ON THE OUTER COEFFICIENT OF QUASICONFORMALITY OF A CYLINDRICAL MAP OF A CONVEX DIHEDRAL WEDGE

KARI HAG and MARJATTA NÄÄTÄNEN

1. Introduction

Let $D$ and $D'$ be domains in $\mathbb{R}^3$ and let $f: D \rightarrow D'$ be a homeomorphism. With each $f$ we can associate two numbers, the inner and outer dilatation of $f$,

\[ K_I(f) = \sup_I \frac{M(fI)}{M(I)}, \quad K_O(f) = \sup_I \frac{M(I)}{M(fI)}, \]

which measure how far $f$ is from being conformal. Here $M(I)$ and $M(fI)$ are the moduli of the curve families $I$ and $fI$, the suprema being taken over all families in $D$; see [6]. Further, the inner and outer coefficients of quasiconformality of $D$ with respect to $D'$ are defined as

\[ K_I(D, D') = \inf_f K_I(f), \quad K_O(D, D') = \inf_f K_O(f). \]

The coefficients of quasiconformality have been calculated only for very few domains. For example, only $K_O(D, B^3)$ has been determined when $D$ is an infinite cylinder or an infinite convex cone. On the other hand, only $K_I(D, B^3)$ is known when $D$ is a convex dihedral wedge. More precisely, Gehring and Väisälä ([3], [6; p. 134]) have found that $K_I(D, D') = \beta/\alpha$ in the case of convex dihedral wedges with angles $\alpha$, $\beta$ and $\alpha=\beta$. It is claimed by Syčev [4] that also $K_O(D, D') = \beta/\alpha$, but no proof is given. In Section 3 we show that this result follows easily for a subclass of mappings which satisfy a cylindrical condition; see 2.1. Taari [5] has obtained the same result for a subclass of mappings satisfying two local conditions; see 4.1. In Section 4 we use a mapping $f$ satisfying Taari's conditions to get a cylindrical map $h$ with $K_O(f) \equiv K_O(h)$. Hence Taari's result follows from our simpler argument.

2. Basic notation

2.1. Definitions. Let $(t, \psi, \varphi)$ be spherical coordinates in $\mathbb{R}^3$, where the polar angle $\varphi$ is measured from the positive $x_3$-axis. A domain in $\mathbb{R}^3$ is a convex dihedral wedge of angle $\alpha$, $0 < \alpha \leq \pi$, if it can be mapped by a similarity transformation onto the domain $D_\alpha = \{(t, \psi, \varphi) | t > 0, \, 0 < \psi < \alpha, \, 0 < \varphi < \pi \}$. 

Let $F$ denote the class of homeomorphisms $f : D_\alpha \to D_\pi$, $0 < \alpha \leq \pi$, whose restrictions $f|D_\alpha : D_\alpha \to D_\pi$ are quasiconformal, and for sufficiently small radii map the intersection of $D_\alpha$ and a circular infinite cylinder with axis the $x_3$-axis onto the intersection of $D_\alpha$ and a similar cylinder. We call such a mapping $f \in F$ cylindrical.

Let $F_L$ denote the class of homeomorphisms $f : D_\alpha \to D_\pi$, $0 < \alpha \leq \pi$, whose restrictions $f|D_\alpha : D_\alpha \to D_\pi$ are quasiconformal, $f(0) = 0$, and which satisfy the following condition at the origin:

$$
\lim_{\delta_1 \to 0} \max_{D_\delta_1 \cap D_\delta_2} r'(x)/\min_{D_\delta_1 \cap D_\delta_2} r'(x) = 1,
$$

where $(r', \psi', z') = f(r, \psi, z)$ in terms of cylindrical coordinates, and the maximum and minimum are taken over $x \in S(\delta_1, \delta_2) = \{(r, \psi, z) : r = \delta_1, |z| \leq \delta_2 \} \cap \overline{D}_\alpha$. We call such a mapping $f \in F_L$ cylindrical at the origin.

2.2. Remark. We can state the definitions for closures of wedges without loss of generality since every cylindrical quasiconformal mapping $f : D_\alpha \to D_\pi$, $0 < \alpha \leq \pi$, can be extended to a homeomorphism $f^* : D_\alpha \to D_\pi$, such that $f^*(x) \to \infty$ as $x \to \infty$.

3. The outer coefficient for cylindrical mappings

3.1. Theorem. For the class $F$ of cylindrical mappings

$$
\inf \{K_0(f) : f \in F\} = \frac{\pi}{\alpha}.
$$

Proof. We show first that $K_0(f) \geq \frac{\pi}{\alpha}$ for $f \in F$. Using cylindrical coordinates, let $r_2 > r_1$ and $G_i = \{(r, \psi, z) : 0 < r < r_i, 0 < \psi < \alpha, z_1 < z < z_2\}$ for $i = 1, 2$, and let $\Gamma$ be the family of curves joining the set $\{(r, \psi, z) : r = r_1, \psi = 0, z_1 \leq z \leq z_2\}$ to a similar set with $\psi = \alpha$, in the closure of $G_2 - G_1$. Then as in Gehring [2; Lemma 1],

$$
M(\Gamma) = (r_1^{-1} - r_2^{-1})(z_2 - z_1)\alpha^{-2}.
$$

Let $\Gamma_1$ be the family of curves joining the sets $\overline{G}_i \cap (r, \psi, z) | z = z_i$, $i = 1, 2$, in $\overline{G}_1$. Then by Väisälä [6; 7.2],

$$
M(\Gamma_1) = \alpha \pi r_i^2 (2\pi(z_2 - z_1))^2).
$$

By (1) and (2),

$$
M(\Gamma)^2 M(\Gamma_1) = \frac{1}{2} \alpha^{-3} (1 - r_1/r_2)^2.
$$

Let $r'_i$ be the radius of $f(G_i)$, $i = 1, 2$, and let $\bar{z}_i$ and $\bar{z}'_i$ be the maximal and minimal $z$-coordinates in the $f$-image of the set $\overline{G}_i \cap \{(r, \psi, z) : z = z_i\}$ $i = 1, 2$, respectively. Denote by $\Gamma'$ and $\Gamma'_1$ the images of $\Gamma$ and $\Gamma_1$ under $f$. Then

$$
M(\Gamma') \leq (1/r'_1 - 1/r'_2)(\bar{z}_2 - \bar{z}_1)^{\alpha^{-2}},
$$

$$
M(\Gamma'_1) \leq \frac{1}{2} \pi (r'_1)^2 (\bar{z}_2 - \bar{z}_1)^{-2}.
$$

and

\[ M(\Gamma')^2M(\Gamma) \equiv \frac{1}{2} \pi^{-3}(1-r_1'/r_2)^3(\bar{z}_2-\bar{z}_1)^2(\bar{z}_2-\bar{z}_1)^{-2}. \]

Next we consider $\bar{z}_2-\bar{z}_1$. We extend $f$ to a quasiconformal mapping $\hat{f}$ of $R^8$ using the same foldings as Taari [5]: Let $g: D_x \rightarrow D_n$ denote the folding given by $g(r, \psi, z) = (r, \pi \psi/\alpha, z)$. Next we extend $f \circ g^{-1}: D_\pi \rightarrow D_\pi$ to a quasiconformal mapping $f_1: R^8 \rightarrow R^8$ by reflection. Finally, let $f_2: R^8 \rightarrow R^8$ be $f_2(r, \psi, z) = (r, \psi', z)$, where

\[
\psi' = \begin{cases} \alpha \psi / \pi & \text{for } 0 \leq \psi \leq \pi \\ \alpha + (2\pi - \alpha) \pi^{-1}(\psi - \pi) & \text{for } \pi \leq \psi \leq 2\pi. \end{cases}
\]

Then $\hat{f} = f_1 \circ f_2^{-1}: R^8 \rightarrow R^8$ is quasiconformal and $\hat{f}|D_\pi = f$. Lemma 8.1 of [3] applies to $f^{-1}\{(r, \psi, z)|0 < r < r_1^2\}$ followed by the map $h(r, \psi, z) = (t, \psi, \varphi)$ with $t = e^z$, $\varphi = \pi r/(2r_2)$. (We can assume $f$ to be normalized in such a way that $f(0, 0, z) \to +\infty$ as $z \to +\infty$.) We get

\[ 0 \leq \bar{z}_2-\bar{z}_1 \leq r_2^2 A K_1(h \circ f^{-1}), \]

where $A$ is an absolute constant.

On the other hand, by (3) and (4),

\[ \alpha^{-3}(1-r_1/r_2)^2 \leq K_0(f)^3 \pi^{-3}(\bar{z}_2-\bar{z}_1)^2(\bar{z}_2-\bar{z}_1)^{-2}. \]

Letting $z_2 \to \infty$ we get by (5)

\[ \alpha^{-3}(1-r_1/r_2)^2 \leq K_0(f)^3 \pi^{-3}. \]

From this with $r_1 \to 0$ we see that $K_0(f) \equiv \pi/\alpha$. On the other hand, if $f$ is the cylindrical map

\[ f(r, \psi, z) = (r, \pi \psi/\alpha, \pi z/\alpha) \]

we have equality so the bound is sharp.

The above result can be extended to locally cylindrical mappings as follows:

3.2. Theorem. For the class $F_L$ of mappings cylindrical at the origin,

\[ \inf \{K_0(f)|f \in F_L\} = \pi/\alpha. \]

Proof. Since the mapping (6) is cylindrical at the origin it suffices, by Theorem 3.1, to show that for each $f \in F_L$ there exists a $g \in F$ such that $K_0(f) \equiv K_0(g)$. As in the proof of Theorem 3.1, each $f \in F_L$ can be extended to a quasiconformal mapping $\hat{f}: R^8 \rightarrow R^8$. Next, let $\hat{g}_n: R^8 \rightarrow R^8$ be the sequence defined by $\hat{g}_n(x) = a_n f(x/n)$, where $a_n$ is chosen in such a way that $|\hat{g}_n(e_1)| = 1$. Since $\hat{g}_n(0) = 0$ we conclude by [6; 19.4, 20.5] that $\{\hat{g}_n\}$ is a normal family and there is a subsequence $\hat{g}_j$, $j \in J \subset N$, converging to a limit function $\hat{g}: R^8 \rightarrow R^8$. The convergence is uniform on compact subsets of $R^8$ and $\hat{g}$ is a homeomorphism since $\hat{g}(0) = 0$, $|\hat{g}(e_1)| = 1$; see [6; 21.4]. By [6; 37.2] $\hat{g}$ is quasiconformal, $\hat{g}(D_\pi) = D_\pi$, and $K_0(f) \equiv K_0(g)$,
where \( g = \hat{g}|\overline{D}_x \). It follows that for each \( r_0 > 0 \) \( g \) maps the compact sets \((r, \psi, z) | r = r_0, |z| \leq M \) into some cylindrical surface. Also, \( g(0) = 0 \) and, by a simple topological argument, \( g \in F \).

3.3. Remark. So far we have only considered the problem of determining \( K_0(D_\alpha, D_\beta), 0 < \alpha \leq \beta < \pi \), for \( \beta = \pi \). Composing \( f: D_\alpha \rightarrow D_\beta \) with the standard folding (6), \( \alpha \) replaced by \( \beta \), we deduce that \( K_0(D_\alpha, D_\beta) = \beta / \alpha \).

4. Application

As an application we construct in Theorem 4.3 a cylindrical map \( h \) from Taari’s functions and show that Taari’s result follows from our result.

4.1. Taari’s conditions. Taari [5] considers the subclass of homeomorphisms \( f: \overline{D} \rightarrow \overline{D}_x \), \( 0 < \alpha \leq \pi, f(0) = 0 \), whose restrictions \( f|D_\alpha \) are quasiconformal mappings onto \( D_\alpha \) and which satisfy the following conditions at the origin:

Condition A. There is a polar angle \( \varphi_0 \), \( 0 < \varphi_0 < \pi / 2 \) such that the limit
\[ \lim_{r \to 0^+} f(te)/t = k(e) \neq 0, \infty \]
exists for every \( e \in \overline{D}_x \) with \( 0 \leq e, e_3 \leq \varphi_0 \), where \( (e, e_3) \) denotes the acute angle between the vectors \( e \) and \( e_3 \).

Condition B. For \( e \in \overline{D}_x \)
\[ \lim_{t \to 0^+} \max_{e_3} (k(e), k(e_3))/\min_{e_3} (k(e), k(e_3)) = 1, \]
where the maximum and minimum are over all vectors \( e \) such that \( (e, e_3) = \varepsilon \).

4.2. Theorem. Let \( f \) be a mapping satisfying Taari’s conditions stated in 4.1. Then there is a cylindrical map \( h \) such that \( K_0(f) \equiv K_0(h) \).

Proof. Given \( f \) we construct, stepwise, a map \( \hat{h}: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) whose restriction \( h = \hat{h}|\overline{D}_x \) will be cylindrical and satisfy \( K_0(f) \equiv K_0(h) \).

The first step is carried out in [5]. Using Condition A Taari constructs a mapping \( g: \overline{D}_x \rightarrow \overline{D}_x, g(0) = 0 \), such that for each \( e \in \overline{D}_x \), \( 0 \leq e, e_3 \leq \varphi_0 \), the restriction of \( g \) to the ray \( \{te | t \geq 0\} \) is linear, i.e. \( g(te) = tg(e) \). Furthermore, using additional rotation and stretching we can assume \( g(e_3) = e_3 \). Also, \( K_0(g) \equiv K_0(f) \).

Next the mapping \( g \) is extended to a quasiconformal map \( \hat{g}: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) as in the proof of Theorem 3.1. We observe, for later use, that if \( 0 \leq (e, e_3) \leq \varphi_0 \), the restriction of \( \hat{g} \) to the ray \( \{te | t \geq 0\} \) is linear. Now, let \( T_n \) denote translation by \( ne_3 \) and let \( \hat{h}_n = T_{-n} \circ \hat{g} \circ T_n \). For \( t \equiv -n \), \( \hat{h}_n(te_3) = T_{-n}(\hat{g}(t+n)e_3)) = te_3 \); hence \( \{\hat{h}_n\} \) is a normal family and has a subsequence \( \hat{h}_j, j \in J \subset \mathbb{N} \), converging to a quasiconformal mapping \( \hat{h}: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). The convergence is uniform on compact subsets of \( \mathbb{R}^3 \), \( \hat{h}(D_x) = D_x \) and \( K_0(h) \equiv K_0(g) \), where \( h = \hat{h}|\overline{D}_x \). Hence \( K_0(h) \equiv K_0(f) \).
To complete the proof we will show that if $C$ is a circular cylinder (domain) with axis the $x_3$-axis, then $\tilde{h}(C)$ is a similar cylinder. For such a cylinder $C$ let $C_j$ be the cone determined by the circle $\partial C \cap \{x_3=0\}$ and the point $(-j)e_3$, and put $C'_j = \tilde{h}(C_j)$. By Theorem 3 of Gehring [1] we find that $\tilde{h}(C) = \text{Ker } C'_j$ since $C = \text{Ker } C_j$. Next we show that $\text{Ker } C'_j$ is a cylinder, i.e. that if $x_0 \in \text{Ker } C'_j$, then $\{x_0 + te_3 \mid t \in \mathbb{R}\} \subseteq \text{Ker } C'_j$. Let $(r_0, \psi_0, z_0)$ be the cylindrical coordinates of $x_0$ and let $x_0 \in \text{Ker } C'_j$. Then there exists an $\varepsilon > 0$, $\varepsilon < r_0$, such that the cylindrical neighborhood $N$ with $|r-r_0|<\varepsilon$, $|\psi-\psi_0|<\varepsilon$, $|z-z_0|<\varepsilon$ is in $C'_j$ for $j \geq j_0$. Next consider $C'_j$. For large $j$, $\tilde{g}$ maps $C_j + je_3$ in such a way that each ray through the origin is mapped onto a ray through the origin. This means that $C'_j$ is a union of rays through $-je_3$; in particular, all rays through $-je_3$ meeting $N$ are in $C'_j$. In the case $t > 0$ take $j$ so large that also $z_0 + j > 0$ and

$$
\text{arc tan } \frac{r_0-\varepsilon}{z_0+j+\varepsilon} \leq \text{arc tan } \frac{\varepsilon/2}{t-\varepsilon/2}.
$$

Then the cylindrical neighborhood of $x_0 + te_3$ with $|r-r_0|<\varepsilon/2$, $|\psi-\psi_0|<\varepsilon/2$, $|z-(z_0+t)|<\varepsilon/2$ is in $C'_j$. Hence $\{x_0 + te_3 \mid t \geq 0\} \subseteq \text{Ker } C'_j$. Similarly, $\{x_0 + te_3 \mid t < 0\} \subseteq \text{Ker } C'_j$. Finally we use Condition B to show that $\tilde{h}(C)$ is circular with axis the $x_3$-axis. Since $k(e) = g(e)$, we see by Condition B that for all $j$

$$
\lim_{\delta \to 0^+} \max_{x \in \delta} (\tilde{g}(T_j x), e_3) \min_{x \in \delta} (\tilde{g}(T_j x), e_3) = 1,
$$

where maximum and minimum are taken over all $x$ such that $(T_j x, e_3) = \delta$. Let $r > 0$ and put $S = \{(r, \psi, z) \mid r = r_0, z = z_0\}$. For $j \geq j_1$ we have by the above

$$
(7) \quad \max_{x \in S} (\tilde{g}(T_j x), e_3) \min_{x \in S} (\tilde{g}(T_j x), e_3) < 1 + \varepsilon.
$$

The continuity of $\tilde{g}$ at $e_3$, the linearity of $\tilde{g}$ on rays close to the $x_3$-axis, and the fact that $\tilde{g}(e_3) = e_3$, together imply that

$$
(8) \quad 1 - \varepsilon < |\tilde{g}(x + je_3)|/|x + je_3| < 1 + \varepsilon
$$

for $j \geq j_2$, $x \in S$. Suppose that $j \geq j_1, j_2$ and let $\varphi_j = \min_{x \in S} (\tilde{g}(T_j x), e_3)$, $\bar{\varphi}_j = \max_{x \in S} (\tilde{g}(T_j x), e_3)$. By (7)

$$
(9) \quad \bar{\varphi}_j < (1 + \varepsilon) \varphi_j.
$$

From (8) we get for $x \in S$

$$
(10) \quad (1 - \varepsilon) \sqrt{r_0^2 + (z_0 + j)^2} < |\tilde{g}(x + je_3)| < (1 + \varepsilon) \sqrt{r_0^2 + (z_0 + j)^2}.
$$

Then by (9) and (10)

$$
(11) \quad \frac{\max_{x \in S} r(\tilde{g}(x + je_3))}{\min_{x \in S} r(\tilde{g}(x + je_3))} \leq \frac{(1 + \varepsilon) \sin (1 + \varepsilon) \varphi_j}{(1 - \varepsilon) \sin \varphi_j}.
$$
Since \( h_j(x) = \hat{g}(x + je_3) - je_3 \), we can replace \( r(\hat{g}(x + je_3)) \) by \( r(\hat{h}_j(x)) \) in (11). Since \( S \) is compact, \( \hat{h}_j \to \hat{h} \) uniformly on \( S \). Also, \( \varrho_j \to 0 \) as \( j \to \infty \), and the upper bound in (11) can be replaced by \((1+\varepsilon)^3/(1-\varepsilon)\). Since \( \varepsilon \to 0 \) was arbitrary,

\[
\max r(\hat{h}(x))/\min r(\hat{h}(x)) = 1
\]

for \( x \in S \).

References


University of Michigan
Department of Mathematics
Ann Arbor, Michigan 48109
USA

University of Trondheim
Department of Mathematics, NTH
N-7034 Trondheim
Norway

University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

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