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ON THE OUTER COEFFICIENT OF QUASICONFORMALITY OF A CYLINDRICAL MAP OF A CONVEX DIHEDRAL WEDGE

KARI HAG and MARJATTA NÄÄTÄNEN

1. Introduction

Let D and D' be domains in $\boldsymbol{R}^3$ and let $f\colon D \to D'$ be a homeomorphism. With each f we can associate two numbers, the inner and outer dilatation of f,

$$K_I(f) = \sup_{\Gamma} \frac{M(f\Gamma)}{M(\Gamma)}, \quad K_O(f) = \sup_{\Gamma} \frac{M(\Gamma)}{M(f\Gamma)},$$

which measure how far f is from being conformal. Here $M(\Gamma)$ and $M(f\Gamma)$ are the moduli of the curve families Γ and $f\Gamma$, the suprema being taken over all families in D; see [6]. Further, the inner and outer coefficients of quasiconformality of D with respect to D' are defined as

$$K_I(D, D') = \inf_{f} K_I(f), \quad K_O(D, D') = \inf_{f} K_O(f).$$

The coefficients of quasiconformality have been calculated only for very few domains. For example, only $K_O(D, B^3)$ has been determined when D is an infinite cylinder or an infinite convex cone. On the other hand, only $K_I(D, B^3)$ is known when D is a convex dihedral wedge. More precisely, Gehring and Väisälä ([3], [6; p. 134]) have found that $K_I(D, D')=\beta/\alpha$ in the case of convex dihedral wedges with angles α, β and $\alpha \leqq \beta$. It is claimed by Syčev [4] that also $K_O(D, D')=\beta/\alpha$, but no proof is given. In Section 3 we show that this result follows easily for a subclass of mappings which satisfy a cylindrical condition; see 2.1. Taari [5] has obtained the same result for a subclass of mappings satisfying two local conditions; see 4.1. In Section 4 we use a mapping f satisfying Taari's conditions to get a cylindrical map h with $K_O(f) \geqq K_O(h)$. Hence Taari's result follows from our simpler argument.

2. Basic notation

2.1. *Definitions.* Let (t, ψ, φ) be spherical coordinates in $\boldsymbol{R}^3$, where the polar angle φ is measured from the positive x_3-axis. A domain in $\boldsymbol{R}^3$ is a *convex dihedral wedge* of angle α, $0<\alpha\leqq\pi$, if it can be mapped by a similarity transformation onto the domain $D_\alpha = \{(t, \psi, \varphi) | t>0,\ 0<\psi<\alpha,\ 0<\varphi<\pi\}$.

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Let F denote the class of homeomorphisms f: $\bar{D}_\alpha \to \bar{D}_\pi$, $0<\alpha\leqq\pi$, whose restrictions $f|D_\alpha$: $D_\alpha \to D_\pi$ are quasiconformal, and for sufficiently small radii map the intersection of D_α and a circular infinite cylinder with axis the x_3-axis onto the intersection of D_π and a similar cylinder. We call such a mapping $f\in F$ *cylindrical.*

Let F_L denote the class of homeomorphisms f: $\bar{D}_\alpha \to \bar{D}_\pi$, $0<\alpha\leqq\pi$, whose restrictions $f|D_\alpha$: $D_\alpha \to D_\pi$ are quasiconformal, $f(0)=0$, and which satisfy the following condition at the origin:

$$\lim_{\substack{\delta_1\to 0\\ \delta_2\to 0}} \max r'(x)/\min r'(x) = 1,$$

where $(r', \psi', z')=f(r, \psi, z)$ in terms of cylindrical coordinates, and the maximum and minimum are taken over $x\in S(\delta_1, \delta_2)=\{(r, \psi, z)|r=\delta_1, \ |z|\leqq\delta_2\}\cap\bar{D}_\alpha$. We call such a mapping $f\in F_L$ *cylindrical at the origin.*

2.2. Remark. We can state the definitions for closures of wedges without loss of generality since every cylindrical quasiconformal mapping f: $D_\alpha\to D_\pi$, $0<\alpha\leqq\pi$, can be extended to a homeomorphism f^*: $\bar{D}_\alpha\to\bar{D}_\pi$, such that $f^*(x)\to\infty$ as $x\to\infty$.

3. The outer coefficient for cylindrical mappings

3.1. Theorem. *For the class F of cylindrical mappings*

$$\inf\{K_O(f)|f\in F\} = \pi/\alpha.$$

Proof. We show first that $K_O(f)\geqq\pi/\alpha$ for $f\in F$. Using cylindrical coordinates, let $r_2>r_1$ and $G_i=\{(r, \psi, z)|0<r<r_i, \ 0<\psi<\alpha, \ z_1<z<z_2\}$ for $i=1, 2$, and let Γ be the family of curves joining the set $\{(r, \psi, z)|r_1\leqq r\leqq r_2, \ \psi=0, \ z_1\leqq z\leqq z_2\}$ to a similar set with $\psi=\alpha$, in the closure of G_2-G_1. Then as in Gehring [2; Lemma 1],

$$M(\Gamma) = (r_1^{-1}-r_2^{-1})(z_2-z_1)\alpha^{-2}. \tag{1}$$

Let Γ_1 be the family of curves joining the sets $\bar{G}_1\cap(r, \psi, z)|z=z_i\}$, $i=1, 2$, in $\bar{G}_1$. Then by Väisälä [6; 7.2],

$$M(\Gamma_1) = \alpha\pi r_1^2/(2\pi(z_2-z_1)^2). \tag{2}$$

By (1) and (2),

$$M(\Gamma)^2 M(\Gamma_1) = \frac{1}{2}\alpha^{-3}(1-r_1/r_2)^2. \tag{3}$$

Let r_i' be the radius of $f(G_i)$, $i=1, 2$, and let $\bar{z}_i$ and $\underline{z}_i$ be the maximal and minimal z-coordinates in the f-image of the set $\bar{G}_2\cap\{(r, \psi, z)|z=z_i\}$ $i=1, 2$, respectively. Denote by Γ' and Γ_1' the images of Γ and Γ_1 under f. Then

$$M(\Gamma') \leqq (1/r_1'-1/r_2')(\bar{z}_2-\underline{z}_1)\pi^{-2},$$

$$M(\Gamma_1') \leqq \frac{1}{2}\pi(r_1')^2(\underline{z}_2-\bar{z}_1)^{-2},$$

and

(4) $$M(\Gamma')^2 M(\Gamma_1') \leqq \frac{1}{2}\pi^{-3}(1-r_1'/r_2')^2(\bar{z}_2-\underline{z}_1)^2(\underline{z}_2-\bar{z}_1)^{-2}.$$

Next we consider $\bar{z}_2-\underline{z}_2$. We extend f to a quasiconformal mapping $\hat{f}$ of $\boldsymbol{R}^3$ using the same foldings as Taari [5]: Let $g\colon D_\alpha \to D_\pi$ denote the folding given by $g(r, \psi, z) = (r, \pi\psi/\alpha, z)$. Next we extend $f\circ g^{-1}\colon D_\pi \to D_\pi$ to a quasiconformal mapping $f_1\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ by reflection. Finally, let $f_2\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be $f_2(r, \psi, z) = (r, \psi', z)$, where

$$\psi' = \begin{cases} \alpha\psi/\pi & \text{for} \quad 0 \leqq \psi \leqq \pi \\ \alpha+(2\pi-\alpha)\pi^{-1}(\psi-\pi) & \text{for} \quad \pi \leqq \psi \leqq 2\pi. \end{cases}$$

Then $\hat{f}=f_1\circ f_2^{-1}\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is quasiconformal and $\hat{f}|\bar{D}_\alpha = f$. Lemma 8.1 of [3] applies to $\hat{f}^{-1}|\{(r, \psi, z)|0<r<r_2'\}$ followed by the map $h(r, \psi, z) = (t, \psi, \varphi)$ with $t=e^z$, $\varphi = \pi r/(2r_2)$. (We can assume f to be normalized in such a way that $f(0, 0, z) \to +\infty$ as $z \to +\infty$.) We get

(5) $$0 \leqq \bar{z}_2 - \underline{z}_2 \leqq r_2' A K_I(h\circ \hat{f}^{-1}),$$

where A is an absolute constant.

On the other hand, by (3) and (4),

$$\alpha^{-3}(1-r_1/r_2)^2 \leqq K_O(f)^3\pi^{-3}(\bar{z}_2-\underline{z}_1)^2(\underline{z}_2-\bar{z}_1)^{-2}.$$

Letting $z_2 \to \infty$ we get by (5)

$$\alpha^{-3}(1-r_1/r_2)^2 \leqq K_O(f)^3\pi^{-3}.$$

From this with $r_1 \to 0$ we see that $K_O(f) \geqq \pi/\alpha$. On the other hand, if f is the cylindrical map

(6) $$f(r, \psi, z) = (r, \pi\psi/\alpha, \pi z/\alpha)$$

we have equality so the bound is sharp.

The above result can be extended to locally cylindrical mappings as follows:

3.2. Theorem. *For the class F_L of mappings cylindrical at the origin,*

$$\inf\{K_O(f)|f\in F_L\} = \pi/\alpha.$$

Proof. Since the mapping (6) is cylindrical at the origin it suffices, by Theorem 3.1, to show that for each $f\in F_L$ there exists a $g\in F$ such that $K_O(f) \geqq K_O(g)$. As in the proof of Theorem 3.1, each $f\in F_L$ can be extended to a quasiconformal mapping $\hat{f}\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$. Next, let $\hat{g}_n\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be the sequence defined by $\hat{g}_n(x) = a_n f(x/n)$, where a_n is chosen in such a way that $|\hat{g}_n(e_1)|=1$. Since $\hat{g}_n(0)=0$ we conclude by [6; 19.4, 20.5] that $\{\hat{g}_n\}$ is a normal family and there is a subsequence $\hat{g}_j$, $j\in J\subset \boldsymbol{N}$, converging to a limit function $\hat{g}\colon \boldsymbol{R}^3 \to \boldsymbol{R}^3$. The convergence is uniform on compact subsets of $\boldsymbol{R}^3$ and $\hat{g}$ is a homeomorphism since $\hat{g}(0)=0$, $|\hat{g}(e_1)|=1$; see [6; 21.3]. By [6; 37.2] $\hat{g}$ is quasiconformal, $\hat{g}(D_\alpha) = D_\pi$, and $K_O(f) \geqq K_O(g)$, where $g=\hat{g}|\bar{D}_\alpha$. It follows that for each $r_0>0$ g maps the compact sets $\{(r, \psi, z)|r=r_0,\ |z|\leqq M\}\cap\bar{D}_\alpha$ into some cylindrical surface. Also, $g(0)=0$ and, by a simple topological argument, $g\in F$.

3.3. Remark. So far we have only considered the problem of determining $K_O(D_\alpha, D_\beta)$, $0<\alpha\leqq\beta<\pi$, for $\beta=\pi$. Composing $f\colon D_\alpha\to D_\beta$ with the standard folding (6), α replaced by β, we deduce that $K_O(D_\alpha, D_\beta)=\beta/\alpha$.

4. Application

As an application we construct in Theorem 4.3 a cylindrical map h from Taari's functions and show that Taari's result follows from our result.

4.1. *Taari's conditions.* Taari [5] considers the subclass of homeomorphisms $f\colon \bar{D}_\alpha\to\bar{D}_\pi$, $0<\alpha\leqq\pi, f(0)=0$, whose restrictions $f|D_\alpha$ are quasiconformal mappings onto D_π and which satisfy the following conditions at the origin:

Condition A. There is a polar angle φ_0, $0<\varphi_0<\pi/2$ such that the limit

$$\lim_{t\to 0+} f(te)/t = k(e) \neq 0, \infty$$

exists for every $e\in\bar{D}_\alpha$ with $0\leqq(e, e_3)\leqq\varphi_0$, where (e, e_3) denotes the acute angle between the vectors e and e_3.

Condition B. For $e\in\bar{D}_\alpha$

$$\lim_{\varepsilon\to 0+} \max\big(k(e), k(e_3)\big)/\min\big(k(e), k(e_3)\big) = 1,$$

where the maximum and minimum are over all vectors e such that $(e, e_3)=\varepsilon$.

4.2. Theorem. *Let f be a mapping satisfying Taari's conditions stated in* 4.1. *Then there is a cylindrical map h such that* $K_O(f)\geqq K_O(h)$.

Proof. Given f we construct, stepwise, a map $\hat{h}\colon \boldsymbol{R}^3\to\boldsymbol{R}^3$ whose restriction $h=\hat{h}|\bar{D}_\alpha$ will be cylindrical and satisfy $K_O(f)\geqq K_O(h)$.

The first step is carried out in [5]. Using Condition A Taari constructs a mapping $g\colon \bar{D}_\alpha\to\bar{D}_\pi$, $g(0)=0$, such that for each $e\in\bar{D}_\alpha$, $0\leqq(e, e_3)\leqq\varphi_0$, the restriction of g to the ray $\{te|t\geqq 0\}$ is linear, i.e. $g(te)=tg(e)$. Furthermore, using additional rotation and stretching we can assume $g(e_3)=e_3$. Also, $K_O(g)\leqq K_O(f)$.

Next the mapping g is extended to a quasiconformal map $\hat{g}\colon \boldsymbol{R}^3\to\boldsymbol{R}^3$ as in the proof of Theorem 3.1. We observe, for later use, that if $0\leqq(e, e_3)\leqq\varphi_0$, the restriction of $\hat{g}$ to the ray $\{te|t\geqq 0\}$ is linear. Now, let T_n denote translation by ne_3 and let $\hat{h}_n=T_{-n}\circ\hat{g}\circ T_n$. For $t\geqq -n$, $\hat{h}_n(te_3)=T_{-n}\big(\hat{g}((t+n)e_3)\big)=te_3$; hence $\{\hat{h}_n\}$ is a normal family and has a subsequence $\hat{h}_j$, $j\in J\subset\boldsymbol{N}$, converging to a quasiconformal mapping $\hat{h}\colon \boldsymbol{R}^3\to\boldsymbol{R}^3$. The convergence is uniform on compact subsets of $\boldsymbol{R}^3$, $\hat{h}(D_\alpha)=D_\pi$ and $K_O(h)\leqq K_O(g)$, where $h=\hat{h}|\bar{D}_\alpha$. Hence $K_O(h)\leqq K_O(f)$.

To complete the proof we will show that if C is a circular cylinder (domain) with axis the x_3-axis, then $\hat{h}(C)$ is a similar cylinder. For such a cylinder C let C_j be the cone determined by the circle $\partial C \cap \{x | x_3=0\}$ and the point $(-j)e_3$, and put $C'_j = \hat{h}_j(C_j)$. By Theorem 3 of Gehring [1] we find that $\hat{h}(C) = \text{Ker } C'_j$ since $C = \text{Ker } C_j$. Next we show that Ker C'_j is a cylinder, i.e. that if $x_0 \in \text{Ker } C'_j$, then $\{x_0 + te_3 | t \in \boldsymbol{R}\} \subset \text{Ker } C'_j$. Let (r_0, ψ_0, z_0) be the cylindrical coordinates of x_0 and let $x_0 \in \text{Ker } C'_j$. Then there exists an $\varepsilon > 0$, $\varepsilon < r_0$, such that the cylindrical neighborhood N with $|r - r_0| < \varepsilon$, $|\psi - \psi_0| < \varepsilon$, $|z - z_0| < \varepsilon$ is in C'_j for $j \geqq j_0$. Next consider C'_j. For large j, $\hat{g}$ maps $C_j + je_3$ in such a way that each ray through the origin is mapped onto a ray through the origin. This means that C'_j is a union of rays through $-je_3$; in particular, all rays through $-je_3$ meeting N are in C'_j. In the case $t > 0$ take j so large that also $z_0 + j > 0$ and

$$\arc \tan \frac{r_0 - \varepsilon}{z_0 + j + \varepsilon} \leqq \arc \tan \frac{\varepsilon/2}{t - \varepsilon/2}.$$

Then the cylindrical neighborhood of $x_0 + te_3$ with $|r - r_0| < \varepsilon/2$, $|\psi - \psi_0| < \varepsilon/2$, $|z - (z_0 + t)| < \varepsilon/2$ is in C'_j. Hence $\{x_0 + te_3 | t \geqq 0\} \subset \text{Ker } C'_j$. Similarly, $\{x_0 + te_3 | t < 0\} \subset \text{Ker } C'_j$. Finally we use Condition B to show that $\hat{h}(C)$ is circular with axis the x_3-axis. Since $k(e) = g(e)$, we see by Condition B that for all j

$$\lim_{\delta \to 0+} \max \big(\hat{g}(T_j x), e_3\big) / \min \big(\hat{g}(T_j x), e_3\big) = 1,$$

where maximum and minimum are taken over all x such that $(T_j x, e_3) = \delta$. Let $r > 0$ and put $S = \{(r, \psi, z) | r = r_0, z = z_0\}$. For $j \geqq j_1$ we have by the above

$$\max_{x \in S} \big(\hat{g}(T_j x), e_3\big) / \min_{x \in S} \big(\hat{g}(T_j x), e_3\big) < 1 + \varepsilon. \tag{7}$$

The continuity of $\hat{g}$ at e_3, the linearity of $\hat{g}$ on rays close to the x_3-axis, and the fact that $\hat{g}(e_3) = e_3$ together imply that

$$1 - \varepsilon < |\hat{g}(x + je_3)| / |x + je_3| < 1 + \varepsilon \tag{8}$$

for $j \geqq j_2$, $x \in S$. Suppose that $j \geqq j_1, j_2$ and let $\underline{\varphi}_j = \min_{x \in S} \big(\hat{g}(T_j x), e_3\big)$, $\bar{\varphi}_j = \max_{x \in S} \big(\hat{g}(T_j x), e_3\big)$. By (7)

$$\bar{\varphi}_j < (1 + \varepsilon) \underline{\varphi}_j. \tag{9}$$

From (8) we get for $x \in S$

$$(1 - \varepsilon) \sqrt{r_0^2 + (z_0 + j)^2} < |\hat{g}(x + je_3)| < (1 + \varepsilon) \sqrt{r_0^2 + (z_0 + j)^2}. \tag{10}$$

Then by (9) and (10)

$$\frac{\max_{x \in S} r\big(\hat{g}(x + je_3)\big)}{\min_{x \in S} r\big(\hat{g}(x + je_3)\big)} < \frac{(1 + \varepsilon) \sin (1 + \varepsilon) \underline{\varphi}_j}{(1 - \varepsilon) \sin \underline{\varphi}_j}. \tag{11}$$

Since $\hat{h}_j(x)=\hat{g}(x+je_3)-je_3$, we can replace $r(\hat{g}(x+je_3))$ by $r(\hat{h}_j(x))$ in (11). Since S is compact, $\hat{h}_j\to\hat{h}$ uniformly on S. Also, $\underline{\varphi}_j\to 0$ as $j\to\infty$, and the upper bound in (11) can be replaced by $(1+\varepsilon)^3/(1-\varepsilon)$. Since $\varepsilon>0$ was arbitrary,

$$\max r(\hat{h}(x))/\min r(\hat{h}(x)) = 1$$

for $x\in S$.

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University of Michigan
Department of Mathematics
Ann Arbor, Michigan 48109
USA

University of Trondheim
Department of Mathematics, NTH
N-7034 Trondheim
Norway

University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

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