2 \) the following estimates hold:

\[
N(f, A^\phi_\lambda(t)) \leq d(n, K, \varphi) \log \lambda,
\]

where

\[
d(n, K, \varphi) \leq a(n, K) \cos^{-2n} \varphi.
\]

**Proof.** Fix \( b \in \partial B^n, \varphi \in (0, \pi/2) \), and \( t \in (0, \cos \phi) \). We first consider the case \( \lambda = 2 \). By elementary geometry \( A^\phi_2(t) \subseteq B^n(x, r) \), where

\[
\begin{align*}
x &= \left(1 - \frac{3t}{4 \cos \varphi}\right) b \\
r &= \frac{t}{4} (9 \tan^2 \varphi + 1)^{1/2}.
\end{align*}
\]

Then \( N(f, A^\phi_2(t)) \leq N(f, B^n(x, r)) \) and by Lemma 2.12

\[
N(f, B^n(x, r)) \leq c(n, K, v_\varphi),
\]

\[
v_\varphi = r/|b-x| = (\sin^2 \varphi + (1/9) \cos^2 \varphi)^{1/2}.
\]

Let us now consider the case \( \lambda > 2 \). Fix \( \lambda > 2 \). Define

\[
m = \min \{k \in N: 2^{-k} t \equiv t/\lambda \} \geq 2.
\]

Thus \( 2^{-m} \equiv 1/\lambda \equiv 2^{-m+1} \equiv 2^{-m/2} \) and hence \( m \equiv \log \lambda / \log \sqrt{2} \). Using the estimate obtained in the case \( \lambda = 2 \) we get

\[
N(f, A^\phi_\lambda(t)) \leq \sum_{j=1}^m N(f, A^\phi_\lambda(2^{-j+1}t)) \leq c(n, K, v_\varphi) \log \lambda / \log \sqrt{2}.
\]

These estimates hold for \( \lambda = 2 \) as well. Hence for all \( \lambda \geq 2 \) we may choose \( d(n, K, \varphi) = c(n, K, v_\varphi) / \log \sqrt{2} \). Since \( 1 - v_\varphi \geq (4/9) \cos^2 \varphi \), the desired estimate with

\[
a(n, K) = (9b(n)/4\psi(n, K))^\nu / \log \sqrt{2}
\]

follows from Lemma 2.12.

2.15. **Remark.** We shall now show by investigating the mapping \( f \) of Example 2.11 that the upper bound of Lemma 2.14 is of the correct order of magnitude for this mapping. By the construction of the automorphic mapping \( f: B^3 \to T \) there exist \( a \in T \) and a sequence \((u_k)\) in \((0, 1)\) with \( \lim u_k = 1 \) such that \( f(u_k e_1) = a \) for all \( k = 1, 2, \ldots \) and a number \( M \in (0, \infty) \) such that \( q(u_k e_1, u_{k+1} e_1) < M \) for all \( k = 1, 2, \ldots \). Fix \( \varphi \in (0, \pi/2) \). After relabeling if necessary we may assume that \( 1-u_4 < \cos \varphi \) and \( 1-u_4 < 1/2 \). For \( \lambda \geq 2 \) let \( A^\phi_\lambda(K(e_1, \varphi) \cap R(e_1, 1-u_4, (1-u_4)/\lambda)) \). Define

\[
p = \min \left\{ k \in N: M(k+1) > q\left( e_1 u_4, e_1 \left(1 - \frac{1-u_4}{\lambda}\right)\right)\right\}.
\]
Then $N(f, A^q_{\lambda}) \equiv N(a, f, A^q_{\lambda}) \equiv p$. Since for $0 \leq v < w < 1$

$$q(e_1 v, e_1 w) = \frac{1}{2} \log \frac{1+w}{1-w} \cdot \frac{1-v}{1+v},$$

we get the estimates

$$q \left( e_1 u_1, e_1 \left( 1 - \frac{1-u_1}{\lambda} \right) \right) \leq \frac{1}{2} \log \left( \lambda - \frac{1}{3} \right) \leq \frac{1}{4} \log \lambda$$

for $\lambda \geq 2$, where we have used the fact $1-u_1 < 1/2$. Hence if $\lambda \geq 2$ is large enough, then $p \geq 1$, and hence $M(p+1) \equiv 2Mp$, which together with the above estimates yields

$$N(f, A^q) \equiv \frac{1}{8M} \log \lambda.$$

We have thus shown that the dependence on $\lambda$ in the upper bound of Lemma 2.14 is the best possible when $\varphi$ is fixed.

For $\theta \in (0, \pi/2)$ let $C(\theta) = \{x \in \mathbb{R}^n: (x|e_n)| = |x| \cos \theta \}$. If $A \subset \mathbb{R}^n$ we write $A_+ = \{x \in \mathbb{R}^n: x_n > 0 \}$. In the next lemma we construct a quasiconformal mapping $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $f$ maps the truncated cone $C(\theta) \cap R(1, s)$ onto $B^3$ for given $\theta \in (0, \pi/2)$ and $s \in (0, 1)$ and such that we get an appropriate upper bound for $K(f)$. The numerical value of this upper bound is probably not the best possible.

**2.16. Lemma.** Let $n=3$, $\theta_0 \in (0, \pi/2)$, and $s \in (0, 1)$. Then there exists a constant $Q(3, \theta_0, s) \equiv 1$ and a $Q(3, \theta_0, s)$-quasiconformal mapping $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ which maps $C(\theta_0) \cap R(1, s)$ onto $B^3$. Moreover, $Q(3, \theta_0, s) \equiv Q(3, \theta_0, r)$ when $\theta_0 \in (0, \pi/2)$ and $0 < s \leq r < 1$, and $Q(3, \theta_0, s) \equiv Q(3, \theta, s)$ when $0 < \theta_0 \equiv \theta < \pi/2$ and $s \in (0, 1)$.

**Proof.** The proof makes use of some ideas of Gehring and Väisälä [3] (cf. Lemma 8.2 in [3] and the proof of Lemma 3.4 in Martio—Srebro [13]).

Let $(R, \varphi, \theta)$ be the spherical coordinates in $\mathbb{R}^3$, where $\varphi \in [0, 2\pi]$ is measured from the direction of $e_1$ to the direction of $e_2$ and $\theta \in [0, \pi]$ is measured from the direction of $e_3$ (cf. [19, 16.4]). Let $f_1: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the mapping defined by $f_1(\infty) = \infty$ and

$$f_1(R, \varphi, \theta) = \begin{cases} R, \varphi, \frac{\pi}{2\theta_0} \theta, & \text{if } 0 \leq \theta < \theta_0, \\ R, \varphi, \frac{\pi \theta}{2(\pi - \theta_0)} + \frac{\pi(\pi - 2\theta_0)}{2(\pi - \theta_0)}, & \text{if } \theta_0 \leq \theta \leq \pi. \end{cases}$$

Then $f_1$ is quasiconformal and maps $C(\theta_0)$ onto $R^3_+$ and $C(\theta_0) \cap R(1, s)$ onto $R(1, s)_+$. It follows from [19, 16.4, 35.1] that

$$K(f_1) \leq \left( \frac{\pi}{\theta_0} - 1 \right)^2 \sin^{-2} \theta_0.$$
Let $f_2: \mathbb{R}^3 \to \mathbb{R}^3$ be the Möbius transformation defined by $f_2(-e_1) = \infty$, $f_2(\infty) = -e_1$, and

$$f_2(x) = \frac{x + e_1}{|x + e_1|^2} - e_1,$$

if $x \in \mathbb{R}^3 \setminus \{-e_1\}$.

Then $f_2$ maps $R(1, s)_+ \to \{x \in \mathbb{R}^3_+: x_1 > 0\} \setminus \bar{B}$, where $B = B^2((1 + s^2)e_1/(1 - s^2), 2s/(1 - s^2))$ and $f_2 B^2(s) = B$. Let $T$ be the tangent plane of $B$ which contains the $x_2$-axis and passes through the point $e_1 + (2s/(1 - s^2))e_3 \in \partial B$. Denote by $\alpha_4$ the acute angle between $T$ and $e_3$. Then $\alpha_4 = \arctan((1 - s^2)/2s)$. Let $(r, \varphi, x_3)$ be the cylindrical coordinates in $\mathbb{R}^3$ with the $x_3$-axis as the symmetry axis, where $\varphi$ is measured from the direction of $e_3$. Let $f_3: \mathbb{R}^3 \to \mathbb{R}^3$ be a quasiconformal folding defined by $f_3(\infty) = \infty$ and

$$f_3(r, \varphi, x_3) = \begin{cases} (r, \varphi, x_3), & \text{if } 0 \leq \varphi < \frac{\pi}{2} - \alpha_4, \\ \left(r, \frac{\pi + 2x_3}{2x_3} + \frac{\pi}{2}, \frac{2x_3 - \pi}{2x_3}, x_3\right), & \text{if } \frac{\pi}{2} - \alpha_4 \leq \varphi < \frac{\pi}{2}, \\ \left(r, \frac{2}{3}, \varphi + \frac{2\pi}{3}, x_3\right), & \text{if } \frac{\pi}{2} \leq \varphi < 2\pi. \end{cases}$$

Then $f_3$ maps $\{x \in \mathbb{R}^3_+: x_1 > 0\} \setminus \bar{B}$ onto $\mathbb{R}^3_+ \setminus \bar{B}$ and it follows from [19, 16.3, 35.1] that

$$(2.18) \quad K(f_3) = \max \left\{ \left(\frac{3}{2}\right)^2, \left(\frac{\pi}{2x_3} + 1\right)^2 \right\}; \quad \alpha_3 = \arctan \frac{1 - s^2}{2s}.$$

Let $f_4: \mathbb{R}^3 \to \mathbb{R}^3$ be the Möbius transformation defined by $f_4(a) = \infty$, $a = ((1 + s)/(1 - s))e_1$, $f_4(\infty) = a$, and

$$f_4(x) = e^2 \frac{x - a}{|x - a|^2} + a, \quad \text{if } x \in \mathbb{R}^3 \setminus \{a\}; \quad e^2 = \frac{4s}{(s - 1)^2}.$$

Then $f_4$ maps $B$ onto $\{x \in \mathbb{R}^3_+: x_1 > 0\}$ in such a way that $B_+$ is mapped onto $\{x \in \mathbb{R}^3_+: x_1 > 0\}$. Let $(r, \varphi, x_2)$ be the same cylindrical coordinate system as above and define $f_5: \mathbb{R}^3 \to \mathbb{R}^3$ by $f_5(\infty) = \infty$ and

$$f_5(r, \varphi, x_2) = \begin{cases} (r, 2\varphi, x_2), & \text{if } 0 \leq \varphi < \frac{\pi}{2}, \\ \left(r, \frac{\varphi}{2} + \frac{3\pi}{4}, x_2\right), & \text{if } \frac{\pi}{2} \leq \varphi < \frac{3\pi}{2}, \\ (r, \varphi, x_2), & \text{if } \frac{3\pi}{2} \leq \varphi < 2\pi. \end{cases}$$

Then $f_5$ maps $f_4(\{x \in \mathbb{R}^3_+: x_1 > 0\} \setminus \bar{B}) = \{x \in \mathbb{R}^3_+: x_1 > 0\}$ onto $\mathbb{R}^3_+$ and $f_5$ is $2^2$-quasiconformal (cf. [19, 16.3, 35.1]). The mapping $f_6 = f_5 \circ f_4 \circ f_2 \circ f_3: \mathbb{R}^3 \to \mathbb{R}^3$ is quasiconformal with $f_6(C(\theta_0) \cap R(1, s)) = \mathbb{R}^3_+$, and the assertion follows from this in view of (2.17) and (2.18).
2.19. Lemma. Let \( n \geq 3 \) and \( \theta_0 \in (0, \pi/2) \). Then there exists a constant \( Q(n, \theta_0) \geq 1 \) depending only on \( n \) and \( \theta_0 \) such that if \( \lambda \geq 2 \) and \( \theta \in (\theta_0, \pi/2) \), there exists a \( Q(n, \theta_0) \)-quasiconformal mapping \( f: \mathbb{R}^n \to \mathbb{R}^n \) with \( f(C(0) \cap R(1, 1/\lambda)) = B^\alpha \).

Proof. Since the constant \( Q(3, \theta, s) \) of Lemma 2.16 is increasing as a function of \( s \) and decreasing as a function of \( \theta \) we may choose \( Q(n, \theta_0, 1/2) \). The proof of the general case \( n \geq 3 \) can be carried out by generalization of Lemma 2.16 in the \( n \)-dimensional case.

2.20. Remark. Martio and Srebro have studied in [13] the problem of mapping strictly star shaped domains of \( \mathbb{R}^n \) onto \( B^n \) by means of bi-lipschitzian quasiconformal mappings of \( \mathbb{R}^n \). Note that the domains in 2.19 need not be star shaped.

The next lemma is one variant of a symmetry principle for the modulus (cf. Gehring—Väisälä [3, Lemma 3.3] and also [20, 4.3]).

2.21. Lemma. Let \( D \) be a domain in \( \mathbb{R}^n, n \geq 2 \), and suppose that there exists a quasiconformal mapping \( f: \mathbb{R}^n \to \mathbb{R}^n \) with \( fD = B^n \). If \( E \) and \( F \) are two subsets of \( D \), then

\[
M(\Delta(E, F; D)) \leq M(\Delta(E, F; R^n))/2K(f)^2.
\]

Proof. By quasiconformality and [20, 4.3] we obtain

\[
M(\Delta(E, F; R^n)) \leq M(\Delta(fE, fF; R^n))K(f) \leq M(\Delta(fE, fF; fD))2K(f)
\]

\[
\leq M(\Delta(E, F; D))2K(f)K(f^{-1}) = M(\Delta(E, F; D))2K(f)^2.
\]

Hereafter we shall use Lemma 2.21 when \( D \) is a truncated cone as in Lemma 2.19.

3. The main result

In [20, 5.5, 5.6] we proved theorems about quasiconformal mappings of \( B^n, n \geq 2 \), which are analogous to a theorem of J. L. Doob [2, Theorem 4] regarding angular limits of bounded analytic functions. Using the results of Section 2, we shall now prove a related theorem for \textit{locally K-quasiconformal} mappings of \( B^n, n \geq 3 \). As will be pointed out in Section 5, the same proof works in the case of somewhat more general mappings as well.

3.1. Theorem. Let \( f: B^n \to \mathbb{R}^n \) be locally \( K \)-quasiconformal, let \( b \in \partial B^n \), and let \( C(f, b) \subset \partial fB^n \). For \( \varepsilon > 0 \) let \( \phi_\varepsilon \in (0, \pi/2), \ E_\varepsilon = K(b, \phi_\varepsilon ) \cap f^{-1}B^n(\varepsilon), \) and \( \delta_\varepsilon = \text{cap dens}(E_\varepsilon, b) \). Moreover, let \( \phi_\varepsilon \) be decreasing and \( \delta_\varepsilon \) increasing. If \( n \geq 3 \) and

\[
\lim_{\varepsilon \to 0} \sup \cos^{n} \phi_\varepsilon \delta_\varepsilon^{(n-1)} \left( \log \frac{1}{\varepsilon} \right)^{n-1} = \infty,
\]

then \( f \) has angular limit 0 at \( b \).
Proof. Suppose that this is not the case. Then there is $\varphi_0 \in (0, \pi/2)$ and a sequence $(b_k)$ in $K(b, \varphi_0) \cap \mathcal{B}_n$ with $b_k \to b$, $f(b_k) \to \beta \neq 0$. Fix $r_0 \in (0, 1)$ such that $\beta \in \mathcal{R}_n \setminus \mathcal{B}_n(r_0)$. Since $b_k \in K(b, \varphi_0)$ and $b_k \to b$ there is an integer $k_1$ such that $1 - |b_k| \geq |b_k - b| (\cos \varphi_0)/2$ for $k \geq k_1$. Since $C(f, b) \subset \partial f \mathcal{B}_n$ there is by Lemma 2.10 an integer $k_0 \geq k_1$ such that $fD(b_k, 1) \subset \mathcal{R}_n \setminus \mathcal{B}_n(r_0)$ for $k \geq k_0$. If $A$ is a proper subset of $\mathcal{B}_n$ and $r > 0$, we abbreviate $A(r) = \mathcal{B}_n(b, r) \cap A$. Let $E = K(b, \varphi_0) \cap (\bigcup_{k=k_0} D(b_k, 1))$. By (2.9) $\mathcal{B}_n(b_k, (1 - |b_k|)) \subset D(b_k, 1)$ for all $k$ and by [20, 1.10] or [19, 10.12] we get the estimate

$$M(E([b_k - b], [b_k - b], b) \equiv C(n, \varphi_0) = c_n \log (1 + (\tanh 1 \cos \varphi_0)/2)$$

for $k \geq k_0$. From Lemma 2.5 it follows that for $\epsilon \in (0, r_0)$ there is an integer $k_\epsilon \geq k_0$ such that $\varrho_\epsilon = |b_{k_\epsilon} - b| \leq \min \{\cos \varphi_0, \cos \varphi_\epsilon\}$ and $M(E_\epsilon(\varrho_\epsilon), \varrho_\epsilon, b) \equiv \delta_\epsilon/2$, where $\delta_\epsilon = \text{cap dens } (E_\epsilon, b)$. Write

$$t_\epsilon = 3^{-n} \min \{\delta_\epsilon/2, c(n, \varphi_0), c_n \log 2\} > 0$$

for $\epsilon \in (0, r_0)$ and let $\lambda_\epsilon \equiv 2$ be defined by $\alpha_\epsilon = \max \{2, \varrho_\epsilon\}$, where $\lambda_\epsilon > 1$ satisfies

$$\log \lambda_\epsilon = (t_\epsilon/6\omega_{n-1})^{1/(1-n)}.$$

Let $F_\epsilon = E_\epsilon(\varrho_\epsilon) \setminus \mathcal{B}_n(b, \varrho_\epsilon/\lambda_\epsilon)$, $F = E(\varrho, \varrho_\epsilon/\lambda_\epsilon)$, and $\Gamma_\epsilon = \Delta (F, F_\epsilon; \mathcal{R}_n)$, when $\epsilon \in (0, r_0)$. It follows from Corollary 2.7 that $M(\Gamma_\epsilon) \equiv 5t_\epsilon/6$. For $\epsilon \in (0, r_0)$ let $D_\epsilon = K(b, \varphi_\epsilon) \cap R(b, \varrho_\epsilon, \varrho_\epsilon/\lambda_\epsilon)$, where $\varphi_\epsilon^* = \max \{\varphi_0, \varphi_\epsilon\}$, and let $\Gamma_\epsilon = \Delta (F, F_\epsilon; D_\epsilon)$. Observe that $F, F_\epsilon \subset D_\epsilon$. Since $\lambda_\epsilon \geq 2$ we get then by Lemmas 2.19 and 2.21

$$M(\Gamma_\epsilon) \equiv t_\epsilon/3Q(n, \varphi_\epsilon^*) \equiv t_\epsilon/3Q(n, \varphi_0^2)$$

for $\epsilon \in (0, r_0)$. By (2.3) we obtain

$$M(f\Gamma_\epsilon) \equiv \omega_{n-1} \left(\log \frac{r_0}{\epsilon}\right)^{1-n}$$

for $\epsilon \in (0, r_0)$. From the modulus inequality [8, 3.2] it follows that

$$M(\Gamma_\epsilon) \equiv K \omega_{n-1} f(f\Gamma_\epsilon).$$

Since $\varrho_\epsilon = \min \{\cos \varphi_0, \cos \varphi_\epsilon\}$ and $\lambda_\epsilon \equiv 2$ we get by Lemma 2.14 and by the above inequalities

$$t_\epsilon/3Q(n, \varphi_0^2) \equiv K d(n, K, \varphi_\epsilon^*) \log \lambda_\epsilon \omega_{n-1} \left(\log \frac{r_0}{\epsilon}\right)^{1-n}$$

$$\equiv K a(n, K) \cos^{-2\pi/3 \left(\varphi_\epsilon^* \log \lambda_\epsilon \omega_{n-1} \left(\log \frac{r_0}{\epsilon}\right)^{1-n}}$$

for $\epsilon \in (0, r_0)$. Since $\varphi_\epsilon$ is decreasing, the limit $\lim_{\epsilon \to 0^+} \varphi_\epsilon = \theta$ exists. Below we shall assume that $\theta = \pi/2$: the slightly easier case $\theta < \pi/2$ can be dealt with by means of a similar reasoning. Then there exists $r_1 \in (0, r_0)$ such that $\varphi_\epsilon \in (\varphi_0, \pi/2)$ for $\epsilon \in (0, r_1)$ and so $\varphi_\epsilon = \varphi_\epsilon$ for $\epsilon \in (0, r_1)$. Since $\delta_\epsilon$ is increasing, the limit $\lim_{\epsilon \to 0^+} \delta_\epsilon = \delta$
exists. Suppose that \( d > 0 \). Choose \( r_2 \in (0, r_1) \) in such a way that \( \delta_\varepsilon \geq d/2 \) for \( \varepsilon \in (0, r_2) \). Then (3.4) yields
\[
\cos^2 \varphi_\varepsilon \left( \log \frac{1}{\varepsilon} \right)^{n-1} \leq C_1
\]
for \( \varepsilon \in (0, r_2) \), where \( C_1 \in (0, \infty) \) is independent of \( \varepsilon \) in view of (3.2) and (3.3). Letting \( \varepsilon \to 0^+ \) yields a contradiction. Hence \( d = 0 \), i.e. \( \delta_\varepsilon \to 0 \) when \( \varepsilon \to 0^+ \) and by (3.2) and (3.3) there is a number \( r_3 \in (0, r_1) \) such that \( t_\varepsilon \geq 3^{-n-1} \delta_\varepsilon \) and \( \tilde{\lambda}_\varepsilon = \lambda_\varepsilon \) for \( \varepsilon \in (0, r_3) \). Then (3.2) and (3.4) yield
\[
\cos^2 \varphi_\varepsilon \delta_\varepsilon^{n/(n-1)} \left( \log \frac{1}{\varepsilon} \right)^{n-1} \leq C_2
\]
for \( \varepsilon \in (0, r_2) \), where \( C_2 \in (0, \infty) \) does not depend on \( \varepsilon \). Letting \( \varepsilon \to 0^+ \) yields a contradiction.

3.5. Corollary. Let \( f: B^n \to \mathbb{R}^n \) be locally \( K \)-quasiconformal, let \( b \in \partial B^n \), let \( \varphi_0 \in (0, \pi/2) \), and let \( C(f, b) \subset \partial f B^n \). For \( \varepsilon > 0 \) let \( E_\varepsilon = K(b, \varphi_0) \cap f^{-1} B^n (\varepsilon) \) and \( \delta_\varepsilon = \text{cap dens} (E_\varepsilon, b) \). If \( n \geq 3 \) and
\[
\limsup_{\varepsilon \to 0} \delta_\varepsilon^{n/(n-1)} \left( \log \frac{1}{\varepsilon} \right)^{n-1} = \infty,
\]
then \( f \) has angular limit \( 0 \) at \( b \).

3.6. Corollary. Let \( f: B^n \to \mathbb{R}^n \) be locally \( K \)-quasiconformal, let \( b \in \partial B^n \), let \( \varphi_0 \in (0, \pi/2) \), and let \( C(f, b) \subset \partial f B^n \). Suppose that there is a set \( E \subset K(b, \varphi_0) \cap B^n \) such that \( \lim_{x \to b, x \in E} f(x) = 0 \). If \( n \geq 3 \) and \( \text{cap dens} (E, b) = \delta > 0 \), then \( f \) has angular limit \( 0 \) at \( b \).

**Proof.** The proof follows from Corollary 3.5 since here \( \delta_\varepsilon \geq \delta > 0 \) for all \( \varepsilon > 0 \).

4. Further results

In this section we shall study the situation of Theorem 3.1 if the assumption \( C(f, b) \subset \partial f B^n \) is dropped. Now one cannot use Lemma 2.10, on which a central part of the proof of Theorem 3.1 was based, and we shall employ here a different method. For this purpose we shall prove the following lemma, where an appropriate upper bound for the absolute value of a quasiregular mapping is found. More specifically, we consider a quasiregular mapping \( f: B^n \to \mathbb{R}^n \), wishing to find an upper bound for \( |f(x)| \) when \( x \) belongs to a ball \( B^n (r) \), \( r \in (0, 1) \), containing a sufficiently large portion of the set where \( |f| \) is small. This method enables us to prove that Corollary 3.6 holds without the assumption \( C(f, b) \subset \partial f B^n \).
4.1. Lemma. Let $f: B^n \to \mathbb{R}^n$ be quasiregular, let $E_\varepsilon = f^{-1}B^n(\varepsilon)$ for $\varepsilon \in (0, 1)$, let $t \in (0, 1)$, and let $\theta > 1$ with $\theta r < 1$. If $\delta' = M(E_\varepsilon, t, r, 0) > 0$ and $N^{\theta r} = N(f, B^n(\theta r))$, then for $x \in \overline{B}^n(r)$

$$|f(x)| \equiv \varepsilon \exp \left( c\delta'_1/N^{\theta r}\right)^{(1-n)},$$

where $c$ is a positive constant depending only on $n$, $K_0(f)$, and $\theta$.

Proof. Fix $x \in \overline{B}^n(r)$. If $|f(x)| \leq \varepsilon$, there is nothing to prove and we may assume $|f(x)| > \varepsilon$. Let $\beta: [0, \infty) \to \mathbb{R}^n$ be the path $\beta(t) = f(x)(1 + t)$, $t \in [0, \infty)$, and let $\gamma: [0, c) \to B^n$ be a maximal lifting of $\beta$, starting at $x$. Then $\gamma(t) \to \partial B^n$ when $t \to c$ and, in particular, $|\gamma| \cap \partial B^n(\theta r) \neq \emptyset$ (cf. [10, 3.12, 3.11]). Let $\Gamma = \Delta(E_\varepsilon, |\gamma|; B^n(\theta r))$. If we write $F_1 = E_\varepsilon \cap B^n(\theta r)$, $F_2 = |\gamma| \cap B^n(\theta r)$, $F_3 = S^{n-1}(\theta r)$, and $\Gamma_{ij} = \Delta(F_i, F_j; R^n)$, then we get by the comparison principle of Lemma 2.6 and by Lemma 2.21

$$M(\Gamma) \equiv 2^{-1}3^{-n} \min \{M(\Gamma_{13}), M(\Gamma_{23}), c_n \log 2\}.$$

Since $F_2 \cap S^{n-1}(\theta r) \neq \emptyset \neq F_3 \cap S^{n-1}(\theta r)$ it follows from [20, 1.10] or [19, 10.12] that $M(\Gamma_{23}) \equiv c_n \log (2 - \theta^{-1})$. Let $A = (\log 2/\log 2\theta)^{n-1}$. By (2.4) we obtain

$$M(\Gamma) \equiv 3^{-n-1} \min \{c_n \log (2 - \theta^{-1}), A\delta'_r\} \equiv a\delta'_r,$$

where $a = A \cdot 3^{-n-1} \min \{1, c_n \log (2 - \theta^{-1})/(\omega_{n-1}(\log 2)^{1-n})\}$ and the upper bound (2.3) for $\delta'_r$ has been used. Since $f|_{\gamma|} \subset R^n \setminus B^n(|f(x)|)$ we obtain by (2.3)

$$M(f\Gamma) \equiv \omega_{n-1} \left( \frac{|f(x)|}{\varepsilon} \right)^{1-n}.$$ 

The modulus inequality in [8, 3.2] yields

$$M(\Gamma) \equiv K_0(f) N^{\theta r} M(f\Gamma).$$

The asserted inequality follows from the above estimates with the constant $c = a(K_0(f) \omega_{n-1}) > 0$.

4.2. Theorem. Let $f: B^n \to \mathbb{R}^n$ be locally $K$-quasiconformal, let $b \in \partial B^n$, $\varphi_0 \in (0, \pi/2)$, and let $E \subset K(b, \varphi_0) \cap B^n$ be a set with cap dens $(E, b) = \delta > 0$. If $n \geq 3$ and the limit $\lim_{x \to b, x \in E} f(x) = 0$ exists, then $f$ has angular limit 0 at $b$.

Proof. Fix $\varphi \in (\varphi_0, \pi/2)$. Let $\varepsilon \in (0, 1)$. Choose $t_\varepsilon \in (0, \cos \varphi)$ such that $E \cap \overline{B}(b, t_\varepsilon) \subset E_\varepsilon = f^{-1}B^n(\varepsilon)$ and $M(E_\varepsilon, s, b) \equiv 2\delta/3$ for all $s \in (0, t_\varepsilon)$. Let $\lambda \equiv 3$ be such that (cf. (2.3))

$$M(\overline{B}(b, s/\lambda), s, b) \equiv \delta/3$$

for all $s > 0$ and let $B^n(x_\varepsilon, r_\varepsilon)$ be the smallest ball containing $A_\varepsilon^b(s) = K(b, \varphi) \cap R(b, s, s/\lambda)$ when $s \in (0, t_\varepsilon)$. Then

$$\begin{align*}
x_\varepsilon &= \left( 1 - \frac{s(1+1/\lambda)}{2 \cos \varphi} \right) b \\
r_\varepsilon &= \frac{s}{2} ((1+1/\lambda)^2 \tan^2 \varphi + (1-1/\lambda)^2)^{1/2}.
\end{align*}$$
If we use the notation $F_1 = E_0 \cap \tilde{A}_0^c(s), F_2 = S^{n-1}(x_s, |x_s-b|) \cap B^n(b, s), F_3 = S^{n-1}(b, 2s)$, and $\Gamma_i = A(F_i, F_j; R^o), 1 \leq i, j \leq 3$, we get by the comparison principle of Lemma 2.6, in view of the choice of $\lambda$ and (2.3),

$$M_{|x_s-b|}(E_0 \cap \tilde{A}_0^c(s), r_s, x_s) \equiv M(\Gamma_{12}) \equiv 3^{-n} \min \{M(\Gamma_{13}), M(\Gamma_{23}), c_n \log 2\} \equiv 3^{-n} \min \{\delta/3, c_n \log 2\} \equiv a\delta$$

for all $s \in (0, t\varepsilon)$; here $a = 3^{-n-1} \min \{1, c_n(\log 2)^n/\omega_{n-1}\}$ and the upper bound (2.3) for $\delta$ has been used as in the proof of Lemma 4.1. We have also used here the lower bound $M(\Gamma_{23}) \geq c_n \log 2$, which follows from [20, 1.10] or [19, 10.12] because $S^{n-1}(b, u) \cap F_2 \neq \emptyset$ for $u \in (0, s)$. Since $\lambda \equiv 3$ it follows that $2r_s \geq |x_s-b| > r_s$ and hence we get in view of (2.4)

$$M(E_0 \cap \tilde{A}_0^c(s), r_s, x_s) \equiv (\log \frac{|x_s-b|}{r_s})/\log 2)^n-1 a\delta = d_1$$

for all $s \in (0, t\varepsilon)$; here, as in what follows, $d_j, j=1, 2, \ldots$, denotes a positive constant depending only on some of the numbers $n, K, \varphi, \delta$. Let $\theta = (|x_s-b|+r_s)/2r_s = d_2 > 1$. Since $n \equiv 3$ it follows from Lemma 2.12 that $N^{\theta} = N\left(f, B^n(x_s, \theta r_s)\right) = d_3$. If we now apply Lemma 4.1 to the mapping $f|B^n(x_s, |x_s-b|)$, we get for $u \in B^n(x_s, r_s)$ the estimate

$$|f(u)| \equiv \varepsilon d_4$$

for all $s \in (0, t\varepsilon)$. Since

$$K(b, \varphi) \cap B^n(b, t) \subset \bigcup_{s \in (0, t\varepsilon)} B^n(x_s, r_s)$$

and since $\varepsilon \in (0, 1)$ was arbitrary, the proof follows now from the definition of an angular limit.

4.3. Corollary. Let $f : B^n \rightarrow \mathbb{R}^n$ be locally $K$-quasiconformal and let $b \in \partial B^n$. If $n \equiv 3$ and the radial limit $\lim_{t \rightarrow 1} f(tb) = 0$ exists, then $f$ has angular limit 0 at $b$.

Proof. By [19, 10.12] the conditions in Theorem 4.2 are satisfied by $\delta = c_n \log 2$.

The next example shows that the condition on the set $E$ in Theorem 4.2 is in a sense the best possible.

4.4. Examples. (1) The locally $K$-quasiconformal mapping in Example 2.11 shows that the conditions $\cap \text{dens}(E, b) > 0$ and $\lim_{x \rightarrow b, x \in E} f(x) = 0$ cannot be replaced by the requirement that there exist a sequence $(b_k)$ in $K(b, \varphi_0) \cap B^n$ with $b_k \rightarrow b, f(b_k) \rightarrow 0$, and $\lim \sup \rho(b_k, b_{k+1}) < \infty$. Note that the situation is different for a quasiconformal mapping $g : B^n \rightarrow G'$, $n \equiv 2$, since then the inclusion $C(g, b) \subset \partial fB^n = \partial G'$ in Lemma 2.10 holds (cf. [20, 6.5, 6.7]).

(2) Theorem 4.2 and Corollary 4.3 fail to hold for the dimension $n = 2$. A counterexample is provided by the analytic function $h : B^2 \rightarrow \mathbb{R}^2 \setminus \{0\}, h(z) = \exp \left(-\left(1 - z^{-4}\right)\right), z \in B^2$, which is a local homeomorphism and has a radial limit 0 at $z=1$. On the other hand the function $h$ does not have any angular limit at 1, since $h(z) \rightarrow \infty$ when
$z \to 1$ through the line $y = -x + 1$. An essential feature of this function $h$ is that, in view of Lemma 4.7 and Corollary 5.5, it does not satisfy condition (5.1) below.

(3) From the example given in [20, 6.6] it follows that the condition $\text{cap dens}(E, b) = 0$ in Theorem 4.2 cannot be replaced by $\text{cap dens}(E, b) > 0$.

According to Example 4.4 (1) the set $E$ in Theorem 4.2 cannot be replaced, in general, by a sequence with the property in 4.4 (1). Following an idea of Bagemihl and Seidel [1, Lemma 1] we shall now show that $E$ can be replaced by a sufficiently dense sequence of the same type, provided that the mapping in question is normal. The next lemma completes the result in [20, 6.3] and Lemma 2.10 above, both proved under the assumption $\alpha \in \partial fB^n$, which is not needed here.

4.5. Lemma. Suppose that $f : B^n \to R^n$ is normal, $b \in \partial B^n$, and $(b_k)$ is a sequence in $B^n$ with $b_k \to b$ and $f(b_k) \to \alpha$. If $(r_k)$ is a sequence of positive numbers with $\lim r_k = 0$, then $f(x) \to \alpha$ as $x$ approaches $b$ through the set $E = \cup D(b_k, r_k)$.

Proof. Suppose that this is not the case. Then there is a sequence $(a_k)$ in $E$ with $a_k \to b$, $f(a_k) \to \beta \neq \alpha$. After relabeling, if necessary, we may assume that $\varphi(a_k, b_k) = r_k$, $k = 1, 2, \ldots$. Let $h_k : B^n \to B^n$ be a conformal self-mapping of $B^n$ with $h_k(b_k) = 0$. Choose a subsequence of $(f \circ h_k^{-1})$, denoted again by $(f \circ h_k^{-1})$, converging $\epsilon$-uniformly towards $g : B^n \to \mathbb{R}^n$. Since $f(b_k) = f(h_k^{-1}(0)) \to \alpha$, $g(0) = \alpha$. On the other hand $h_k(a_k) \to 0$ and $f(a_k) = f(h_k^{-1}(h_k(a_k))) \to \beta \neq \alpha$, which contradicts the continuity of $g$ at 0. The proof is complete.

4.6. Theorem. Let $f : B^n \to R^n$ be a locally $K$-quasiconformal mapping omitting one finite point, let $b \in \partial B^n$, $\varphi \in (0, \pi/2)$, and let $(b_k)$ be a sequence in $B^n \cap K(b, \varphi)$ with $b_k \to b$ and $f(b_k) \to \alpha$. If $n \equiv 3$ and $\lim \varphi(b_k, b_{k+1}) = 0$, then $f$ has angular limit $\alpha$ at $b$.

Proof. From [10, 2.9] and [19, 20.4] it follows that $f$ is normal, and hence by Lemma 4.5 $f(x)$ tends to $\alpha$ as $x$ approaches $b$ through a curve $C \subset K(b, \varphi) \cap B^n$ terminating at $b$ and consisting of the geodesics of the hyperbolic metric, joining $b_k$ to $b_{k+1}$, for each $k = 1, 2, \ldots$. The proof now follows from Theorem 4.2.

The next result follows from the proof of [7, 5.8].

4.7. Lemma. Let $f : B^n \to R^n$ be a normal quasiregular mapping having a radial limit $\alpha$ at $b \in \partial B^n$. Then $f$ has angular limit $\alpha$ at $b$.

4.8. Remark. Lemma 4.7 was pointed out to the author by Prof. O. Martio. From 4.7 and [10, 2.10] we get an alternative proof for Corollary 4.3 under the additional assumption $R^n \setminus fB^n \neq 0$. 
5. Concluding remarks

As was pointed out in Example 4.4 (2), Theorem 4.2 fails to hold when the dimension $n=2$. In this final section we shall show how Theorems 4.2 and 3.1 can be generalized to cover the case $n=2$ as well and how the condition on the mapping can be weakened. As a byproduct we obtain a normality criterion for quasiregular mappings, which is related to a problem of W. K. Hayman concerning meromorphic functions.

Let $f: B^n \rightarrow R^n$, $n \geq 2$, be a quasiregular mapping with the following property: there exist numbers $p \in [1, \infty)$ and $t \in (0, \infty)$ such that

$$N(f, D(x, t)) \leq p \quad \text{for all} \quad x \in B^n. \quad (5.1)$$

Then one can easily prove for $f$ estimates of the same type as those in Lemmas 2.12 and 2.14, where only locally $K$-quasiconformal mappings of $B^n$, $n \geq 3$, were considered. This observation, together with the fact that Lemma 2.19 holds for $n=2$ as well, is all that is needed to carry over the proofs of Theorems 3.1 and 4.2 to the case of quasiregular mappings satisfying (5.1). Thus we have

5.2. Theorem. Theorems 3.1 and 4.2 hold for a quasiregular mapping $f: B^n \rightarrow R^n$, $n \geq 2$, with property (5.1).

Condition (5.1) has also some interest in the theory of normal functions. In [16, Problem 3.5] W. K. Hayman asked whether there exists a non-normal meromorphic function $f: B^2 \rightarrow R^2$ with property (5.1). This question was answered in the affirmative by Lappan [5], who constructed a non-normal analytic function satisfying (5.1) for some $t>0$ and for $p=1$. From the next result, based on a theorem of Rickman [18], it follows that such a function cannot omit any point in $R^2$.

5.3. Theorem. A quasiregular mapping $f: B^n \rightarrow R^n \setminus \{d\}$, $n \geq 2$, $d \in R^n$, with property (5.1) is normal.

Proof. Let $(h_j)$ be a sequence of conformal self-mappings of $B^n$. By Ascoli's theorem [19, Chapter 19] it will be enough to show that $(f_j)$, $f_j = f \circ h_j$, is equi-continuous at 0. Fix $s>0$ with $B^n(s) \subset D(0, t)$. Then by (5.1) $N(f_j, B^n(s)) \leq p$ and $K_1(f_j) = K_1(f)$ in the notation of [18]. By [18, 4.4] there are constants $C_1>0$ and $C_2>0$ depending only on $K_1(f)$, $n$, and $d$ such that

$$q(f_j B^n(s/\lambda)) \leq C_1 \left( \exp \left( C_2 p^{n+1} \left( \log \lambda \right)^{-n} \right) - 1 \right)$$

for all $\lambda>1$, where $q$ is the spherical metric. Letting $\lambda \rightarrow \infty$ yields the desired conclusion.

For the next result the reader is referred to [10, 2.10].
5.4. Corollary. A locally $K$-quasiconformal mapping $f: \mathbb{B}^n \rightarrow \mathbb{R}^n \setminus \{0\}$ is normal if $n \geq 3$.

Proof. Condition (5.1) is satisfied by [10, 2.3].
The following corollary seems also to be well-known.

5.5. Corollary. An analytic function $f: \mathbb{B} \rightarrow \mathbb{R}^2 \setminus \{0\}$ with property (5.1) is normal.

5.6. Remarks. (1) For the remark that Theorem 5.3 is related to Hayman's problem, the author wishes to thank Prof. P. Lappan.

(2) Some results in [14] concerning locally $K$-quasiconformal mappings of $\mathbb{B}^n$, $n \geq 3$, can be generalized in the spirit of this section.

References

On the boundary behavior of locally $K$-quasiconformal mappings in space


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