ON THE VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS WITH A DEFICIENT VALUE

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1. Introduction

Let \( F \) be a family of functions meromorphic in the complex plane \( \mathbb{C} \) and \( S \) a subset of \( \mathbb{C} \). We call \( S \) a Picard set for \( F \) if every transcendental \( f \in F \) assumes every complex value with at most two exceptions infinitely often in \( \mathbb{C} - S \). We use the usual notation of the Nevanlinna theory and denote

\[
M(\delta) = \{ f : f \text{ meromorphic in } \mathbb{C}, \delta(\infty, f) \equiv \delta \}.
\]

Anderson and Clunie [1] have proved the following

Theorem A. Suppose that \( q \) and \( \delta \) are given with \( q > 1 \), \( 0 < \delta \leq 1 \). Then, if the complex sequence \( \{a_n\} \) satisfies

\[
(1) \quad \left| \frac{a_{n+1}}{a_n} \right| \equiv q, \quad n = 1, 2, \ldots
\]

there exist a constant \( K = K(q) \) such that, if

\[
(2) \quad \log \frac{1}{d_n} > K\delta^{-2} \log \frac{2}{\delta} (\log |a_n|)^2.
\]

the set

\[
(3) \quad S = \bigcup_{n=1}^{\infty} D_n,
\]

where \( D_n = \{ z : |z - a_n| < d_n \} \), is a Picard set for \( M(\delta) \).

We shall consider the dependence of the constant

\[
K(q, \delta) = K(q)\delta^{-2} \log \frac{2}{\delta}
\]

on \( \delta \) in the condition (2) and prove

Theorem 1. For any \( q > 1 \) there exists a constant \( K = K(q) \) depending only on \( q \), not on \( \delta \), such that Theorem A still holds if the condition (2) is replaced by

\[
(4) \quad \log \frac{1}{d_n} > K (\log |a_n|)^2.
\]

Furthermore, Anderson and Clunie [1] proved

Theorem B. Suppose that the sequence \( \{a_n\} \) satisfies (1) and

\[
\log \frac{1}{d_n} = K \frac{(\log |a_n|)^2}{\log q}.
\]

for some \( K > 1/2 \). Then \( S \), defined by (3), is a Picard set for entire functions.

We shall show that for any \( q > 1 \), the condition \( K > 1/2 \) here is the best possible, i.e. the constant \( K \) in (5) cannot be replaced by \( 1/2 \). We prove

Theorem 2. Let \( q > 1 \), \( a_n = (-1)^{n+1} q^n \), and

\[
\log \frac{1}{d_n} = \frac{(\log |a_n|)^2}{2 \log q}.
\]

Then the set \( S \), defined by (3), is not a Picard set for entire functions.

For other results on Picard sets we refer to [2], [4], and [5].

2. Lemmas

Matsumoto [3] has proved the following

Lemma C. There exists an absolute constant \( A > 0 \) such that if \( t \geq 3 \) and \( f \) is analytic in the annulus \( 1 < |z| < e^t \) and omits the values \( 0 \) and \( 1 \), then the spherical diameter of the image of \( |z| = e^{t/2} \) under \( f \) is at most \( Ae^{-t/2} \).

Let \( g \) be meromorphic in the annulus \( 1 < |z| < e^t \) and omit there three values, \( 0, 1 \) and \( c \). If \( c \neq \infty \), we set

\[
f(z) = \frac{(c-1)g(z)}{c(g(z)-1)}
\]

and applying Lemma C we get

Lemma 3. Let \( g \) be as above. There exists a constant \( A_c \) depending only on \( c \) such that the spherical diameter of the image of \( |z| = e^{t/2} \) under \( g \) is at most \( A_c e^{-t/2} \).

3. Proof of Theorem 1

Let \( S \) satisfy the hypotheses of Theorem 1 and let us suppose that \( f \) is meromorphic and non-rational in the plane such that \( \delta(\infty, f) = \delta > 0 \) and

\[
f^{-1}([0, 1, c]) \subset S \cup \{z: |z| = r_0\}
\]

for some \( r_0 > 0 \), \( 0 \neq c \neq 1 \).

Using Schottky's theorem if \( c = \infty \), and Lemma 1 of Anderson and Clunie [1] if \( c \neq \infty \), we see that \( |f(z)| > 5 \) on the circles \( \Gamma_n: |z| = q^{1/2} |a_n| \) and \( \gamma_n: |z| = \)
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$q^{-1/2}|a_n|$ if $n$ is sufficiently large. Applying Lemma 3 in the annulus $d_n <|z-a_n| < 1$ we note that, for all large values of $n$, either

(i) $|f(z)| \leq 3$ for any $z \in C_n = \{ \zeta : |\zeta - a_n| = \sqrt{d_n} \}$ or

(ii) $f(C_n) \subset U(a, 1/100) = \{ w : |w-a| < 1/100 \}$ for some $a$ satisfying $|a| < 4$.

Let $E_n$ be the region bounded by $\Gamma_n, \gamma_n$ and $C_n$. Let us suppose that the case (ii) happens. Then $f(E_n)$ is a region whose boundary is contained in

$$U(a, 1/100) \cup \{ w : |w| > 5 \},$$

and connecting $C_n$ to $\Gamma_n$ by a path we see that $f(E_n)$ contains at least one point from the set

$$\{ w : |w| < 5 \} - U(a, 1/100).$$

Therefore $f(E_n)$ contains the set $\{ w : |w| < 5 \} - U(a, 1/100)$, and we see that $f$ takes at least one of the values 0 and 1 in $E_n$. We are led to a contradiction and therefore the case (i) happens for all large $n$. Applying the minimum principle we see now that

(iii) $|f(z)| \equiv 3$

if $|z|$ is sufficiently large, say $|z| > q$, and $z$ lies outside the union of the discs $|\zeta - a_n| < \sqrt{d_n}$.

We denote by $B_1, B_2, \ldots$ positive constants depending only on $q$. We choose $B_1 > 0$ and a sequence $q < r_1 < r_2 < \ldots$ such that

$$er_n \equiv r_{n+1} < e^{q/8} r_n$$

and that the ring domain

$$r_n (1-B_1) - 1 < |z| < r_n (1+B_1) + 1$$

does not contain any of the points $a_p$. We set

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{1}{f(re^{i\theta})} \right| \frac{r_n^2 - r_2^2}{r_n^2 - 2r_n r \cos (\theta - \phi) + r_2^2} d\phi.$$

Then $u$ is harmonic in $|z| < r_n$, $u(0) = -m(r_n, \infty)$,

$$-u(z) \equiv \frac{101}{100} m(r_n, \infty)$$

on $|z| = r_n^{1/100} = t_n$ (if $n$ is large enough), and if $|a_p| < r_n$, then

$$-u(z) \equiv B_2 m(r_n, \infty)$$

on $C_p$. Let $C_p, \ p = s_n, s_n + 1, \ldots, k_n$, be those of the discs $C_p$ which satisfy

$$C_p \cap \{ z : t_n < |z| < r_n \} \neq \emptyset$$

and let $H_n$ be the bounded domain bounded by $|z| = t_n, |z| = r_n$ and $C_p, \ p = s_n, \ldots, k_n$. The function

$$g(z) = \sum_{p=s_n}^{k_n} \frac{\log (2r_n |z-a_p|)}{\log (2r_n \sqrt{d_p})}$$
is harmonic and positive in $H_n$ and $g(z) \equiv 1$ on the boundary component $C_p$. The function

$$w(z) = \frac{\log(r_n/|z|)}{\log(r_n/t_n)}$$

is harmonic and positive in $H_n$ and $w(z) = 1$ on $|z| = t_n$. Then the function

$$v(z) = u(z) + m(r_n, \infty) \left( \frac{101}{100} w(z) + B_5 g(z) \right),$$

which is harmonic in $H_n$, satisfies $v(z) \equiv 0 > -\log|f(z)|$ on the boundary circles $|z| = t_n$ and $C_p$, $p = s_n, s_n + 1, \ldots, k_n$, and $v(z) \equiv u(z) = -\log|f(z)|$ on $|z| = r_n$. From the maximum principle it follows that $-\log|f(z)| \equiv v(z)$ in $H_n$, and therefore

(iv) \[ -m(r_n-1, \infty) \equiv \frac{1}{2\pi} \int_0^{2\pi} v(r_n-1 e^{i\varphi}) \, d\varphi. \]

Because $u$ is harmonic in $|z| < r_n$ we have

$$\frac{1}{2\pi} \int_0^{2\pi} u(r_n-1 e^{i\varphi}) \, d\varphi = u(0) = -m(r_n, \infty).$$

The function $w$ satisfies

$$w(z) \equiv \frac{900}{792 \log r_n}$$

on $|z| = r_n-1$. It follows from (1) that $k_n \equiv B_3 \log r_n$, and from (4) we see that if $s_n \equiv p \leq k_n$, then

$$\log(2r_n/\sqrt{d_p}) > 1/2 \, K(\log t_n)^2 = \frac{K(\log r_n)^2}{20000}.$$

The sequence $r_n$ was chosen such that $\log(2r_n/|z-a_p|) \equiv B_4$ for any $p$ on $|z| = r_n-1$, and we get the estimate

$$g(z) \equiv \frac{B_5}{K \log r_n}$$

on $|z| = r_n-1$. Let $K = K(q)$ in the condition (4) be chosen such that

(v) \[ K = 100 B_2 B_5. \]

Then we get from (iv)

$$-m(r_n-1, \infty) \equiv m(r_n, \infty) \left( -1 + \frac{909}{792 \log r_n} + \frac{1}{100 \log r_n} \right),$$

and this implies that

$$m(r_n, \infty) \equiv m(r_n-1, \infty) \left( 1 + \frac{3}{2 \log r_n} \right)$$

for all large values of $n$, say for $n \geq n_0$. Therefore

$$m(r_{n_0+p}, \infty) \equiv m(r_{n_0}, \infty) \prod_{k=1}^{p} \left( 1 + \frac{3/2}{k \log r_{n_0}} \right) \equiv m(r_{n_0}, \infty) p^{3/2}.$$
and because \( p \geq \log r_{n_0 + p} \) we have \( m(r, \infty) = O(\log r)^{3/2} \). We assumed that \( \delta(\infty, f) \geq 0 \) and we get
\[
\text{vi) } T(r, f) = O(\log r)^{3/2}.
\]

Let \( z_k \) and \( b_k, \ k = 1, 2, \ldots \), be the zeros and poles of \( f \) and let \( f \) have \( u_n \) zeros and \( v_n \) poles in \( |z-a_n|<1/\|a_n\| \). We choose \( z \in C \) such that \( |z-b_k| \equiv \sqrt{d_n/v_n} \) for any \( k \), and applying the Poisson—Jensen formula with \( R = |a_n|\sqrt{q} \) we get
\[
\log 3 \equiv \log |f(z)|
\]
\[
\leq m(R, \infty) \frac{R+|z|}{R-|z|} + \sum_{|z_k| \leq R} \log \frac{|R(z-z_k)|}{|R^2 - \bar{z}_k z|} - \sum_{|b_k| \leq R} \log \frac{|R(z-b_k)|}{|R^2 - \bar{b}_k z|}
\]
\[
\leq O(T(R, f)) + (u_n-v_n) \log \sqrt{d_n+v_n} \log v_n + O(n(R, \infty) \log R).
\]

It follows from (vi) that \( n(r, \infty) = O(\log r)^{1/2} \), and we see now from (4) that
\[
(u_n-v_n)(\log R)^2 \leq O(\log R)^{3/2}.
\]

Therefore \( u_n \equiv v_n \) for all large values of \( n \), and we get
\[
n(r, 0) \equiv (1 + o(1)) n(r, \infty)
\]
for \( |a_n| + 1/|a_n| < r < |a_{n+1}| - 1/|a_{n+1}|, \ n \geq n_0, \). This implies that
\[
N(r, 0) \equiv (1 + o(1)) N(r, \infty)
\]
and therefore \( \delta(0, f) > 0 \). However, the growth condition (vi) guarantees that \( f \) has at most one deficient value. We are led to a contradiction and we see that if \( K=K(q) \) is chosen by (v), then any non-rational \( f \) for which \( \delta(\infty, f) > 0 \) takes at least one of the three values 0, 1 and \( c \) infinitely often in \( C-S \). By means of a linear transformation, we conclude now that any transcendental \( g \) with \( \delta(\infty, g) > 0 \) takes every value with at most two exceptions infinitely often in \( C-S \), and Theorem 1 is proved.

4. Proof of Theorem 2

Let \( q, a_n \) and \( d_n \) be as in Theorem 2. We shall consider the function
\[
f(z) = z^4 \prod_{n=1}^{\infty} (1 - z/a_n).
\]

If \( n \) is large we see easily that \( |f(z)| \geq 3 \) on the circle \( |z|=|a_n|q^{1/2} \). Therefore it follows from Rouché's theorem that \( f \) has exactly one 1-point on the ring domain \( |a_n|q^{-1/2} \leq |z| \leq |a_n|q^{1/2} \). We denote this 1-point by \( b_n \). In order to prove Theorem 2 it is sufficient to prove that \( b_n \) lies on the open segment \( I_n = (a_n-d_n, a_n+d_n) \). From the definition of \( f \) we see that \( f(a_n-d_n)f(a_n+d_n) < 0 \), and therefore it is sufficient to show that \( |f(a_n \pm d_n)| > 1 \).
Let \( x = a_n + d_n \) or \( x = a_n - d_n \). We denote by \( m_1, m_2, \ldots \) positive constants depending only on \( q \). We see easily that
\[
\left| \prod_{k=0}^{\infty} (1 - x/a_k) \right| \equiv m_1.
\]
We denote \( h(z) = \prod_{k=1}^{n-1} (1 - z/a_k) \). We have
\[
|h(x)| = \left( \frac{|x| + q^{-n+1}}{q^{-n+1}} \cdot \frac{|x| - q^{-n+2}}{q^{-n+2}} \right) \cdot \left( \frac{|x| + q^{-n-3}}{q^{-n-3}} \cdot \frac{|x| - q^{-n-4}}{q^{-n-4}} \right),
\]
where
\[
R_k(x) = (|x| + q^{-2k+1})(|x| - q^{-2k}) = x^2 + q^{-2k}((q-1)|x| - q^{-2k+1}) \equiv x^2
\]
if \( k \geq m_2 \) and \( R_k(x) \equiv m_3 x^2 \) for any \( k \). Therefore
\[
\log |h(x)| \equiv (n-1) \log |x| - \frac{1}{2} n (n-1) \log q - m_4,
\]
and we see that
\[
\log |f(x)| \equiv \log |d_n/a_n| + (n+3) \log |x| - \frac{1}{2} n (n-1) \log q - m_5.
\]
Here \( n \log q = \log |a_n|, \) \( \log |x| \equiv \log |a_n| - 1/|a_n| \) and
\[
\log |d_n/a_n| = -\frac{(\log |a_n|)^2}{2 \log q} - \log |a_n| = -\left( 1 + \frac{n}{2} \right) \log |a_n|,
\]
and we note that \( \log |f(x)| \equiv 2 \log |a_n| - m_6 > 0 \) for all large \( n \). This completes the proof of Theorem 2.

References


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