LIPSCHITZ AND QUASICONFOMAL TUBULAR NEIGHBOURHOODS OF SPHERES IN CODIMENSION TWO

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In this paper it is shown that if \( X \) is a codimension 2 sphere in \( S^n, n \neq 4, 5, 6 \), then \( X \) has either a Lipschitz or a quasiconformal tubular neighbourhood if \( X \) is either locally Lipschitz flat or locally quasiconformally flat.

The notation of this paper is the same as that established in [GV]. In particular \( C \) denotes either of the categories LIP or QC. Theorem 3.3 of [GV] tells us that if \( X \) is a locally \( C \)-flat codimension 2 sphere in \( S^n, n \neq 4, 6 \), and if \( X \) is homotopically unknotted in \( S^n \), then \((S^n, X)\) is \( C \)-homeomorphic to \((S^n, S^{n-2})\). In this paper we consider the case where \( X \) might be knotted, obtaining the following result.

**Theorem 1.** Let \( X \subseteq S^n \) be a locally \( C \)-flat \( \text{TOP} (n-2) \)-sphere in \( S^n \). If \( n \neq 4, 5 \) or 6 then there is a neighbourhood \( N \) of \( S^{n-2} \) in \( S^n \) and a \( C \)-embedding \((N, S^{n-2}) \to (S^n, X)\).

Analogously with Theorem 3.4 of [GV], we have the following result.

**Theorem 2.** Let \( g: S^{n-2} \to S^n \) be a locally \( C \)-flat embedding. If \( n \neq 4 \) or 5 then \( g \) extends to a \( C \)-embedding of a neighbourhood of \( S^{n-2} \) in \( S^n \).

**Proof of Theorem 1.** Encasing as in the proof of Theorem 3.3 of [GV], since \( n \neq 6 \Rightarrow n-2 \neq 4 \) we may assume that only two \( C \)-encasings are necessary to exhibit the local \( C \)-flatness of \( X \).

Now transfer everything to \( \mathbb{R}^n \). Using the \( C \)-Schoenflies theorem we may extend one of the encasings to a \( C \)-homeomorphism of \( \mathbb{R}^n \). If we replace \( X \) by its inverse image under this homeomorphism, we see that it may be assumed that one of the two \( C \)-encasings is the inclusion. By reflection, we may assume that we have the following situation: \( X \cap [\mathbb{R}^n \setminus B^n(a)] = \mathbb{R}^{n-2} \setminus B^{n-2}(a) \) for some \( a < 1 \), and there is a \( C \)-embedding \( h: B^n \to \mathbb{R}^n \) with \( h^{-1}X = B^{n-2} \) and \( X \cap B^n \subset hB^{n-2} \). Thus the knotted part of \( X \) is trapped inside \( B^n \) where it is encased by a single \( C \)-encasing. Assume that the norm on \( \mathbb{R}^n \) is \( |(x_i)| = \max \{|x_i|\} \) rather than the pythagorean norm so that \( B^n \) is a cubic ball rather than a round ball, thus allowing PL methods.

Choose \( \alpha: (V, \mathbb{R}^{n-2}) \to (\mathbb{R}^n, X) \), a topological embedding where \( V \) is a neighbourhood of \( \mathbb{R}^{n-2} \) in \( \mathbb{R}^n \). The existence of \( \alpha \) follows from the topological local flatness of \( X \) by [KS₂], \( X \) admits a normal disc bundle in \( \mathbb{R}^n \) since \( n \neq 4 \); as noted in [K],

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the 2-disc bundles are classified by \( H^2(X; \mathbb{Z}) \) which, when \( n \neq 4 \), is the trivial group. Thus \( X \) has a trivial normal disc bundle in \( \mathbb{R}^n \) thereby providing us with the embedding \( \alpha \). Since \( \alpha \V \) is a neighbourhood of \( \overline{\mathbb{R}^n} \setminus \mathbb{B}^{n-2} = X \setminus \mathbb{R}^n \setminus B^n \), we may assume that \( \overline{\mathbb{R}^n} \setminus \mathbb{B}^n \subset \alpha \V \) so, using the relative TOP-Schoenflies theorem, \([B] \) and \([GV] \), we may extend \( \alpha |\overline{\mathbb{R}^n} \setminus B^n(b) \) for some \( b \in (a, 1) \) to a homeomorphism \( \beta \) of \( \mathbb{R}^n \) so that \( \beta \overline{\mathbb{R}^n} \setminus \mathbb{B}^{n-2} = \overline{\mathbb{R}^n} \setminus \mathbb{B}^{n-2} \). Choose \( r > 0 \) sufficiently small so that \( \overline{\mathbb{B}^{n-2} \times B^2(r)} \subset h \mathbb{B}^n \). Let \( \gamma = \alpha \beta |\overline{\mathbb{R}^n} \setminus \mathbb{B}^{n-2} \times B^2(r) \). Then the embedding \( \gamma \) satisfies the following properties: \( \gamma \subset \mathbb{B}^n \cap h \mathbb{B}^n \); \( \gamma \) is the identity on a neighbourhood of \( S^{n-3} \times B^2(r) \); \( \gamma |\overline{\mathbb{R}^n} \setminus \mathbb{B}^{n-2} \times 0 \cap X \cap \mathbb{B}^n \). 

Suppose we can construct a \( C \)-embedding

\[
\delta: \overline{\mathbb{B}^{n-2} \times B^2(r/2)} \to \mathbb{B}^n
\]

which is the identity on a neighbourhood of \( S^{n-3} \times B^2(r/2) \) and satisfies \( X \cap \mathbb{B}^n \subset \delta |\overline{\mathbb{B}^{n-2} \times 0} \). Let

\[
N = [\overline{\mathbb{R}^n} \setminus \mathbb{B}^n] \cup [\overline{\mathbb{B}^{n-2} \times B^2(r/2)}]
\]

and extend \( \delta \) over \( N \) by the identity. Then \( N \) is a neighbourhood of \( \overline{\mathbb{R}^n} \setminus \mathbb{B}^{n-2} \) in \( \mathbb{R}^n \) and \( \delta \) is a \( C \)-embedding. Moreover \( \delta \overline{\mathbb{R}^n} \setminus \mathbb{B}^{n-2} = \overline{\mathbb{R}^n} \setminus \mathbb{B}^{n-2} \cup \delta |\overline{\mathbb{B}^{n-2} \times 0} = X \). Thus, apart from the change of scenery from \( S^n \) to \( \mathbb{R}^n \), \( \delta \) is the required \( C \)-embedding. Thus it is sufficient to construct the \( C \)-embedding \( \delta \) as above.

Consider the TOP handle \( \gamma \): this is PL straight on \( \partial \overline{\mathbb{B}^{n-2} \times B^2(r)} = S^{n-3} \times B^2(r) \), being the inclusion there. Since \( n \neq 4 \) or \( 5 \), either \( n \leq 3 \) or \( n = 2 \neq 3 \) and \( n \geq 3 \). Using \([M]\) in the former case and \([KS]\) in the latter case, we may straighten \( \gamma \). More precisely, there is an isotopy \( \gamma_t: \overline{\mathbb{B}^{n-2} \times B^2(r)} \to \overline{\mathbb{B}^n} \) \( (0 \leq t \leq 1) \) with \( \gamma_0 = \gamma, \gamma_1 |\overline{\mathbb{B}^{n-2} \times B^2(r/2)} \) PL and \( \gamma_t = \gamma \) on a neighbourhood of

\[
[S^{n-3} \times B^2(r)] \cup [\overline{\mathbb{B}^{n-2} \times (B^2(r) \setminus B^2(s))}]
\]

for some \( s < r \). Let

\[
Y = (\overline{\mathbb{R}^n} \setminus h^{-1} \gamma |\overline{\mathbb{B}^{n-2} \times 0}] \cup h^{-1} \gamma_1 |\overline{\mathbb{B}^{n-2} \times 0}.
\]

It is claimed that \( Y \) is a locally \( C \)-flat TOP \( (n-2) \)-sphere in \( \mathbb{R}^n \) with \( \mathbb{R}^n \setminus Y \) homotopy equivalent to \( S^1 \).

(i) \( Y \) is a TOP \( (n-2) \)-sphere: this follows from the fact that \( \gamma |\overline{\mathbb{B}^{n-2} \times 0} \) is homeomorphic to \( \gamma_1 |\overline{\mathbb{B}^{n-2} \times 0} \) by a homeomorphism which is the identity on the boundary. Note that \( \gamma_1 |\overline{\mathbb{R}^n} \setminus \mathbb{B}^n \setminus B^n = \emptyset \), since \( \gamma_1 |\overline{\mathbb{R}^n} \setminus \mathbb{B}^n \times B^2(r) \) \( \subset B^n \), so that \( h^{-1} \gamma_1 |\overline{\mathbb{B}^{n-2} \times 0} \cap (\overline{\mathbb{R}^n} \setminus h^{-1} \gamma |\overline{\mathbb{B}^{n-2} \times 0}) = \emptyset \).

(ii) \( Y \) is locally \( C \)-flat: at points of \( \overline{\mathbb{R}^n} \setminus h^{-1} \gamma |\overline{\mathbb{B}^{n-2} \times 0} \) this is immediate; at points of \( h^{-1} \gamma_1 |\overline{\mathbb{B}^{n-2} \times 0} \) this follows from the fact that \( h^{-1} \gamma_1 |\overline{\mathbb{B}^{n-2} \times B^2(r/2)} \) is a \( C \)-embedding; at the remaining points of \( Y \), viz \( h^{-1} [S^{n-3} \times 0] \), it follows from
the fact that $\gamma$ and $\gamma_1$ are the inclusion on a neighbourhood of $S^{n-3} \times B^2(r)$, so that in a neighbourhood of $h^{-1}[S^{n-3} \times 0]$, $Y$ is still $\tilde{R}^{n-2}$.

(iii) $\tilde{R}^{n} \setminus Y$ is homotopy equivalent to $S^1$: in fact $\gamma_1$ provides an isotopy of $\tilde{R}^{n}$ throwing $\tilde{R}^{n-2}$ onto $Y$, so $Y$ is even topologically unknotted.

Now apply Theorem 3.3 of [GV] to $(\tilde{R}^{n}, Y)$. Since $n \neq 4$ or 6, there is a $C$-homeomorphism $f: (\tilde{R}^{n}, \tilde{R}^{n-2}) \rightarrow (\tilde{R}^{n}, Y)$. Moreover, because of the way the $C$-homeomorphism was constructed in [GV], we may assume that $f$ is the identity on a neighbourhood of $\tilde{R}^{n-2} \setminus h^{-1}\gamma[B^{n-2} \times 0]$. Let $\delta = hf^{-1}h^{-1}\gamma_1[\tilde{B}^{n-2} \times B^2(r/2)]$. We check the required properties of $\delta$.

(a) $\delta$ is a $C$-embedding:

$$\gamma_1[\tilde{B}^{n-2} \times B^2(r/2)] \subset \gamma_1[\tilde{B}^{n-2} \times B^2(r)] = \gamma[\tilde{B}^{n-2} \times B^2(r)] \subset hB^n,$$

so $h^{-1}\gamma_1[\tilde{B}^{n-2} \times B^2(r/2)]$ is a $C$-embedding as, therefore, is $f^{-1}h^{-1}\gamma_1[\tilde{B}^{n-2} \times B^2(r/2)]$. Making $r$ smaller if necessary, we can be sure that

$$f^{-1}h^{-1}\gamma_1[\tilde{B}^{n-2} \times B^2(r/2)] \subset B^n,$$

so that $\delta$ is a well-defined $C$-embedding.

(b) $\delta$ is the identity on a neighbourhood of $S^{n-3} \times \tilde{B}^2(r/2)$: this follows from the facts that $\gamma_1$ is the inclusion on such a set and $f$ is the identity on a neighbourhood of $\tilde{R}^{n-2} \setminus h^{-1}\gamma[B^{n-2} \times 0]$ hence on a neighbourhood of

$$h^{-1}\gamma[S^{n-3} \times B^2(r/2)] = h^{-1}\gamma_1[S^{n-3} \times B^2(r/2)]$$

provided $r$ is small enough.

(c) $X \cap \tilde{B}^n \subset \delta[\tilde{B}^{n-2} \times 0]$: in fact,

$$f^{-1}h^{-1}\gamma_1[\tilde{B}^{n-2} \times 0] = h^{-1}\gamma[\tilde{B}^{n-2} \times 0],$$

so

$$\delta[\tilde{B}^{n-2} \times 0] = \gamma[\tilde{B}^{n-2} \times 0] = X \cap \tilde{B}^n.$$

This completes the construction of $\delta$ and hence completes the proof of Theorem 1. \qed

**Proof of Theorem 2.** The proof of Theorem 2 is much the same as that of Theorem 1 but one uses instead $(C, g)$-encasing and [GV, Theorem 3.4] neither of which requires the restriction $n \neq 6$. \qed

**References**


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