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ON THE VALUE DISTRIBUTION OF MEROMORPHIC FUNCTIONS WITH A DEFICIENT VALUE

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1. Introduction

Let F be a family of functions meromorphic in the complex plane C and S a subset of C. We call S a Picard set for F if every transcendental $f\in F$ assumes every complex value with at most two exceptions infinitely often in $C-S$. We use the usual notation of the Nevanlinna theory and denote

$$M(\delta) = \{f\colon f \text{ meromorphic in } C,\ \delta(\infty, f) \geqq \delta\}.$$

Anderson and Clunie [1] have proved the following

Theorem A. *Suppose that q and δ are given with $q>1$, $0<\delta\leqq 1$. Then, if the complex sequence $\{a_n\}$ satisfies*

$$\left|\frac{a_{n+1}}{a_n}\right| \geqq q, \tag{1}$$

$n=1, 2, \ldots,$ *there exist a constant $K=K(q)$ such that, if*

$$\log\frac{1}{d_n} > K\delta^{-2}\log\frac{2}{\delta}(\log|a_n|)^2, \tag{2}$$

the set

$$S = \bigcup_{n=1}^{\infty} D_n, \tag{3}$$

where $D_n=\{z\colon |z-a_n|<d_n\}$, is a Picard set for $M(\delta)$.

We shall consider the dependence of the constant

$$K(q, \delta) = K(q)\delta^{-2}\log\frac{2}{\delta}$$

on δ in the condition (2) and prove

Theorem 1. *For any $q>1$ there exists a constant $K=K(q)$ depending only on q, not on δ, such that Theorem* A *still holds if the condition* (2) *is replaced by*

$$\log\frac{1}{d_n} > K(\log|a_n|)^2. \tag{4}$$

Furthermore, Anderson and Clunie [1] proved

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Theorem B. *Suppose that the sequence* $\{a_n\}$ *satisfies* (1) *and*

$$\log\frac{1}{d_n} \geqq \frac{K(\log|a_n|)^2}{\log q}. \tag{5}$$

for some $K>1/2$. *Then* S, *defined by* (3), *is a Picard set for entire functions.*

We shall show that for any $q>1$, the condition $K>1/2$ here is the best possible, i.e. the constant K in (5) cannot be replaced by $1/2$. We prove

Theorem 2. *Let* $q>1$, $a_n=(-1)^{n+1}q^n$, *and*

$$\log\frac{1}{d_n} = \frac{(\log|a_n|)^2}{2\log q}. \tag{6}$$

Then the set S, *defined by* (3), *is not a Picard set for entire functions.*

For other results on Picard sets we refer to [2], [4], and [5].

2. Lemmas

Matsumoto [3] has proved the following

Lemma C. *There exists an absolute constant* $A>0$ *such that if* $t\geqq 3$ *and* f *is analytic in the annulus* $1<|z|<e^t$ *and omits the values* 0 *and* 1, *then the spherical diameter of the image of* $|z|=e^{t/2}$ *under* f *is at most* $Ae^{-t/2}$.

Let g be meromorphic in the annulus $1<|z|<e^t$ and omit there three values, 0, 1 and c. If $c\neq\infty$, we set

$$f(z) = \frac{(c-1)\,g(z)}{c(g(z)-1)}$$

and applying Lemma C we get

Lemma 3. *Let* g *be as above. There exists a constant* A_c *depending only on* c *such that the spherical diameter of the image of* $|z|=e^{t/2}$ *under* g *is at most* $A_c e^{-t/2}$.

3. Proof of Theorem 1

Let S satisfy the hypotheses of Theorem 1 and let us suppose that f is meromorphic and non-rational in the plane such that $\delta(\infty,f)=\delta>0$ and

$$f^{-1}(\{0,1,c\})\subset S\cup\{z\colon |z|<r_0\}$$

for some $r_0>0$, $0\neq c\neq 1$.

Using Schottky's theorem if $c=\infty$, and Lemma 1 of Anderson and Clunie [1] if $c\neq\infty$, we see that $|f(z)|>5$ on the circles Γ_n: $|z|=q^{1/2}|a_n|$ and γ_n: $|z|=$

$q^{-1/2}|a_n|$ if n is sufficiently large. Applying Lemma 3 in the annulus $d_n<|z-a_n|<1$ we note that, for all large values of n, either

(i) $|f(z)|\geqq 3$ for any $z\in C_n=\{\zeta\colon |\zeta-a_n|=\sqrt{d_n}\}$ or

(ii) $f(C_n)\subset U(a, 1/100)=\{w\colon |w-a|<1/100\}$ for some a satisfying $|a|<4$.

Let E_n be the region bounded by Γ_n, γ_n and C_n. Let us suppose that the case (ii) happens. Then $f(E_n)$ is a region whose boundary is contained in

$$U(a, 1/100)\cup\{w\colon |w|>5\},$$

and connecting C_n to Γ_n by a path we see that $f(E_n)$ contains at least one point from the set

$$\{w\colon |w|<5\}-U(a, 1/100).$$

Therefore $f(E_n)$ contains the set $\{w\colon |w|<5\}-U(a, 1/100)$, and we see that f takes at least one of the values 0 and 1 in E_n. We are led to a contradiction and therefore the case (i) happens for all large n. Applying the minimum principle we see now that

(iii) $$|f(z)|\geqq 3$$

if $|z|$ is sufficiently large, say $|z|>\varrho$, and z lies outside the union of the discs $|\zeta-a_n|<\sqrt{d_n}$.

We denote by $B_1, B_2, \ldots$ positive constants depending only on q. We choose $B_1>0$ and a sequence $\varrho<r_1<r_2<\ldots$ such that

$$er_n\leqq r_{n+1}<e^{9/8}r_n$$

and that the ring domain

$$r_n(1-B_1)-1<|z|<r_n(1+B_1)+1$$

does not contain any of the points a_p. We set

$$u(re^{i\alpha})=\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{1}{f(r_ne^{i\varphi})}\right|\frac{r_n^2-r^2}{r_n^2-2r_nr\cos(\alpha-\varphi)+r^2}\,d\varphi.$$

Then u is harmonic in $|z|<r_n$, $u(0)=-m(r_n,\infty)$,

$$-u(z)\leqq\frac{101}{100}m(r_n,\infty)$$

on $|z|=r_n^{1/100}=t_n$ (if n is large enough), and if $|a_p|<r_n$, then

$$-u(z)\leqq B_2m(r_n,\infty)$$

on C_p. Let C_p, $p=s_n, s_n+1, \ldots, k_n$, be those of the discs C_p which satisfy

$$C_p\cap\{z\colon t_n<|z|<r_n\}\neq\emptyset$$

and let H_n be the bounded domain bounded by $|z|=t_n$, $|z|=r_n$ and C_p, $p=s_n, \ldots, k_n$. The function

$$g(z)=\sum_{p=s_n}^{k_n}\frac{\log(2r_n/|z-a_p|)}{\log(2r_n/\sqrt{d_p})}$$

is harmonic and positive in H_n and $g(z)\geqq 1$ on the boundary component C_p. The function

$$w(z)=\frac{\log(r_n/|z|)}{\log(r_n/t_n)}$$

is harmonic and positive in H_n and $w(z)=1$ on $|z|=t_n$. Then the function

$$v(z)=u(z)+m(r_n,\infty)\left(\frac{101}{100}w(z)+B_2 g(z)\right),$$

which is harmonic in H_n, satisfies $v(z)\geqq 0>-\log|f(z)|$ on the boundary circles $|z|=t_n$ and C_p, $p=s_n, s_n+1, \ldots, k_n$, and $v(z)\geqq u(z)=-\log|f(z)|$ on $|z|=r_n$. From the maximum principle it follows that $-\log|f(z)|\leqq v(z)$ in H_n, and therefore

$$\text{(iv)} \qquad -m(r_{n-1},\infty)\leqq\frac{1}{2\pi}\int_0^{2\pi}v(r_{n-1}e^{i\varphi})\,d\varphi.$$

Because u is harmonic in $|z|<r_n$ we have

$$\frac{1}{2\pi}\int_0^{2\pi}u(r_{n-1}e^{i\varphi})\,d\varphi=u(0)=-m(r_n,\infty).$$

The function w satisfies

$$w(z)\leqq\frac{900}{792\log r_n}$$

on $|z|=r_{n-1}$. It follows from (1) that $k_n\leqq B_3\log r_n$, and from (4) we see that if $s_n\leqq p\leqq k_n$, then

$$\log(2r_n/\sqrt{d_p})>\frac{1}{2}K(\log t_n)^2=\frac{K(\log r_n)^2}{20\,000}.$$

The sequence r_n was chosen such that $\log(2r_n/|z-a_p|)\leqq B_4$ for any p on $|z|=r_{n-1}$, and we get the estimate

$$g(z)\leqq\frac{B_5}{K\log r_n}$$

on $|z|=r_{n-1}$. Let $K=K(q)$ in the condition (4) be chosen such that

$$\text{(v)} \qquad K=100B_2B_5.$$

Then we get from (iv)

$$-m(r_{n-1},\infty)\leqq m(r_n,\infty)\left(-1+\frac{909}{792\log r_n}+\frac{1}{100\log r_n}\right),$$

and this implies that

$$m(r_n,\infty)\leqq m(r_{n-1},\infty)\left(1+\frac{3}{2\log r_n}\right)$$

for all large values of n, say for $n\geqq n_0$. Therefore

$$m(r_{n_0+p},\infty)\leqq m(r_{n_0},\infty)\prod_{k=1}^{p}\left(1+\frac{3/2}{k+\log r_{n_0}}\right)\leqq m(r_{n_0},\infty)p^{3/2}.$$

and because $p \leqq \log r_{n_0+p}$ we have $m(r, \infty) = O(\log r)^{3/2}$. We assumed that $\delta(\infty, f) > 0$ and we get

(vi) $$T(r,f) = O(\log r)^{3/2}.$$

Let z_k and b_k, $k = 1, 2, \dots$, be the zeros and poles of f and let f have u_n zeros and v_n poles in $|z - a_n| < 1/|a_n|$. We choose $z \in C_n$ such that $|z - b_k| \geqq \sqrt{d_n}/v_n$ for any k, and applying the Poisson—Jensen formula with $R = |a_n|\sqrt{q}$ we get

$$\begin{aligned}
\log 3 &\leqq \log |f(z)| \\
&\leqq m(R, \infty) \frac{R+|z|}{R-|z|} + \sum_{|z_k|<R} \log \left| \frac{R(z-z_k)}{R^2 - \bar{z}_k z} \right| - \sum_{|b_k|<R} \log \left| \frac{R(z-b_k)}{R^2 - \bar{b}_k z} \right| \\
&\leqq O\big(T(R,f)\big) + (u_n - v_n) \log \sqrt{d_n} + v_n \log v_n + O\big(n(R, \infty) \log R\big).
\end{aligned}$$

It follows from (vi) that $n(r, \infty) = O(\log r)^{1/2}$, and we see now from (4) that

$$(u_n - v_n)(\log R)^2 \leqq O(\log R)^{3/2}.$$

Therefore $u_n \leqq v_n$ for all large values of n, and we get

$$n(r, 0) \leqq \big(1 + o(1)\big) n(r, \infty)$$

for $|a_n| + 1/|a_n| < r < |a_{n+1}| - 1/|a_{n+1}|$, $n \geqq n_0$. This implies that

$$N(r, 0) \leqq \big(1 + o(1)\big) N(r, \infty)$$

and therefore $\delta(0, f) > 0$. However, the growth condition (vi) quarantees that f has at most one deficient value. We are led to a contradiction and we see that if $K = K(q)$ is chosen by (v), then any non-rational f for which $\delta(\infty, f) > 0$ takes at least one of the three values 0, 1 and c infinitely often in $C - S$. By means of a linear transformation, we conclude now that any transcendental g with $\delta(\infty, g) > 0$ takes every value with at most two exceptions infinitely often in $C - S$, and Theorem 1 is proved.

4. Proof of Theorem 2

Let q, a_n and d_n be as in Theorem 2. We shall consider the function

$$f(z) = z^4 \prod_{n=1}^{\infty} (1 - z/a_n).$$

If n is large we see easily that $|f(z)| > 3$ on the circle $|z| = |a_n| q^{1/2}$. Therefore it follows from Rouché's theorem that f has exactly one 1-point on the ring domain $|a_n| q^{-1/2} \leqq |z| \leqq |a_n| q^{1/2}$. We denote this 1-point by b_n. In order to prove Theorem 2 it is sufficient to prove that b_n lies on the open segment $I_n = (a_n - d_n, a_n + d_n)$. From the definition of f we see that $f(a_n - d_n) f(a_n + d_n) < 0$, and therefore it is sufficient to show that $|f(a_n \pm d_n)| > 1$.

Let $x=a_n+d_n$ or $x=a_n-d_n$. We denote by $m_1, m_2, \ldots$ positive constants depending only on q. We see easily that

$$\left|\prod_{k=n+1}^{\infty}(1-x/a_k)\right| \geqq m_1.$$

We denote $h(z)=\prod_{k=1}^{n-1}(1-z/a_k)$. We have

$$|h(x)| = \left(\frac{|x|+q^{n-1}}{q^{n-1}}\cdot\frac{|x|-q^{n-2}}{q^{n-2}}\right)\cdot\left(\frac{|x|+q^{n-3}}{q^{n-3}}\cdot\frac{|x|-q^{n-4}}{q^{n-4}}\right)\cdots,$$

where

$$\begin{aligned} R_k(x) &= (|x|+q^{n-2k+1})(|x|-q^{n-2k}) \\ &= x^2+q^{n-2k}\big((q-1)|x|-q^{n-2k+1}\big) \geqq x^2 \end{aligned}$$

if $k\geqq m_2$ and $R_k(x)\geqq m_3x^2$ for any k. Therefore

$$\log|h(x)| \geqq (n-1)\log|x| - \frac{1}{2}n(n-1)\log q - m_4,$$

and we see that

$$\log|f(x)| \geqq \log|d_n/a_n| + (n+3)\log|x| - \frac{1}{2}n(n-1)\log q - m_5.$$

Here $n\log q=\log|a_n|$, $\log|x|\geqq\log|a_n|-1/|a_n|$ and

$$\log|d_n/a_n| = -\frac{(\log|a_n|)^2}{2\log q} - \log|a_n| = -\left(1+\frac{n}{2}\right)\log|a_n|,$$

and we note that $\log|f(x)|\geqq 2\log|a_n|-m_6>0$ for all large n. This completes the proof of Theorem 2.

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