NON-HOMOGENEOUS COMBINATIONS OF COEFFICIENTS OF UNIVALENT FUNCTIONS

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Introduction

G. Schober communicated us in 1978 the following problem: Determine \( \max_{f \in S} \Re (a_3 + i a_2) \) (see also [4], p. 84). In this paper we consider the general problem of finding \( \max \Re (a_3 + \lambda a_2) \) for an arbitrary complex parameter \( \lambda \) and for functions \( f \in S(b) \). Löwner’s parametric method shall be extensively used in the following considerations.

The case of \( S_R(b) \)

Let \( b \in (0, 1) \) and let \( A = \{z \mid |z| < 1\} \). The class \( S_R(b) \) consists of the univalent functions \( f: A \to A \) for which \( f(z) = b(z + a_2z^2 + a_3z^3 + \ldots) \) with \( a_k \in \mathbb{R} \). The problem reduces to the study of \( a_3 + \lambda a_2 \) for \( \lambda \in \mathbb{R} \). In [2] pp. 8, 9, 10 we have derived the following sharp estimates for functions \( f \in S_R(b) \):

\[
a_3 \equiv a_2^2 - (1 - b^2),
\]

\[
a_3 \equiv 1 - b^2 + a_2^2 \left(1 + \frac{1}{\log b}\right) \quad \text{if} \quad |a_2| \equiv -2b \log b,
\]

\[
a_3 \equiv a_2^2 + 1 - b^2 - 2(\sigma^2 - b^2) + 4\sigma^2 \log \sigma \quad \text{if} \quad |a_2| > -2b \log b.
\]

The parameter \( \sigma \) is determined by \( \sigma - \sigma \log \sigma = b + |a_2|/2 \).

Taking into account that \( |a_2| \leq 2(1 - b) \) we immediately obtain

\[
\min (a_3 + \lambda a_2) = \begin{cases} 
-(1 - b^2) - \frac{1}{4} \lambda^2 & \text{if} \quad |\lambda| \leq 4(1 - b),
\end{cases}
\]

\[
3 - 8b + 5b^2 - 2(1 - b)|\lambda| & \text{if} \quad |\lambda| > 4(1 - b).
\]

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The upper bound is more complicated. We have to distinguish between the following two cases:

1) \(1 + \frac{1}{\log b} \leq 0\), 2) \(1 + \frac{1}{\log b} > 0\).

In both these cases we have to deal separately with the possibilities \(|a_2| \leq -2b \log b\) and \(|a_2| \geq -2b \log b\). After an elementary but rather long calculation we arrive at the following result.

Case 1: \(e^{-1} \leq b < 1\).

\[
\max(a_3 + \lambda a_2) = \begin{cases} 
1 - b^2 - \frac{\log b}{4 + \log b} & \text{if } |\lambda| \leq 4b(1 + \log b), \\
1 - b^2 + |\lambda| \left(\frac{\sigma - \frac{1}{4} |\lambda|}{4} \right) + 2(\sigma - b)^2 & \text{if } 4b(1 + \log b) < |\lambda| < 4b, \\
3 - 8b + 5b^2 + 2(1 - b)|\lambda| & \text{if } |\lambda| \geq 4b.
\end{cases}
\]

The number \(\sigma \in [b, 1]\) is determined by \(\sigma \log \sigma + b = |\lambda|/4\).

Case 2: \(0 < b < e^{-1}\).

\[
\max(a_3 + \lambda a_2) = \begin{cases} 
1 - b^2 + |\lambda| \left(\frac{\sigma - \frac{1}{4} |\lambda|}{4} \right) + 2(\sigma - b)^2 & \text{if } |\lambda| < 4b, \\
3 - 8b + 5b^2 + 2(1 - b)|\lambda| & \text{if } |\lambda| \geq 4b.
\end{cases}
\]

The number \(\sigma \in [b, 1]\) is determined by \(\sigma \log \sigma + b = |\lambda|/4\).

The general case \(S(b)\)

As usual, for \(b \in (0, 1)\), \(S(b)\) consists of the univalent functions \(f: A \to A\) for which \(f(z) = b \{z + a_2 z^2 + a_3 z^3 + \ldots\}\). Instead of \(S\) we shall sometimes write \(S(0)\). We shall consider the dense subclass of slit-functions. For these the following Löwner expressions hold:

\[
a_2 = -2 \int_b^1 \chi(u) \, du, \quad a_3 = a_2^2 - 2 \int_b^1 u \chi^2(u) \, du,
\]

where \(\chi(u) = e^{i\vartheta(u)}\) is a continuous function. For a piecewise continuous \(\vartheta\) the formulae (1) still define coefficients of functions \(f \in S(b), b \in [0, 1]\).

For a given number \(\lambda = \mu + iv\) we have

(2a) \(\text{Re}(a_3 + \lambda a_2) = 4 \left(\int_b^1 \cos \vartheta(u) \, du\right)^2 - 4 \left(\int_b^1 \sin \vartheta(u) \, du\right)^2 - 2 \left(\int_b^1 u \cos 2\vartheta(u) \, du\right) - 2 \mu \int_b^1 \cos \vartheta(u) \, du + 2v \int_b^1 \sin \vartheta(u) \, du\).
Consider first the case \( v=0 \), i.e. \( \lambda \in \mathbb{R} \). In this case we have

\[
\text{Re} (a_3 + \mu a_2) \equiv 4 \left( \int_b^1 \cos \theta(u) \, du \right)^2 - 2 \int_b^1 u \cos \theta u \, du - 2 \int_b^1 \cos \theta(u) \, du.
\]

For the Löwner functions \( f \in S_R(b) \) we have, according to [5] p. 10,

\[
a_2 = -2 \int_b^1 \cos \theta(u) \, du, \quad a_3 = a_2^2 - 2 \int_b^1 u \cos \theta(u) \, du.
\]

Thus in this case the maximum is attained in the subclass \( S_R(b) \), for which the solution was determined above.

From now on we assume that

\( v \neq 0 \).

By considering \( f(z) \) instead of \( f(z) \) we see that \( v \int_b^1 \sin \theta(u) \, du \equiv 0 \) in the maximum case. Similarly, by considering \( -f(-z) \) instead of \( f(z) \) we find \( \mu \int_b^1 \cos \theta(u) \, du \equiv 0 \) in the maximum case. For brevity, let us normalize

\[
v \geq 0, \quad \mu \equiv 0; \quad \int_b^1 \sin \theta(u) \, du \equiv 0, \quad \int_b^1 \cos \theta(u) \, du \equiv 0.
\]

Rewriting (2a) we obtain

(2) \[
\text{Re} (a_3 + \lambda a_2) = 1 - b^2 + 4 \left( \int_b^1 \cos \theta(u) \, du \right)^2 - 4 \left( \int_b^1 \sin \theta(u) \, du \right)^2
\]

\[
- 4 \int_b^1 u \cos^2 \theta(u) \, du - 2 \mu \int_b^1 \cos \theta(u) \, du + 2v \int_b^1 \sin \theta(u) \, du.
\]

Let us replace the maximizing \( \theta \) by \( \tilde{\theta} \) which is obtained from \( \theta \) by changing \( \theta \) into \( \pi - \theta \) on an arbitrary subinterval \( l \) of \( [b, 1] \). The functional (2) is then altered in such a way that

\[
\text{Re} (a_3 + \lambda a_2) - \text{Re} (\tilde{a}_3 + \lambda \tilde{a}_2)
\]

\[
= 16 \left( \int_b^1 \cos \theta(u) \, du - \frac{\mu}{4} \right) \int_b^1 \cos \theta(u) \, du - 16 \int_b^1 \cos \theta(u) \, du
\]

We deduce from this that if \( \cos \theta(u) \neq 0 \) at some point, then \( \cos \theta(u) \geq 0 \), i.e.

\( \cos \theta(u) \equiv 0 \)

in the maximum case.

Similarly, we can deduce from (2) that in the maximum case \( \cos \theta \) is decreasing (and hence \( |\sin \theta| \) is increasing), since the only part depending on the arrangement of the values of \( \cos \theta \) is \( - \int_b^1 u \cos^2 \theta(u) du \).

The part in (2) which depends explicitly on \( \sin \theta \) is

\[
- 4 \left( \int_b^1 \sin \theta(u) \, du \right)^2 + 2v \int_b^1 \sin \theta(u) \, du.
\]
If $v \geq 4(1-b)$, it follows from (2) that in the maximum case
\[ \sin \theta(u) \equiv 0. \]
For, if $\sin \theta$ assumed negative values, we could change its sign without affecting $\cos \theta$ and thus increase $\int_b^1 \sin \theta(u)du$, and by doing so, we would increase the above mentioned part determined by $\sin \theta$. Therefore, if in the maximum case $\sin \theta$ assumes negative values, we must have $v < 4(1-b)$ and $\int_b^1 \sin \theta(u)du = v/4$.

A necessary condition for the function $\theta$ to be extremal is that the first order variation of (2) is zero. This leads to the condition
\[
\left( \int_b^1 \cos \theta(u) \, du - \frac{1}{4} \right) \sin \theta(u) + \left( \int_b^1 \sin \theta(u) \, du - \frac{1}{4} \right) \cos \theta(u) = u \sin \theta(u) \cos \theta(u).
\]
If $\theta$ has to give rise to the maximum, then the second order variation has to be non-positive. This leads to the following condition: For all piecewise continuous functions $\phi$ we have
\[
(3a) \quad \left( \int_b^1 \phi(u) \sin \theta(u) \, du \right)^2 - \left( \int_b^1 \phi(u) \cos \theta(u) \, du \right)^2 + \left( \int_b^1 \sin \theta(u) \, du - \frac{1}{4} \right) \int_b^1 \phi^2(u) \sin \theta(u) \, du - \left( \int_b^1 \cos \theta(u) \, du - \frac{1}{4} \right) \int_b^1 \phi^2(u) \cos \theta(u) \, du + \int_b^1 u\phi^2(u) \cos^2 \theta(u) \, du - \int_b^1 u\phi^2(u) \sin^2 \theta(u) \, du \equiv 0.
\]

**The perfect square representation**

Let $C$ be an arbitrary parameter. The identity
\[ a_3 - a_2^2 - Ca_2 + \frac{1}{2} C^2 \log b = -2 \int_b^1 A^2(u) \, du; \]
\[ A(u) = \sqrt{u \left( \kappa(u) - \frac{C}{2u} \right)} \]
follows from the formulae (1). Hence
\[ \text{Re} \left( a_3 - a_2^2 - Ca_2 + \frac{1}{2} C^2 \log b \right) = -2 \int_b^1 \text{Re} A^2(u) \, du = 2 \int_b^1 |A(u)|^2 \, du - 4 \int_b^1 (\text{Re} A(u))^2 \, du = 1 - b^2 - \frac{1}{2} |C|^2 \log b + \text{Re} (Ca_2) - 4 \int_b^1 (\text{Re} A(u))^2 \, du. \]
Let us make use of the choice

\[ C = - \left( a_2 + \frac{1}{2} \lambda \right) \]

which gives

\[ \text{Re} \left( a_3 + \lambda a_2 \right) = 1 - b^2 + \frac{1}{4} v^2 - \frac{1}{4} \mu^2 \log b - \left( \text{Im} a_2 + \frac{1}{2} \right)^2 \]

\[-(1 + \log b) \left( \text{Re} a_2 \right)^2 - \mu \text{Re} a_2 \log b - 4 \int_b^1 \left( \text{Re} A(u) \right)^2 du.\]

If \( b \neq e^{-1} \), this can be written as

\[ \text{Re} \left( a_3 + \lambda a_2 \right) = 1 - b^2 + \frac{1}{4} v^2 - \frac{1}{4} \mu^2 \frac{\log b}{1 + \log b} - \left( \text{Im} a_2 + \frac{1}{2} v \right)^2 \]

\[-(1 + \log b) \left( \text{Re} a_2 + \frac{\mu \log b}{2(1 + \log b)} \right)^2 - 4 \int_b^1 \left( \text{Re} A(u) \right)^2 du.\]

We can also rewrite (4) in the form

\[ \text{Re} \left( a_3 + \lambda a_2 \right) = 1 - b^2 + \frac{1}{4} v^2 - \left( \text{Re} a_2 + \frac{1}{2} \mu \right)^2 \log b \]

\[-\left( \text{Im} a_2 + \frac{1}{2} v \right)^2 - 4 \int_b^1 \left( \text{Re} A(u) \right)^2 du.\]

The representation (4) is closely related to those used by Haario and Jokinen in [1].

**Extremals of type 2; 2**

Suppose that \( e^{-1} < b < 1 \) (hence \( 1 + \log b > 0 \)) and obtain from (4a)

\[ \text{Re} \left( a_3 + \lambda a_2 \right) \approx 1 - b^2 + \frac{1}{4} v^2 - \frac{1}{4} \mu^2 \frac{\log b}{1 + \log b}.\]

Equality is possible if and only if

i) \( \text{Im} a_2 + \frac{1}{2} v = 0, \)

ii) \( \text{Re} a_2 + \frac{\mu \log b}{2(1 + \log b)} = 0, \)

iii) \( \text{Re} A(u) = 0 \) i.e. \( \cos \theta(u) = -\frac{\text{Re} a_2 + \mu/2}{2u}. \)

We shall show that (5) is sharp for some numbers \( \lambda = \mu + iv. \)
Let us choose \(|\mu| \leq 4b(1 + \log b)\) and let
\[
\sigma = \frac{|\mu|}{4(1 + \log b)};
\]
therefore \(0 \leq \sigma \leq b\). Define \(\mathcal{G}\) in such a way that
\[
\cos \mathcal{G}(u) = \frac{\sigma}{u},
\]
\[
\sin \mathcal{G}(u) = \begin{cases} 
\sqrt{1 - \frac{\sigma^2}{u^2}} & \text{for } b \leq u \leq c, \\
-\sqrt{1 - \frac{\sigma^2}{u^2}} & \text{for } c < u \leq 1.
\end{cases}
\]

The point \(c\) will be chosen later. For this \(\mathcal{G}\) we have
\[
\text{Re} \, a_2 = -2 \int_b^1 \cos \mathcal{G}(u) \, du = \frac{|\mu| \log b}{2(1 + \log b)} = -\frac{\mu \log b}{2(1 + \log b)};
\]
thus ii) is satisfied. Now it follows that
\[
\text{Re} \, a_2 + \frac{1}{2} \mu = \frac{\mu}{2(1 + \log b)} = -2\sigma,
\]
which means that iii) holds. In order to show i) we choose \(c\) such that \(\text{Im} \, a_2 = -2 \int_b^1 \sin \mathcal{G}(u) \, du = -\nu/2\). This is possible so far as
\[
\frac{1}{2} |\nu| \leq 2 \int_b^1 |\sin \mathcal{G}(u)| \, du,
\]
i.e. \(|\nu| \leq 4\left(\sqrt{1 - \sigma^2} - \sqrt{b^2 - \sigma^2} + \sigma \pi \text{arc} \cos \frac{\sigma}{b} - \sigma \pi \text{arc} \cos \sigma\right)\).

The equality case for \(\mu > 0\) can be handled similarly. Collecting the results we arrive at

**Theorem 1.** Let \(e^{-1} < b < 1\), \(\lambda = \mu + iv\), \(\sigma = |\mu|/4(1 + \log b)\). If
\[
\begin{cases} 
|\mu| \leq 4b(1 + \log b), \\
|\nu| \leq 4\left(\sqrt{1 - \sigma^2} - \sqrt{b^2 - \sigma^2} + \sigma \pi \text{arc} \cos \frac{\sigma}{b} - \sigma \pi \text{arc} \cos \sigma\right),
\end{cases}
\]
then
\[
\max_{f \in S(b)} \text{Re} \left( a_3 + \lambda a_2 \right) = 1 - b^2 + \frac{1}{4} v^2 - \frac{1}{4} \mu^2 \log b \frac{1}{1 + \log b}.
\]

The maximum is reached for a function mapping \(\Delta\) onto \(\Delta\) minus two slits.
Note. If \( b = e^{-1} \), then similar arguments show that to each \( v \) with \( |v| < 4(1-e^{-1}) \) there belongs a one-parametric family of extremal functions parametrized by \( \Re a_2 \in [-2e^{-1}, 2e^{-1}] \).

**Extremals of type 1:2**

Now we take (4b) as a starting point. Let \( |\mu| \leq 4b, \ t = -(\Re a_2 + \mu/2)/2 \). From \( |a_2| \leq 2(1-b) \) it follows that \( 0 \leq t \leq 1 \). In this notation we have

\[
\Re A(u) = \sqrt{u} \left( \cos \vartheta(u) - \frac{t}{u} \right).
\]

For all functions \( \vartheta \) the following holds. If \( t \leq b \), we have the trivial estimate \( |\cos \vartheta(u) - t/u| \geq 0 \) for \( b \leq u \leq 1 \). If \( t > b \), we can say more:

\[
\left| \cos \vartheta(u) - \frac{t}{u} \right| \geq \begin{cases} 
1 - \frac{t}{u} & \text{for } b \leq u \leq t, \\
0 & \text{for } t \leq u \leq 1.
\end{cases}
\]

Therefore, we have

\[
-(\Re A(u))^2 \geq \begin{cases} 
-u \left(1 - \frac{t}{u}\right)^2 & \text{for } b \leq u \leq t, \\
0 & \text{for } t \leq u \leq 1,
\end{cases}
\]

and thus

\[
-4 \int_b^1 (\Re A(u))^2 \, du \equiv 6t^2 - 4t^2 \log t + 2b^2 - 8tb + 4t^2 \log b,
\]

with the equality if and only if

\[
\cos \vartheta(u) = \begin{cases} 
1 & \text{for } b \leq u \leq t, \\
\frac{t}{u} & \text{for } t \leq u \leq 1.
\end{cases}
\]

From (4b) we obtain now

\[
\Re (a_2 + \lambda a_2) \equiv g(t),
\]

where

\[
g(t) = \begin{cases} 
1 - b^2 + \frac{1}{4} v^2 - \frac{1}{4} \mu^2 - 4t^2 - 2t\mu - 4t^2 \log b - \left(\Im a_2 + \frac{1}{2} v\right)^2 & \text{for } 0 \leq t \leq b, \\
1 - b^2 + \frac{1}{4} v^2 - \frac{1}{4} \mu^2 - 4t^2 - 2t\mu + 6t^2 - 4t^2 \log t + 2b^2 - 8tb - \left(\Im a_2 + \frac{1}{2} v\right)^2 & \text{for } b \leq t \leq 1.
\end{cases}
\]

This function \( g \) is differentiable on \([0, 1]\) and

\[
g'(t) = \begin{cases} 
-8t - 2\mu - 8t \log b & \text{for } 0 \leq t \leq b, \\
-2\mu - 8t \log t - 8b & \text{for } b \leq t \leq 1.
\end{cases}
\]

Consider first the case \( e^{-1} < b < 1 \) and take

\[
4b(1 + \log b) \leq |\mu| \leq 4b.
\]
Now \( g'(t) \equiv 0 \) on \([0, b]\) and hence \( g\) has its maximum on \([b, 1]\), where \( g'\) has one zero \( \sigma \). This \( \sigma \) is determined by the condition \(-4\sigma \log \sigma = 4b + \mu\). We obtain

\[
\max_{\sigma \equiv t \equiv 1} g(t) = g(\sigma) = 1 + b^2 + \frac{1}{4} \nu^2 - \frac{1}{4} \mu^2 + 2\sigma^2 - \sigma \mu - 4\sigma b
\]

and thus

\[
\text{Re} (a_3 + \lambda a_2) \equiv 1 + b^2 + \frac{1}{4} \nu^2 - \frac{1}{4} \mu^2 + 2\sigma^2 - \sigma \mu - 4\sigma b,
\]

where the equality occurs if and only if

\[
\begin{align*}
\text{i)} & \quad \cos \theta(u) = \begin{cases} 1 & \text{for } b \leq u \leq \sigma, \\
\frac{\sigma}{u} & \text{for } \sigma \leq u \leq 1, \end{cases} \\
\text{ii)} & \quad \sigma = -\frac{1}{2} \left( \text{Re} a_2 + \frac{1}{2} \mu \right) \\
\text{iii)} & \quad -4\sigma \log \sigma = 4b + \mu, \\
\text{iv)} & \quad \text{Im} a_2 + \frac{1}{2} \nu = 0.
\end{align*}
\]

In order to show that these conditions can be satisfied simultaneously we consider \( \mu \) with

\[ -4b \leq \mu \leq -4b(1 + \log b). \]

There is one \( \sigma \equiv e^{-1} \) with \(-4\sigma \log \sigma = 4b + \mu\). Define

\[
\cos \theta(u) = \begin{cases} 1 & \text{for } b \leq u \leq \sigma, \\
\frac{\sigma}{u} & \text{for } \sigma \leq u \leq 1. \end{cases}
\]

So far the conditions i), ii) and iii) are satisfied. In order to make iv) hold we have to require

\[
\frac{1}{4} |\nu| \leq \int_b^1 |\sin \theta(u)| \, du.
\]

The equality case \( \mu > 0 \) is treated similarly. The results collected give

**Theorem 2.** Let \( e^{-1} < b \leq 1, \lambda = \mu + iv \) and \( \sigma \in [e^{-1}, 1] \) be determined by

\[-4\sigma \log \sigma = 4b - |\mu|.
\]

If

\[
\begin{cases} 4b(1 + \log b) \equiv |\mu| \leq 4b, \\
|\nu| \leq 4(\sqrt{1 - \sigma^2} - \sigma \arccos \cos \sigma),
\end{cases}
\]

we have

\[
\max_{f \in S(b)} \text{Re} (a_3 + \lambda a_2) = 1 + b^2 + \frac{1}{4} \nu^2 - \frac{1}{4} \mu^2 + 2\sigma^2 + \sigma |\mu| - 4\sigma b.
\]

The maximum is reached for a function mapping \( \Delta \) onto \( \Delta \) minus a forked slit.
Next consider the case \( 0 < b \leq e^{-1} \). From (6) we see that \( g'(t) \equiv 0 \) on \([0, b]\). Thus, again, \( g \) has its maximum on \([b, 1]\). Arguments similar to those in the previous case lead to

**Theorem 3.** Let \( 0 < b \leq e^{-1} \), \( \lambda = \mu + iv \) and \( \sigma \in [e^{-1}, 1] \) is determined by

\[
-4\sigma \log \sigma = 4b - |\mu|.
\]

If

\[
\begin{align*}
|\mu| & \leq 4b, \\
|v| & \leq 4(1 - \sigma^2 - \sigma \arccos \sigma),
\end{align*}
\]

we have

\[
\max_{f \in S(b)} \Re (a_3 + \lambda a_2) = 1 + b^2 + \frac{1}{4} v^2 - \frac{1}{4} \mu^2 + 2\sigma^2 + \sigma |\mu| - 4\sigma b.
\]

The maximum is reached for a function mapping \( \Delta \) onto \( \Delta \) minus a forked slit.

**Extremals of type 1:1**

A particular case of extremals of type 1:1 is obtained if \( \mu = 0 \). From (3) we see that for such an extremal we have

\[
\sin \theta(u) \int_b^1 \cos \theta(u) \, du + \cos \theta(u) \int_b^1 \sin \theta(u) \, du - \mu \sin \theta(u) \cos \theta(u) = \frac{1}{4} v \int_b^1 \cos \theta(u) \, du.
\]

Integration over \([b, 1]\) gives

\[
2 \int_b^1 \sin \theta(u) \, du \int_b^1 \cos \theta(u) \, du - \int_b^1 u \cos \theta(u) \cos \theta(u) \, du = \frac{1}{4} v \int_b^1 \cos \theta(u) \, du.
\]

We consider only those cases where \( v \equiv 4(1 - b) \). We know that in the maximum case \( \sin \theta(u) \equiv 0 \) and thus

\[
v \int_b^1 \cos \theta(u) \, du \equiv 8 \int_b^1 \sin \theta(u) \, du \int_b^1 \cos \theta(u) \, du \equiv 8(1 - b) \int_b^1 \cos \theta(u) \, du.
\]

Therefore, if \( \int_b^1 \cos \theta(u) \, du \neq 0 \), we must have \( v \equiv 8(1 - b) \). It is clear that we have even \( v < 8(1 - b) \).

**Theorem 4.** For \( 0 < b < 1 \) assume that \( |v| \equiv 8(1 - b) \). Then

\[
\max_{f \in S(b)} \Re (a_3 + iv a_2) = -3 + 8b - 5b^2 + 2(1 - b)|v|.
\]

The maximum is reached for a function \( f \) for which

\[
b \left( f - \frac{1}{f} \right) = z - \frac{1}{z} \pm 2(1 - b)i.
\]

This function maps \( \Delta \) onto \( \Delta \) minus a rectilinear slit.
If \( \varepsilon^{-1} \leq b < 1 \), we can say more. From (4) we see that
\[
\text{Re} (a_3 + i v a_2) \leq 1 - b^2 + \frac{1}{4} v^2 - \left( \text{Im} a_2 + \frac{1}{2} v \right)^2.
\]

If \( v \geq 4(1 - b) \), it follows from \( |a_2| \leq 2(1 - b) \) that
\[
\left| \text{Im} a_2 + \frac{1}{2} v \right| \leq \frac{1}{2} v - 2(1 - b);
\]
\[
\text{Re} (a_3 + i v a_2) \leq -3 + 8b - 5b^2 + 2(1 - b)v.
\]
The equality sign holds if \( \sin \vartheta(u) \equiv 1 \). Negative values of \( v \) can be treated similarly.

**Theorem 5.** For \( \varepsilon^{-1} \equiv b < 1 \) assume that \( |v| \equiv 4(1 - b) \). Then
\[
\max_{f \in S(b)} \text{Re} (a_3 + i v a_2) = -3 + 8b - 5b^2 + 2(1 - b)|v|.
\]
The maximum is reached for a function \( f \) for which
\[
b \left( f - \frac{1}{f} \right) = z - \frac{1}{z} \pm 2(1 - b)i.
\]
This function maps \( \Delta \) onto \( \Delta \) minus a rectilinear slit.

The general cases 1:1 remain to be discussed. Let
\[
p = \int_b^1 \cos \vartheta(u) \, du - \frac{1}{4} \mu \geq 0, \quad q = \int_b^1 \sin \vartheta(u) \, du - \frac{1}{4} v.
\]
The variational formula (3) thus assumes the form
\[
(7) \quad p \sin \vartheta(u) + q \cos \vartheta(u) = u \sin \vartheta(u) \cos \vartheta(u).
\]
We have to consider four alternatives with respect to \( \sin \vartheta(u) \) and \( \cos \vartheta(u) \).

1° There exists a value \( u \) for which \( \sin \vartheta(u) = 0 \).

From (7) it follows that \( q = 0 \) and hence \( v \equiv 4(1 - b) \). We can say even more. Because \( |\sin \vartheta| \) is increasing, there exists a number \( c \in [b, 1] \) such that \( \sin \vartheta(u) = 0 \) on \( [b, c) \), \( \sin \vartheta(u) \neq 0 \) on \( (c, 1] \). Therefore we see from (7) that
\[
\cos \vartheta(u) = \begin{cases} 
1 & \text{on } [b, c), \\
\frac{p}{u} & \text{on } (c, 1]; \ c \equiv p.
\end{cases}
\]
From \( q = 0 \) it follows further that
\[
\frac{1}{4} v \equiv \int_b^1 |\sin \vartheta(u)| \, du = \int_c^1 \sqrt{1 - p^2/u^2} \, du \leq \int_p^1 \sqrt{1 - p^2/u^2} \, du = \sqrt{1 - p^2} = \arccos p;
\]
thus
\[
v \leq 4(\sqrt{1 - p^2} - p \arccos p).
\]
By using (2) we decide that for a prescribed $p$, $\text{Re} \ (a_3 + \lambda a_2)$ is maximal if
\[
\int_b^1 u \cos^2 \theta(u) \, du = \frac{1}{2} (c^2 - b^2) - p^2 \log c
\]
is minimal, i.e. if $c = p$. Thus the maximizing choice of $c$ and $\cos \theta(u)$ is
\[
\cos \theta(u) = \begin{cases} 
1 & \text{on } [b, p], \\
p & \text{on } [p, 1],
\end{cases}
\]
which gives
\[
p = \int_b^1 \cos \theta(u) \, du - \frac{1}{4} \mu = p - b - p \log p - \frac{1}{4} \mu,
\]
i.e.
\[-4p \log p = 4b + \mu.
\]
If $e^{-1} < b < 1$, the previous condition implies, because $p \in [b, 1]$, that
\[4b(1 + \log b) \leq |\mu| \leq 4b.
\]
Similarly, if $0 < b \leq e^{-1}$, we obtain
\[|\mu| \leq 4b.
\]
Therefore, in the case $1^\circ \ \text{Re} \ (a_3 + \lambda a_2)$ is maximized, according to Theorems 1 and 3, by extremal functions of the type $1:1$.
Next, consider the remaining cases where
$2^\circ \ \sin \theta(u) \neq 0$.
The following alternatives are to be checked.
1) $p = 0$.
\[
0 \equiv \frac{1}{4} \mu = \int_b^1 \cos \theta(u) \, du \geq 0.
\]
Thus
\[
\mu = 0, \quad \cos \theta(u) \equiv 0 \quad \text{and} \quad |\sin \theta(u)| \equiv 1.
\]
There are two possibilities available.
If $\nu \equiv 4(1 - b)$, we know that in the maximum case $\sin \theta(u) \equiv 0$, i.e. $\sin \theta(u) \equiv 1$, and therefore we are led to the cases of Theorem 5, where
\[
\text{Re} \ (a_3 + iv a_2) = -3 + 8b - 5b^2 + 2\nu(1 - b).
\]
If $\nu < 4(1 - b)$, we see from (2) that
\[
\text{Re} \ (a_3 + \lambda a_2) = 1 - b^2 + \frac{1}{4} \nu^2 - \left(\text{Im} \ a_2 + \frac{\nu}{2}\right)^2 \equiv 1 - b^2 + \frac{1}{4} \nu^2,
\]
where the equality, making $\text{Im} a_\theta = -v/2$, is reached for

$$
\sin \theta(u) = \begin{cases}
1 & \text{on } \left[ b, \frac{1+b}{2} + \frac{v}{8} \right], \\
-1 & \text{on } \left( \frac{1+b}{2} + \frac{v}{8}, 1 \right]. 
\end{cases}
$$

This maximum thus belongs to the cases of Theorem 1.

2) $p>0$, $q=0$.

From (7) we see that $\cos \theta(u) = p/u$, $p \equiv b$ and $|\sin \theta(u)| = \sqrt{1 - p^2/u^2}$. Further,

$$
\frac{1}{4} v \equiv \int_b^1 |\sin \theta(u)| du = \int_b^1 \sqrt{1 - p^2/u^2} du
$$

$$
= \sqrt{1 - p^2} - \sqrt{b^2 - p^2} + p \arccos \frac{p}{b} - p \arccos p.
$$

Because $p = \int_b^1 \cos \theta(u) du - \mu/4$, we have

$$
p + \frac{1}{4} \mu = -p \log b; \quad p = \frac{-\mu}{4(1 + \log p)}.
$$

From $p>0$ it follows now that $b \in (e^{-1}, 1)$, and therefore we are in the cases of Theorem 1.

3) $p>0$, $q>0$.

Now we have $\sqrt{1} \leq \int_b^1 \sin \theta(u) du \leq 1 - b$. Here one can repeat the conclusions on pp. 132—134, i.e. changing the signs of $\theta$ properly without affecting $\cos \theta$ we can always diminish $\int_b^1 \sin \theta(u) du$ into the value $\sqrt{1}/4$. This new $\theta$ increases $\text{Re}(a_\theta + ia_\theta)$ to its maximum. Because for the new $\theta$ $q = 0$, we see that 3) is not the maximum case.

3° sin $\theta(u)$ obtains negative values.

According to the remark on p. 134 we know that in the maximum case necessarily $\int_b^1 \sin \theta(u) du = v/4$, i.e. $q = 0$. From (7) we see that we can now go back to the function $\theta$ which was defined in 1° and thus we end up with the same conclusions as in 1°.

4° There exists a value $u$ for which $\cos \theta(u) = 0$.

In this case $p = 0$, but because $\cos \theta(u) \equiv 0$ and $\mu \equiv 0$, we must have $\cos \theta(u) \equiv 0$ and $\mu = 0$. We are led back to the beginning of 2°, where this case was handled under the assumption $p = 0$.

From 1°—4° we decide now that in the cases not handled yet there are

$$
p > 0, \quad q < 0, \quad \cos \theta > 0, \quad \sin \theta > 0.
$$

Rewrite (7) in the form

(7a) $$F(\theta, u) = \frac{p}{\cos \theta} + \frac{q}{\sin \theta} - u = 0.$$
Because \( F_s(\vartheta, u) > 0 \) we see that (7a) determines \( \vartheta \) as a differentiable function of \( u \) and

\[
du = \left( \frac{p \sin \vartheta}{\cos^2 \vartheta} - \frac{q \sin \vartheta}{\sin^2 \vartheta} \right) d\vartheta.
\]

It follows from the Löwner theory [3] that such a function \( \vartheta \) determines a solution of type 1:1.

If we denote

\[
\alpha = -\vartheta(1), \quad \omega = -\vartheta(b),
\]

we obtain from (3) the equations

\[
\begin{aligned}
\begin{cases}
 p \sin \alpha - q \cos \alpha = \sin \alpha \cos \alpha, \\
 p \sin \omega - q \cos \omega = b \sin \omega \cos \omega.
\end{cases}
\end{aligned}
\]

(8a)

Two more equations can be obtained from

\[
\begin{aligned}
\begin{cases}
 p = \frac{1}{b} \cos \vartheta(u) du - \frac{1}{4} \mu = \int_{-\omega}^{u} \cos \vartheta \left( \frac{p \sin \vartheta}{\cos^2 \vartheta} - \frac{q \cos \vartheta}{\sin^2 \vartheta} \right) d\vartheta - \frac{1}{4} \mu, \\
 q = \int_{b}^{1} \sin \vartheta(u) du - \frac{1}{4} v = -p (\tan \alpha - \tan \omega - \alpha + \omega) - q \log \frac{\sin \alpha}{\sin \omega} - \frac{v}{4}.
\end{cases}
\end{aligned}
\]

(8b)

From (2) we see that

\[
\max_{f \in S(b)} \Re (a_3 + \lambda a_2) = 1 - b^2 - \frac{1}{4} \mu^2 + \frac{1}{4} v^2 - p\mu + qv + 4pq (\tan \alpha - \tan \omega)
\]

\[
-2q^2 \left( \frac{1}{\sin^2 \alpha} - \frac{1}{\sin^2 \omega} \right).
\]

For the original problem \( \max_{f \in S} \Re (a_3 + i a_2) \) we have determined the following numerical solution:

\[
\begin{aligned}
\alpha &= -0.528\cdot513\cdot532, \\
\omega &= -0.066\cdot344\cdot080;
\end{aligned}
\]

\[
\max_{f \in S} \Re (a_3 + i a_2) = 3.190\cdot298\cdot109.
\]

Note 1. From (3a) it follows that \( \vartheta = \pi/2 \) gives at least a local maximum if and only if

\[
v = 4(1 - b) + 4 \left( \frac{\int_{b}^{1} \varphi(u) du}{\int_{b}^{1} \varphi^2(u) du} - \int_{b}^{1} u \varphi(u) du \right)
\]
for all piecewise continuous functions \( \varphi \) on \([b, 1]\). If \( b \in [e^{-1}, 1) \), we have from Schwarz's inequality
\[
\left( \int_b^1 \varphi(u) \, du \right)^2 \leq \int_b^1 \frac{du}{u} \int_b^1 u \varphi^2(u) \, du = -\log b \int_b^1 u \varphi^2(u) \, du \leq \int_b^1 u \varphi^2(u) \, du.
\]
The condition for \( v \) is thus in accordance with Theorem 5.

**Note 2.** By solving the system (8) with the aid of power series in the neighbourhood of \( x = \omega = \pi/2 \) we find that for \( |v| \leq 4(1-b) + 4(1-eb)/(e-1) \) the functions with one rectilinear slit give \( \max_{f \in S(b)} \Re(a_3 + i\alpha_a) \).

**Note 3.** The problem of determining \( \min_{f \in S(b)} \Re(a_3 + \lambda a_a) \) is easily reduced to the problem studied here. By considering \( -if(iz) \) instead of \( f(z) \) we see that
\[
\min_{f \in S(b)} \Re(a_3 + \lambda a_a) = -\max_{f \in S(b)} \Re(a_3 + i\lambda a_a).
\]

**Note 4.** By using the same arguments as before one can also determine the part of the coefficient body \((a_2, a_3)\) of \( S(b) \) where the boundary functions are of the type 1:1.

**References**


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